Dynamic Assignment of Objects to Queuing Agents with Private Values *

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Abstract

This paper analyzes the optimal assignment of objects which arrive sequentially to agents organized in a waiting list. Applications include the assignment of social housing and organs for transplants. We analyze the optimal design of probabilistic queuing disciplines, punishment schemes, the optimal timing of applications and information releases. We consider three efficiency criteria: the vector of values of agents in the queue, the probability of misallocation and the expected waste. Under private values, we show that the first-come first-served mechanism dominates a lottery according to the first two criteria but that lottery dominates first-come first-serve according to the last criterion. Punishment schemes accelerate turnover in the queue at the expense of agents currently in the waiting list, application schemes with commitment dominate sequential offers and information release always increases the value of agents at the top of the waiting list.

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1 Introduction

1.1 Dynamic assignment of objects to queuing agents

This paper analyzes the dynamic assignment of objects to agents organized in a closed waiting list. Objects arrive over time, and each time a new object becomes available it is offered to agents according to a fixed sequence. Each agent in the order decides whether to accept the object or not. If the object is rejected by all agents in the queue, it is wasted. If one agent accepts the object, a new agent enters the queue and all agents following the agent who picked the object move up in the waiting list. Agents have a common additive waiting cost. Our objective in this paper is to study how different probabilistic queuing disciplines, different punishment schemes after a rejection, and different policies of information release affect the behavior of agents in the waiting list, their expected welfare, the turnover in the queue and the amount of waste.

The situation we study arises whenever there exists a huge imbalance between demand and supply for an object and monetary transfers cannot be used to match the two sides of the market. Examples include the assignment of social housing, of deceased donor organs for transplant, or of spots in daycare. In all these examples, objects are heterogeneous – apartments which become available have different sizes and locations, organs are harvested on deceased donors of different ages and health conditions, daycares have different staffs and amenities. Agents have preferences over the heterogeneous objects which are assumed to be uncorrelated over time. We also suppose that agents have private values, reflecting idiosyncratic preferences over the different objects.

Because objects are heterogeneous, when an agent receives an offer, he faces an optimal stopping problem. Should he accept the current object or wait to receive a better object in the future? The answer to this question depends on the characteristics of the assignment system, like the queuing discipline, punishment scheme after a rejection or information released to the agents which affect the continuation value. The features of the assignment system also influence the total number of assignments. If the queuing discipline gives an advantage to agents according to their waiting time, agents at the top of the waiting list have a higher continuation value and hence are more likely to be selective and to reject current proposals. This may result in a sequence of rejections, leading to waste for objects with a short lifetime like organs and vacancies and delays for durable objects like apartments.[1]

[1]See the article in the New York Times dated September 20, 2012

The design of optimal assignment schemes must thus balance the values of agents in the waiting list with the inefficiencies resulting from the waste of objects.

Existing assignment mechanisms fall into three categories: first-come first serve mechanism with different priority groups, scoring rules and lotteries. First-come first serve (FCFS) mechanisms with priority groups divide applicants into different priority groups and rank each applicant within a priority according to the date of entry in the waiting list. When an object becomes available, it is first assigned to a priority group, either through absolue priorities – priority groups are ranked and the object is always proposed to priority groups in the same order, or through a rotating or quota scheme – different priority groups are offered the object according to a fixed rotating sequence or the objects are proportionally divided across priority groups. Within each priority group, objects are offered to applicants in sequence according to their order in the waiting list. In the United States, the 1964 Civil Rights Act requires public housing authorities to assign units to applicants according to a Tenant Selection and Assignment Plan (TSAP) which functions as a FCFS mechanism with priority groups. [2] FCFS mechanisms with priority groups are also used to assign hearts and intestines for transplant.[3]. Scoring rules assign points to applicants and establish priorities among applicants according to their number of points. Scoring mechanisms are used to assign council housing in England and Wales and, since the fall of 2014, in Paris.[4] In both cases, waiting time is used as a tie-breaking rule to distinguish between applicants with the same score. Scoring rules are also used to assign deceased donors' kidneys, livers and lungs. For the allocation of kidneys, waiting time

for a discussion of waste in the assignment of kidneys for transplants http://www.nytimes.com/2012/09/20/health/transplant-experts-blame-allocation-system-for-discarding-kidneys.html

[2]The definitions of priority groups and specific assignment of units across priority groups differs across public housing authorities. For example, in the TSAP of the New York Housing Authority, apartments rotate across five large classes of priority groups while the Chicago Housing Authority assigns apartments across priority groups using absolute priorities. The definitions of priority groups are similar but not identical in the two TSAPs. General guidelines for TSAPs of the department of Housing and Urban Development are given in www.hud.gov/offices/pih/rhiip/phguidebooknew.pdf. Up-to-date descriptions of the TSAPs in New York and Chicago can be found at http://www.nyc.gov/html/nycha and http://www.thecha.org/.

[3]See the OPTN Policies 6.5.B and 7.3.A updated 6/24/2015 available at http://optn.transplant.hrsa.gov.

[4]See https://www.gov.uk/council-housing for council housing in England and Wales and http://www.paris.fr/services-et-infos-pratiques/demander-un-logement-social-37 for social housing in Paris.

has a major weight in the computation of the score. In the case of liver and lung transplants, waiting time is only used to break ties among applicants with the same score.[5] Finally, lotteries are used in New York to allocate subsidized housing managed by private and nonprofit developers and listed by NYC Housing connect. Eligible applicants to affordable housing units are chosen by a uniform random draw. [6].

1.2 Overview of the main results

The main result of the paper compares the value of all agents in the waiting list under the FCFS and the lottery schemes. We show that *all agents in the waiting list* including the last one prefer the FCFS scheme to the lottery. This surprising result rests on the following intuition. We first observe that conditional on the fact that the object is picked by him or any agent with higher rank, the expected number of waiting periods for any agent is independent of the queuing discipline. Any agent at rank i will on average wait i periods before obtaining the object. Note that the FCFS maximizes the probability that the object be assigned to agents at higher ranks while the lottery minimizes this probability. Second, we note that the expected value of the object obtained by the agent is the average of the values picked by agents at higher ranks. If the equilibrium behavior results in more selectivity for agents at higher ranks, the value of the objects picked by the agents increases. We prove that agents are more selective under FCFS than under the lottery when the set of values are discrete and for two-agents queuing systems when the set of values is continuous.

We use the theoretical model to discuss other features of assignment rules. Punishment schemes are used in the assignment of social housing to prevent continuous rejections.[7] The guidelines of the department of Housing and Urban Development indicate that applicants who reject apartments without good cause should be taken off the waiting list. In Paris, an applicant who refuses an offer is kept out of the assignment process for six months and regains his rank on the waiting list after this waiting period. We observe that any punishment scheme reduces the value of agents in the queue, making them less selective. An alternative to the sequential offer mechanism is to ask agents to apply and commit ex ante to accepting objects after they observe their value. Housing Connect in New York, most councils in England and

[5]See the OPTN Policies 8.5.D, 9.6.D and 10.4.A updates 6/24/2015 available at http://optn.transplant.hrsa.gov.

[6]See https://a806-housingconnect.nyc.gov

[7]Punishment schemes are not used in the assignment of organs for transplants.

the city of Paris have adopted the latter system, implementing a website where agents apply for specific housing units. This alternative system leads agents to accept offers based on an ex ante expectation of their continuation value rather than an interim expectation after they observe that they have been offered the object. This change in expected continuation values makes agents more selective and increases their values. Finally, we consider the effect of information release about an agent's rank in the sequence of offers and show that this information release increases the value of the top agent in the waiting list but results in ambiguous effects on the other agents.

Our analysis also shows the existence of a tension between the values of agents in the list (the "insiders") and the turnover in the queue which allows "outsiders" to join the waiting list. The FCFS maximizes the value of insiders, but as a consequence induces highly selective behavior. By making agents very selective, the FCFS mechanism also minimizes the probability that an object be allocated to an agent when another agent in the waiting list values it more. But this selectivity comes at a price, as it results in a high fraction of objects being wasted, slowing down the turnover in the queue, and resulting in a loss for agents who are not currently in the waiting list. In order to accelerate turnover in the queue, the mechanism must instead induce low selectivity. The lottery then stands out as the mechanism to be used, as it results in low values for the insiders, and a high rate of acceptance of the objects.

We would like to mention at the outset some limitations of the model. First, we consider closed waiting lists and do not allow for stochastic entry and exit in the waiting list. By fixing the size of the waiting list, we greatly simplify the analysis and focus on the dynamics induced by endogenous acceptance decisions rather than exogenous entry and exit. In applications, the number of agents to which any object can be offered is typically very low. Given their short lifetime, organs can only be offered to a small number of patients. Housing units are offered to a small number of applicants – the New York TSAP for example specifies the length of the queue as a function of the number of available units and the recent history of acceptances and rejections. Hence we believe that the assumption of a closed waiting list reflects real assignment systems and is a reasonable approximation. Second, we assume that agents' values are uncorrelated across time and that all agents have known and identical waiting costs (except for one variant of the model where we study the effect of two known waiting costs). The absence of correlation of values across times seems a good approximation in the case of organ transplant, but is probably less compelling in the social housing application where agents have persistent geographic preferences. However, if we consider waiting lists restricted to specific projects or small geographic areas, the assumption of uncorrelated values becomes more plausible. Third, we restrict attention to private values assuming away correlation across agents. The private values model captures heterogeneity across agents, allowing for idiosyncratic preferences over the objects, whereas the common values model considered in the earlier literature imposes homogeneity across agents, Third, we focus attention on additive waiting costs. This assumption is made both for tractability and because we believe that in applications like social housing and organ transplant, agents experience flow utilities every period, before and after the assignment of the object. The additive waiting cost can be interpreted as the difference between the utility flow associated to the lowest quality object and the utility flow without the object.

1.3 Related literature

The assignment policies for social housing and the management of waiting lists for organs have long been the object of attention in the operations research literature and more recently been discussed in the economics literature on dynamic assignment mechanisms. Kaplan (1986) and (1987) studies tenant assignment policies as FCFS mechanisms with priority groups computing the expected probability of assignment and expected waiting time under different policies. In a similar vein, Zenios (1999) and Zenios, Chertow and Wein (2000) model the kidney transplant waiting list as a FCFS mechanisms with priority groups with a random assignment of organs across policy groups and computes expected waiting times and the expected fraction of agents receiving a kidney as a function of the random assignment policy.

Su and Zenios (2004), (2005) and (2006) and Su, Zenios and Chertow (2004) explicitly introduce patient's choice in the queuing model for kidney transplants. Su and Zenios (2005) explicitly compute the optimal assignment policy for a fixed population of heterogeneous patients, while Su, Zenios and Chertow (2004) simulate the assignments under different policies respecting the individual rationality of patients. Su and Zenios (2006) develop a mechanism design model to take into account the incentive constraints of patients who have private information about the value of kidneys. Closest to our analysis, Su and Zenios (2004) study the effect of the queueing discipline on the assignment of kidneys in a queueing model where agents can reject the offer. In their model, they show that the Last Come First Serve queuing discipline implements the socially optimal outcome and dominates the FCFS mechanism which results in excessive waste. The optimality of the LCFS mechanism is due to the fact that when agents are homogeneous, the agent who enters the system internalizes perfectly the externalities he imposes on the other agents in the LCFS mechanism. (See also Hassin (1985)). Su and Zenios (2004) point out, as we do, the existence of a tension between absolute priorities given to insiders and the negative externalities this behavior imposes on outsiders. Finally, Su and Zenios (2004) consider a family of queuing disciplines, which are convex combinations of LCFS and FCFS mechanisms and show that an increase in the weight on the FCFS rule results in more selective behavior. Our analysis differs from Su and Zenios (2004) in several key dimensions. First, and most importantly, we analyze he case of heterogeneous agents with independent values whereas they assume homogeneous agents. Second, we consider a closed waiting list whereas their analysis rests on the assumption that the waiting list (understood as the number of agents to whom the object may be offered) evolves over time. Third, we consider different efficiency criteria. They focus on a single measure of social welfare – the expected sum of values of all (homogeneous) agents in the waiting list, whereas we distinguish between the values of agents at different ranks in the waiting list, the expected probability of misallocation and the expected probability of waste.

Recent contributions in economics on the assignment of objects to queuing agents with a specific focus on social housing have been proposed by Leshno (2014), Schummer (2015) and Thakral (2015). Leshno (2014) studies dynamic allocation of objects to queueing agents when agent's preferences are unknown. He shows that, in the absence of transfers, a dynamic allocation mechanism where agents are placed in a priority buffer queue is incentive compatible and furthermore that it is optimal to choose uniformly among agents in the buffer queue. While his study is inspired by the same applications to social housing and organ transplant as ours, the two papers differ in many respects. First, in Leshno (2014), agent's preferences are perfectly correlated across time – some agents prefer to live in the North, other in the South, and the mechanism is designed to elicit this persistent type. In our model, there is no persistence of types and hence no information to be elicited from agents. Second, Leshno (2014) only considers two types of objects – there is no vertical quality differentiation – and assumes that there is always an agent who is assigned the object. Hence, there is no waste in his model, and the efficiency discussion is limited to one criterion: the probability of misallocation.

Schummer (2015) considers a model with homogeneous agents receiving heterogeneous objects over time as in Su and Zenios (2004). He considers a planner who can make arbitrary changes in agents' continuation values (by "influencing" them to accept objets of a given value) and analyzes the effect of influence over the values of agents in the waiting list. He shows that all agents are adversely affected by the intervention of the planner, echoing our result that the FCFS mechanism dominates any other mechanism for any agent in the queue. He also highlights, as we do, the tension between the value of agents in the waiting list and the waste of objects. Finally, Schummer (2015) extends the analysis by considering risk-averse agents who are harmed by uncertainty about the waiting time. He shows that an outside party whose objective is to minimize the variance of waiting times will typically prefer to exert influence so that all agents are teated symmetrically, a result which relates to our finding that the lottery minimizes asymmetries in the values of agents in the waiting list.

Thakral (2015)'s analysis is motivated by the rules implemented by public housing authorities. He proposes a new assignment mechanism – the multiple waitlist procedure– , by which agents can choose to reject an object and be placed on a waiting list for a different object. He shows that this mechanism satisfies strategy-proofness, efficiency and the absence of justified envy. Thakral (2015)'s model differs from ours in several dimensions: he assumes that preferences over objects are fixed and that uncertainty relates to the time at which any object becomes available rather than the values of objects. He does not explicitly consider waiting costs dynamically incurred by the agents, but measures the cost of waiting by the maximal number of periods an agent is willing to wait to get into his preferred building.

Finally, our work belongs more distantly to the emerging literature on dynamic matching models, enriching assignment mechanisms with dynamic considerations. Other papers in this literature include Abdulkadiroglu and Loerscher (2007), Unver (2010), Bloch and Houy (2012), Bloch and Cantala (2013) , Kurino (2014), Kennes, Monte, Tumennasan (2014) , Akbarpour, Li, Gharan and Shayan (2014) for models without transfers and Gerkshov and Moldovanu (2009a), (2009b), and Dizdar, Gerkshov and Moldovanu (2011) for models with transfers.

1.4 Contents of the paper

The remainder of the paper unfolds as follows. We present our model in Section 2. In Section 3, we characterize the equilibrium of the discrete model with two agents, illustrating the main forces at work in our model. Section 4 extends the analysis to arbitrary queues in the discrete model. In Section 5, we analyze the continuous model. Section 6 contains extensions of the two-agent model to heterogeneous waiting costs, mechanisms with prior applications, with eviction probabilities and information release about the rank in the sequence. We conclude and give directions for future research in Section 7. All proofs are collected in the Appendix.

2 The Model

2.1 Queues, values and waiting costs

We consider a society with an infinite number of agents, organized in a closed waiting list of size n. We let $i = 1, 2, ..., n$ denote the rank of agents in the waiting list. Time is discrete, and at each period $t = 1, 2, ...$, a new object becomes available. Agents in the waiting list draw a value for the object, $\theta \in \Re$. This value is observed privately by the agent, but not by the other agents nor by the planner. In the discrete model, θ can only take on two values, $\theta \in \{0, 1\}$. In the continuous model, θ is taken from a continuous distribution F with support $[\underline{\theta}, \overline{\theta}]$.

We assume that each object is different, and that there is no persistence in agents' values. The values $\theta_t, \theta_{t'}$ drawn by an agent for the objects available at periods t and $t' \neq t$ are uncorrelated. At any period t, the values drawn by the different agents are independent. Each time an agent waits in the queue, he incurs an additive cost $c > 0$. Assuming that the reservation utility of an agent outside the queue is sufficiently low, we can guarantee that individual rationality constraints are satisfied, so that agents always have an incentive to enter the waiting list.

2.2 Probabilistic queuing disciplines

We let i=1,2,,n denote the rank of agents in the waiting list : the longer the time in the queue, the lower the ranking of an agent. A queuing discipline selects a sequence of agents according to their ranking in the waiting list. A probabliistic queuing discipline assigns a probability distribution p over the finite set R of all $n!$ sequences of agents in the waiting list. We denote by $\rho : N \to N$ a typical sequence in R. For practical reasons, we focus attention on probabilistic queuing disciplines which respect the ranking of agents in the waiting list and do not give higher weight to agents with higher rank.[8] More precisely, we assume that the probabilistic queuing discipline satisfies the following condition:

Assumption 2.1 *For any two agents $i < j$, and any orders ρ, ρ' such that $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) \geq p(\rho')$.*

[8]Note that an agent with higher rank in the waiting list has less priority in our model.

Assumption 2.1 states that, whenever two agents i and j are permuted in the sequences ρ and ρ', then the sequence in which the agent with the lowest rank is chosen first is picked with a probability at least as large as the sequence in which he is chosen last. In the special case where there are only two agents in the waiting list, R only contains two sequences $\rho_1 = 1, 2$ and $\rho_2 = 2, 1$ and assumption 2.1 reduces to: $p(\rho_1) \geq p(\rho_2)$. In general, we can define a partial order in the set of probability distributions satisfying assumption 2.1, $\mathcal{P}$, by letting $p \succeq p'$ if and only if, for any two agents $i < j$, any any sequences ρ, ρ' such that $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) - p(\rho') \geq p'(\rho) - p'(\rho')$. The set of probability distributions satisfying assumption 2.1, $\mathcal{P}$, is a complete lattice and admits a minimal and maximal element. The maximal element is the FCFS mechanism where all probability weight is placed on the order $\hat{\rho}$, where $\hat{\rho}(i) = i$ for all i, that we denote $\hat{p}$. The minimal element is the *lottery* where all sequences ρ in R are chosen with equal probability, $p(\rho) = \frac{1}{n!}$ that we denote $\tilde{p}$.

2.3 Agents' strategies and values

Using the probability distribution $p \in \mathcal{P}$, the planner picks a sequence ρ of agents to whom the object is proposed. If agent i is proposed an object of value θ and accepts it, he collects the value θ and leaves the waiting list. All other agents in the waiting list incur the cost c, a new agent enters the waiting list at position n and all agents whose rank is higher than i move up one position in the waiting list. If no agent accepts the object, the object is wasted, all agents incur the waiting cost c, keep their rank in the waiting list, and no new agent is allowed to enter the waiting list.

In the benchmark model, we suppose that agents are not given any information in their order in the sequence ρ. A Markovian strategy for agent i specifies his acceptance rule for the object of value θ as a function of his rank in the waiting list.

In the discrete model, agent i always accepts the object with value 1. Hence, the only choice of agent i is whether he accepts the object with value 0 or not. A Markovian strategy for agent i is then the probability $q(i) \in [0, 1]$ that he accepts the object with value 0. With this notation, we write the value of agent i as:

$$\begin{aligned} V(i) &= \Pr[\text{object accepted by } j < i](V(i-1) - c) \\ &\quad + \Pr[\text{agent } i \text{ is proposed the object}](\Pr[\theta_i = 1] \\ &\quad + (1 - \Pr[\text{object accepted by } j \leq i])(V(i) - c). \end{aligned}$$

This expression distinguishes between three possible outcomes: either the object is picked by an agent with lower rank than i, and i moves up one position in the waiting list, or agent i picks the object and receives value 1 with probability $\Pr[\theta_i = 1]$, or the object is not picked or picked by an agent with higher rank than i, and i remains in the same position in the waiting list.

In the continuous model, a Markovian strategy for agent i is a threshold value $\hat{\theta}_i$ such that agent i accepts any object of value $\theta \geq \hat{\theta}_i$ and rejects any object of value $\theta < \hat{\theta}_i$. The value of agent i is then given by

$$\begin{aligned} V(i) &= \Pr[\text{object accepted by } j < i](V(i-1) - c) \\ &\quad + Pr[\text{agent } i \text{ is proposed the object}] \int_{\hat{\theta}_i}^{\overline{\theta}} \theta dF(\theta) \\ &\quad + (1 - \Pr[\text{object accepted by } j \leq i](V(i) - c). \end{aligned}$$

Notice that there is a clear distinction between the optimal strategies of agents in the FCFS mechanism and in any other probabilistic queuing discipline. In the FCFS mechanism, player 1's problem is a classical unconstrained optimal stopping problem which does not depend on the behavior of other agents in the waiting list. Given agent 1's threshold, agent 2's threshold can be computed as the solution of an optimal stopping problem, etc.. In the FCFS mechanism, there is a unique vector of optimal strategies which can be computed as the solution of a recursive system of individual optimal stopping problems. By contrast, for any other probabilistic queuing discipline, the value of an agent depends on the strategies of other agents in the waiting list. Agents are playing a game against other agents in the waiting list, and the behavior of agents is characterized by a Markovian equilibrium of the game. Equilibrium is no longer guaranteed to be unique and cannot be computed as the solution to a recursive system.

2.4 Efficiency criteria

We consider a society with a varying population – agents enter and leave the waiting list over time – so that there is no obvious efficiency criterion we can apply to rank assignment mechanisms. Instead, we define three different criteria which capture different facets of the problem. We first consider the *vector of values of agents in the waiting list*, $\mathbf{V} = (V(1), V(2), ..., V(n))$. This criterion captures the welfare of insiders – agents who are currently active in the waiting list. We compare two vectors of values using Pareto dominance.

This is a strong criterion – stronger than the utilitarian criterion used by Su and Zenios (2004) – but it turns out that this criterion can be applied to compare assignment mechanisms in our model. (We note that this is also the criterion used in Schummer (2015)). Our second criterion focusses on the static efficiency of the assignment mechanism. Given that monetary transfers are not allowed and that the assignment mechanism uses a priority rule which is unrelated to the values of objects, the assignment mechanism may result in a (static) misallocation. The object may be picked by an agent i whereas there exists another agent j in the waiting list who would have accepted the object and such that $\theta_j > \theta_i$. We measure this misallocation loss by the expected probability that the object is given to an agent i when there exists another agent j who accepts the object and for whom $\theta_j > \theta_i$. Our third criterion is the probability that the object is rejected by all agents in the waiting list. When an object is rejected by all agents, it is wasted and no new agent is allowed to enter the queue. The expected waste, measured by the probability that any object is rejected by all agents, captures the speed at which the queue is served, and the welfare of outsiders – agents who have not yet entered the waiting list.

3 The two-agent discrete model

We start the analysis with the simple case of a two-agent waiting list and discrete values. This simple case will help us introduce the main results of the paper in a setting where Markovain equilibria can easily be computed and illustrated. When the waiting list only consists of two agents, under Assumption 2.1, a probabilistic queueing discipline is characterized by a single parameter $p \in [\frac{1}{2}, 1]$ denoting the probability that the order $\rho_1 = 1, 2$ is chosen.

3.1 Private values

We first consider the case of private values. Let π denote the (independent) probability that the high value is drawn by any of the two players. We restrict attention to pure strategies and characterize the Markovian equilibria in pure strategies. We say that a player is selective if he rejects object 0. We first note that, whenever $p \geq \frac{1}{2}$, there cannot be an equilibrium where playert 1 is selective whereas player 2 is not.

Lemma 3.1 *In the two agent discrete value model, if $p \neq \tilde{p}$, there is no equilibrium where agent $q(1) = 1$ and $q(2) = 0$.*

We thus consider the three other potential equilibria: one where both agents are selective, $q(1) = q(2) = 0$, one where the top agent in the queue is selective but not the second, $q(1) = 0, q(2) = 1$ and one where both agents accept the low quality object, $q(1) = q(2) = 1$. We index each equilibrium by the number of selective agents, $k = 0, 1, 2$.

3.1.1 Both players are selective

In this equilibrium , when player 1 chooses first, with probability π, he picks the object and player 2 moves up in the waiting list; with probability $(1-\pi)\pi$, player 2 picks the object, and with probability $(1-\pi)^2$, none of the players picks the object which is wasted. If player 2 chooses first, he picks the objects with probability π ; with probability $\pi(1-\pi)$ player 1 picks the object and player 2 moves up in the waiting list, and with probability $(1-\pi)^2$ no agent picks the object. We compute the values of the two agents as

$$\begin{aligned}
V^2(1) &= p[\pi + (1-\pi)(V^2(1) - c)] + (1-p)[\pi(1-\pi) + (1-\pi(1-\pi))(V^2(1) - c)], \\
V^2(2) &= p[\pi(1-\pi) + \pi(V^2(1) - c) + (1-\pi)^2(V^2(2) - c)] \\
&\quad + (1-p)[\pi + \pi(1-\pi)(V^2(1) - c) + (1-\pi)^2(V^2(2) - c)].
\end{aligned}$$

Let $W(i) \equiv V(i) - c$ denote the value of player i at the next period, taking into account the waiting cost. Simplifying, we obtain

$$\begin{aligned}
W^2(1) &= 1 - \frac{c}{\pi(1-\pi+p\pi)}, \\
W^2(2) &= 1 - \frac{2c}{\pi(2-\pi)}
\end{aligned}$$

Notice that the values of both players are decreasing in the waiting cost c and increasing in the probability π. Higher values of p result in higher values of $W^2(1)$ but do not affect the value of the second player. To check for which parameters this equilibrium exists, we write conditions under which both players reject the low value object. Player 1 rejects the low value object if

$$0 \leq W^2(1),$$

At the time he is offered the objet, player 2 ignores his order in the sequence and updates his beliefs about the order in the sequence using the probability p. His continuation value after rejection is given by

$$pW^2(2) + (1-p)(\pi W^2(1) + (1-\pi)W^2(2),$$

and player 2 rejects the low value object if

$$0 \leq pW^2(2) + (1-p)(\pi W^2(1) + (1-\pi)W^2(2)).$$

As $W^2(2) < W^2(1)$, an equilibrium where both agents are selective exists if and only if

$$0 \leq pW^2(2) + (1-p)(\pi W^2(1) + (1-\pi)W^2(2)).$$

Finally, note that in that equilibrium, the expected misallocation is zero , $\mu = 0$– the object will always be assigned to an agent with value 1 – and the expected waste is $\nu = (1-\pi)^2$.

3.1.2 Player 1 is selective, player 2 is not

In this equilibrium, when player 1 chooses first, with probability π he picks the object and player 2 moves up in the seniority queue while with probability $1-\pi$, playert 2 picks the object. When player 2 chooses first, he always picks the object. The values of the two agents are

$$\begin{aligned} V^1(1) &= p[\pi + (1-\pi)(V^1(1) - c)] + (1-p)(V^1(1) - c), \\ V^1(2) &= p[\pi(V^1(1) - c) + \pi(1-\pi)] + (1-p)\pi, \end{aligned}$$

yielding

$$\begin{aligned} W^1(1) &= 1 - \frac{c}{p\pi}, \\ W^1(2) &= \pi(1 + p - p\pi) - 2c. \end{aligned}$$

We observe again that both values are increasing in π and decreasing in c. In addition, both values are increasing in the probability p. The values of both agents are lower than in the equilibrium where both agents are selective, $W^1(1) < W^2(1)$ and $W^1(2) < W^2(2)$. Finally, in this equilibrium, static misallocation arises when the second player receives the object and values it at zero, while the first player has a value of 1, an event which occurs with probability $\mu = \pi(1-\pi)(1-p)$. Hence misallocation is decreasing in p. As player 2 always picks the object, the expected waste in equilibrium is equal to zero, $\nu = 0$. This equilibrium exists if and only if

$$\begin{aligned} 0 &\leq W^1(1), \\ 0 &\geq pW^1(2) + (1-p)(\pi W^1(1) + (1-\pi)W^1(2). \end{aligned}$$

3.1.3 No agent is selective

In this equilibrium, both players immediately pick the object, and the values are given by

$$\begin{aligned} V^0(1) &= p\pi + (1-p)(V^0(1) - c), \\ V^0(2) &= p(V^0(1) - c) + (1-p)\pi, \end{aligned}$$

so that

$$\begin{aligned} W^0(1) &= \pi - \frac{c}{p}, \\ W^0(2) - c &= \pi - 2c. \end{aligned}$$

Both values are increasing in π and decreasing in c, and the value of the first agent is increasing in p. Clearly, the values of both agents are lower than in the equilibrium where agent 1 is selective, $V^0(1) < V^1(1)$ and $V^0(2) < V^1(2)$. In this equilibrium, misallocation occurs with probability $\mu = \pi(1-\pi)$, when the object is allocated to the first player in the order who has a low value when the second player in the order has a high value. This equilibrium exists if and only if $W^0(1) \leq c$ and $pW^0(2) + (1-p)(\pi W^0(1) + (1-\pi)W^0(2) \leq 0$. Given that $W^0(1) > W^0(2)$, the equilibrium exists if and only if

$$c \geq \pi p.$$

3.1.4 Equilibria of the two agent model

We illustrate in Figure 1 the regions of parameters for which the three equilibria exist when the two values are equiprobable, $\pi = \frac{1}{2}$. An equilibrium where both players are selective exists if and only if

$$c \leq \frac{3(1+p)}{2(5+p+2p^2)},$$

an equilibrium where player 1 is selective exists if and only if

$$\frac{p(6-p+p^2)}{8(1+p^2)} \leq c \leq \frac{p}{2},$$

and an equilibrium where both players accept both objects if

$$c \geq \frac{p}{2}.$$

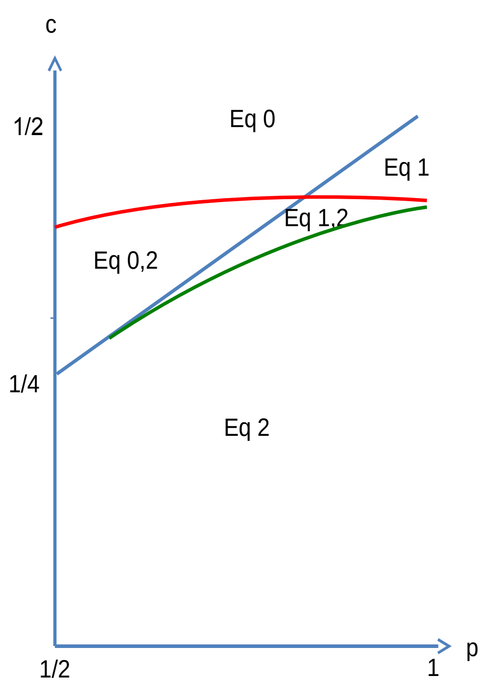

Figure 1: The two-agent discrete model

We first note that there exist two parameter regions where multiple equilibria exist: one where equilibria 0 and 2 coexist and one where equilibria 1 and 2 coexist. The multiplicity of equilibria stems from the fact that players' choices are *strategic complements*: if one player is selective, the probability that the other player is offered the object increases. This increases the value of the other player, making him more selective. Notice however that for high and low values of the waiting cost, equilibrium is unique, with all agents being selective when c is sufficiently low, and no agent being selective for c sufficiently high. We also remark that the asymmetric equilibrium does not exist when p is small. When agents are treated symmetrically in the order, they do not adopt asymmetric selection strategies.

Finally, we analyze the comparative statics effect of an increase in c and p on the equilibrium type. As c increases, agents become less selective, and equilibrium moves from a more selective equilibrium to a less selective equilibrium. When p increases, two effects are at play. First, the values $W^2(1)$, $W^1(1)$ and $W^1(2)$ strictly increase – the other values remain constant – increasing the likelihood that agents are more selective. Second, the continuation value of the second agent after a rejection, $pW(2) + (1-p)(\pi W(1) + (1-\pi)W(2))$, decreases, decreasing the likelihood that the second agent is selective. The two effects result in an ambiguous comparative static effect of p on the selectivity of equilibrium. The function $f(p) \equiv \frac{3(1+p)}{2(5+p+2p^2)}$ is first increasing, then decreasing in p. For example $f(0.9) \sim 0.3818 < 0.375 = f(1)$. Hence, for values of the waiting cost $c \in (0.375, 0.3818)$, there exists a selective equilibrium at $p = 0.9$ but not at $p = 1$. In addition, the value of the second agent is lower at the 1 equilibrium when $p = 1$ than at the 2 equilibrium when $p = 0.9$. (For example for $c = 0.38$, the values are -0.013 and -0.01 respectively.) Hence, we cannot in general state that that the equilibrium value for agent 2 is increasing in p. This ambiguity disappears when one compares the two extreme cases $p = \tilde{p} = \frac{1}{2}$ and $p = \hat{p} = 1$. In both these extreme situations, the second agent's continuation value after a rejection is simply $W(2)$. The second effect disappears and a shift from $p = \frac{1}{2}$ to $p = 1$ increases the selectivity of equilibrium for any c. It is then easy to see that the equilibrium value of the second agent is higher at $p = 1$ than at $p = \frac{1}{2}$, also establishing that the misallocation loss is lower at $p = 1$ than at $p = \frac{1}{2}$ but that the expected waste is higher at $p = 1$ than at $p = \frac{1}{2}$.

4 The general discrete model

4.1 Characterization of equilibrium

We now analyze the general model with discrete private values, when the fixed size of the queue is an arbitrary number n. For any mechanism p, and vector of (pure) strategies $\mathbf{q}$, we compute two vectors: $\gamma = \gamma(1), ..., \gamma(n)$

collects the expected probabilities that agent i picks the object, where the expectation is taken over the realization of the values θ and of the order ρ. The vector $\omega = \omega(1), ..., \omega(n)$ collects the expected probabilities that agent i is proposed the object. Notice that, as at most one agent receives the object at the end of the assignment process, for any realization of θ and ρ, the sum of assignment probabilities is bounded by 1 and hence, $\sum_i \gamma(i) \leq 1$. On the other hand, as agents may reject the object when it is proposed to them, we may very well have $\sum_i \omega(i) > 1$. For example, if $n = 2$, both agents are selective, values are private and the order $\rho = 12$ is chosen with probability p, we compute $\omega(1) = p + (1-\pi)(1-p)$ and $\omega(2) = p(1-\pi) + 1 - p$, so that $\omega(1) + \omega(2) = 2 - \pi > 1$. Notice also that, if $q(i) = 1$, $\gamma(i) = \omega(i)$ as agent i accepts the object whenever it is proposed, whereas when $q(i) = 0$, $\gamma(i) = \pi\omega(i)$ as agent i only accepts the object when it is proposed and of high value. With these notations in hand, we prove the following Lemma.

Lemma 4.1 *Under Assumption 2.1, if $i < j$ and $q(i) = q(j)$, $\omega(i) \geq \omega(j)$ and $\gamma(i) \geq \gamma(j)$.*

Lemma 4.1 establishes that, when the mechanism is consistent with the waiting list and two agents adopt the same strategy, an agent with lower rank in the waiting list has a higher chance of being proposed the object , and hence a higher chance to pick the object than an agent with higher rank. Next, we use this Lemma to show that the equilibrium values of agents are monotonic in the rank in the waiting list:

Lemma 4.2 *Under Assumption 2.1 in any equilibrium of the game, if $i < j$ then $V(i) \geq V(j)$.*

The intuition underlying Lemma 4.2 is clear when one compares either two agents who are selective or two agents who are not selective. By Lemma 4.1, agents with a lower rank in the waiting list are proposed the object more often. Whenever agents adopt the same strategies in equilibrium, this implies that agents with a lower rank obtain a higher value. The only case where this reasoning may fail is when one compares the value of two agents choosing different strategies. But then, we can use a revealed preference argument to show that the agent with the lower rank gets a higher value choosing his equilibrium strategy than by choosing the same strategy as the other agent, and the result follows.

Next we use Lemma 4.2 to show that there exists no equilibrium where agents with a lower rank in the waiting list is more selective than an agent with a higher rank:

Lemma 4.3 *Under Assumption 2.1, if $p \neq \tilde{p}$ there is no equilibrium where $q(i) = 1$ and $q(j) = 0$ for some $j > i$.*

Given Lemma 4.3, we can focus attention on equilibria where the first k agents in the waiting list are selective and the last $(n - k)$ agents are not selective. We call these equilibria *k-equilibria*, with $k = 0, ..., n$. Given these strategies, we compute, for $0 < k < n$,

$$
\begin{aligned}
V^k(1) &= \omega(1)\pi + (1 - \gamma(1))(V^k(1) - c), \\
V^k(i) &= \sum_{t=1}^{i-1} \gamma(t)(V^k(i-1) - c) + \gamma(i) + (1 - \sum_{t=1}^{i})\gamma(t))(V^k(i) - c) \text{ for } i \leq k, \\
V^k(i) &= \sum_{t=1}^{i-1} \gamma(t)(V^k(i-1) - c) + \gamma(i)\pi + (1 - \sum_{t=1}^{i})\gamma(t))(V^k(i) - c) \text{ for } i > k,
\end{aligned}
$$

Solving the recursive system, we obtain

$$
\begin{aligned}
W^k(i) &= 1 - \frac{ic}{\sum_{t=1}^{i} \gamma(t)} \text{ for } i \leq k, \\
W^k(i) &= \frac{\sum_{t=1}^{k} \gamma(t) + \sum_{t=k+1}^{i} \gamma(t)\pi}{\sum_{t=1}^{i} \gamma(t)} - \frac{ic}{\sum_{t=1}^{i} \gamma(t)} \text{ for } i > k.
\end{aligned}
$$

The preceding expression shows that the value of agent i can be decomposed into the expected value of the object and the expected waiting cost. Conditional on the fact that the object is picked by an agent with rank less or equal to i, the expected waiting time is exactly equal to the rank of the agent irrespective of the probabilistic queuing discipline. This result is easy to understand: on expectation, each agent will stay exactly one period at each rank, and the total number of waiting periods is thus equal to the rank of the player. When $i \leq k$, the agent will only accept objects with high value so the expected value of the object is equal to 1. When $i > k$, with some probability, the agent accepts the low value object so the expected value is a convex combination of the high and low values. A careful inspection of the expression also shows that a shift to the FCFS mechanism increases the probability that the object is assigned to an agent with rank lower or equal to i – thereby reducing expected waiting costs, and increases the probability that an agent picks the object when he has lower rank and only accepts high value objects – thereby increasing the expected value of the object picked.

On the other hand, the lottery results in the lowest probability that the object is picked by an agent with lower rank, resulting in the lowest values for all agents. We formalize this observation in the following Proposition:

Proposition 4.4 *Let $\hat{W}^k(i)$ and $\tilde{W}^k(i)$ denote the value of agent i in a k equilibrium under the FCFS and lottery mechanisms. Then, for any mechanism $p \in \mathcal{P}$, $\hat{W}^k(i) \geq W^k(i)$. For $k = 0$ or $k = n$, $W^k(i) \geq \tilde{W}^k(i)$.*

Proposition 4.4 establishes that for a fixed k equilibrium *all agents* prefer the strict seniority order $\hat{p}$ to any other mechanism p and when all agents are selective or nonselective, they prefer any rule to the uniform random order $\tilde{p}$. Proposition 4.4 compares the values of agents for different probabilistic queuing disciplines but for a fixed equilibrium structure. Our next results compares equilibrium values of agents across equilibrium structures. It shows that, for any probabilistic queuing discipline p, all agents prefer an equilibrium where the number of selective players increases.

Proposition 4.5 *The equilibrium values satisfy: For any k, i, $W^{k+1}(i) \geq W^k(i)$.*

Proposition 4.5 compares values of the same agent in two different equilibria – one where k agents are selective and one where $k+1$ agents are selective, and shows that an agent's value is higher in an equilibrium with more selective agents. The intuition underlying this result is based on the probability $\omega(i)$ that an agent is proposed the object. When agent $k+1$ switches from choosing $q(k+1) = 1$ to $q(k+1) = 0$, the probability that all other agents are proposed the object weakly increases whereas his probability of being proposed the object remains constant. We show that an increase in the probability $\omega(i)$ for all $i \neq k+1$ together with a constant $\omega(k+1)$ weakly raises the value of all agents in equilibrium.[9]

Finally, in order to compare the equilibrium values of all agents under the FCFS and lottery mechanisms, we need to consider how a change in the probabilistic queuing affects the degree of selectivity in equilibrium. This step of the argument cannot be established for general queuing disciplines

[9] There are cases where a switch from the k to the $k+1$ equilibrium has no effect on the equilibrium values of the agents. For example in the FCFS mechanism, all agents with rank $i < k$ are unaffected by the switch.The equilibrium value of agent $k+1$ is also the same in both equilibria.However the equilibrium value of agent $k+2$ differs in the two equilibria, as agent $k+2$ has a positive probability of being proposed the object in the $k+1$ equilibrium but not in the k equilibrium. Recursively, the equilibrium values of agents $i > k+2$ are different in the two equilibria.

where the continuation value of agent i is a convex combination of $W(i-1)$ and $W(i)$ with weights depending on the probabilistic queuing discipline. However, for the two extreme cases of FCFS and lottery mechanisms, the continuation value of player i after a rejection is equal to $W(i)$ and using Proposition 4.4, we show that the degree of selectivity of equilibrium must be higher under FCFS than under the lottery. This fact allows us to compare equilibrium values of all agents under the two queuing disciplines:

4.2 FCFS vs. lotteries

We focus attention on the two extreme probabilistic queuing disciplines $\hat{p}$ and $\tilde{p}$ and characterize all equilibrium structures under these mechanisms.

If $p = \hat{p}$, a k equilibrium exists if and only if

$$W^k(k) = 1 - \frac{kc}{\pi[1 + ... + (1-\pi)^{k-1}]} \geq 0$$

$$\frac{\pi[1 + .. + (1-\pi)^k]}{\pi[1 + ... + (1-\pi)^{k-1} + (1-\pi)^k]} - \frac{(k+1)c}{\pi[1 + ... + (1-\pi)^{k-1} + (1-\pi)^k]} \leq 0.$$

or

$$\frac{1-(1-\pi)^{k+1}}{k+1} \leq c \leq \frac{1-(1-\pi)^k}{k}.$$

As the function $f(k) = \frac{1-(1-\pi)^k}{k}$ is decreasing in k, [10] we can partition the positive real line into intervals $A^k = [\frac{1-(1-\pi)^{k+1}}{k+1}, \frac{1-(1-\pi)^k}{k}]$ such that the unique equilibrium of the game is a k equilibrium when $c \in A^k$.

If $p = \tilde{p}$, the equilibrium must treat all agents symmetrically, and the only two candidate equilibria are the 0 and n equilibrium. The 0 equilibrium exists if and only if $c \geq \frac{\pi}{n}$ whereas the n equilibrium exists if and only if $c \leq \frac{1-(1-\pi)^n}{n}$. Notice that the n equilibrium only exists when the n equilibrium exists under the FCFS mechanism.

Proposition 4.6 *The equilibrium values of all agents are higher under the FCFS rule $\hat{p}$ than under the lottery $\tilde{p}$. The misallocation loss is lower under $\hat{p}$ than $\tilde{p}$ and the expected waste is lower under $\tilde{p}$ than under $\hat{p}$.*

[10] To see this consider k as a continuous variable and note that the sign of $f'(k)$ is the same as the sign of $(1-\pi)^k(1-k\log(1-\pi))-1$. Differentiating $g(k) = (1-\pi)^k(1-k\log(1-\pi))$, we obtain $g'(k) = -k(1-\pi)^k(\log(1-\pi))^2 < 0$, and finally observe that $g(0) = 1$.

5 The continuous model

In the continuous model, agents draw their values from a continuous, atomless distribution $F(\cdot)$ with density function $f(\cdot)$.

5.1 Equilibrium in the continuous model

T he value of agent $i > 1$ is given by

$$W(i) = \frac{\sum_{j<i} \gamma(j) W(i-1) + \omega(i) \int_{\theta_i}^{\overline{\theta}} t f(t) dt - c}{\sum_{j \leq i} \gamma(j)}. \tag{1}$$

and

$$W(1) = \frac{\omega(1) \int_{\theta_1}^{\overline{\theta}} t f(t) dt - c}{\gamma(1)} \tag{2}$$

By induction, we compute

$$W(i) = \frac{\sum_{j=1}^{i} \omega(j) \int_{\theta_j}^{\overline{\theta}} t f(t) dt - ic}{\sum_{j=1}^{i} \gamma(j)}. \tag{3}$$

The previous expression decomposes the value of an agent into the expected waiting cost and the expected value of the object picked by the agent. As in the discrete case, given that the object is picked by any agent preceding and including agent i, the expected number of waiting periods of agent i is equal to i. The expected value of the object picked by the agent is easy to compute. Given that the agent expects to spend one period at every state $j = 1, ..., i$, his expected value of the object is the expectation over all possible ranks $j \leq i$, of the average value of the object picked by an agent of rank j.

We next prove that agents with lower seniority ranks are more selective. The proof of this result parallels the proof of Lemma 4.2 in the discrete case. First observe Lemma 4.1 can be directly adapted to the continuous model[11]:

Lemma 5.1 *Under Assumption 2.1, if $i < j$ and $\theta(i) = \theta(j)$, $\omega(i) \geq \omega(j)$ and $\gamma(i) \geq \gamma(j)$.*

Based on Lemma 5.1, we establish

[11] We omit the proof because it is identical to the proof of Lemma 4.1

Lemma 5.2 *Under Assumption 2.1 in any equilibrium of the game, if $i < j$ then $W(i) \geq W(j)$.*

Lemma 5.2 shows that equilibrium values are ranked according to the seniority in the waiting queue. In equilibrium, the optimal stopping rule of agent i leads him to accept any value greater than the threshold $\theta(i)$ which is equal to the continuation value after rejection, a convex combination of $W(i-1)$ and $W(i)$. Hence $W(i) \leq \theta(i) \leq W(i-1)$. Because $W(i) \leq W(i+1)$ by Lemma 5.2, we conclude that $\theta(i+1) \leq \theta(i)$. In addition, except for the special case of the uniform lottery, the inequality is strict as Lemmas 5.1 and 5.2 hold with strict inequality.

In the next step of the analysis, we compute how equilibrium values are affected by a change in the probabilistic sequence, for a fixed set of thresholds $(\theta(1), ..\theta(n))$. We prove an analog to Proposition 4.4, showing that equilibrium values are maximized at the FCFS mechanism and minimized at the lottery.

Proposition 5.3 *Fix a strategy vector $(\theta(1), ..\theta(n))$ with $\theta(1) \geq \theta(2)... \geq \theta(n)$. For any probabilistic sequence $p \in \mathcal{P}$, let V be the value of the strategy vector under p, $\hat{V}$ the value under the FCFS mechanism and $\tilde{V}$ the value under the lottery. Then, for any i,*

$$\hat{V}(i) \geq V(i) \geq \tilde{V}^k(i).$$

Proposition 5.3 does not allow us to conclude that agents prefer the FCFS scheme to the lottery or to any other scheme, because the threshold values $\theta(i)$ differ for different probabilistic queuing disciplines. To understand better how changes in the queuing discipline affect the equilibrium threshold of agents, we focus attention of the FCFS rule. Under FCFS, recall that the optimal threshold of agent i is a function only of the thresholds of the agents preceding i in the waiting list, $j = 1, .., i-1$ and is obtained as the unique solution to the equation:

$$\begin{aligned}\theta(i)[1 - F(\theta(1))..F(\theta(i))] &= \int_{\theta(1)}^{\overline{\theta}} tf(t)dt + F(\theta(1)) \int_{\theta(2)}^{\overline{\theta}} tf(t)dt + \\ &.. \quad +F(\theta(1))..F(\theta(i-1)) \int_{\theta(i)}^{\overline{\theta}} tf(t)dt - ic = 0.\end{aligned}$$

We do the following recursive computation. Start with an arbitrary threshold θ for the first agent, and let $\phi_2(\theta)$ denote the optimal threshold of the second agent given θ. Let $\phi_3(\theta)$ denote the optimal threshold of the third agent given that the top agent adopts the threshold θ, the second agent $\phi_2(\theta)$. Recursively, let $\phi_i(\theta)$ denote the optimal threshold of the agent at rank i given that the top agent chooses θ and all successive agents choose their optimal thresholds. The next Proposition establishes that an increase in the threshold of the top agent results in an increase in the optimal thresholds of all other agents:

Proposition 5.4 *For any $i$and any $\theta \leq \hat{\theta}_1$ where $\hat{\theta}_1$ is the unconstrained optimal threshold of the top agent, $\phi_i(\theta)$ is increasing in θ.*

An immediate consequence of Proposition 5.4 is that when $n = 2$, the optimal threshold of the second agent in the FCFS mechanism is increasing in the threshold of the top agent. The more selective the first agent is, the higher the threshold (and hence the value) of the second agent. We use this fact to obtain sharp results in the continuous model with two-agent queues.

5.2 The two-agent continuous model

When $n = 2$, we compute the values of the two agents as

$$\begin{aligned} W_1(p, \theta_1, \theta_2) &= p[\int_{\theta_1}^{\overline{\theta}} tf(t)dt + F(\theta_1)W_1] \\ &+ (1-p)[F(\theta_2)\int_{\theta_1}^{\overline{\theta}} tf(t)dt + (1 - F(\theta_2)(1 - F(\theta_1)))W_1 - c, \\ &= \frac{\int_{\theta_1}^{\overline{\theta}} tf(t)dt}{1 - F(\theta_1)} - \frac{c}{[p + (1-p)F(\theta_2)](1 - F(\theta_1)}. \end{aligned} \quad (4)$$

and

$$\begin{aligned} W_2(p, \theta_1, \theta_2) &= p[(1 - F(\theta(1))W_1 + F(\theta_1)[\int_{\theta_2}^{\overline{\theta}} t(f)dt + F(\theta_2)W_2]] \\ &+ (1-p)[\int_{\theta_2}^{\overline{\theta}} tf(t)dt + F(\theta_2)[(1 - F(\theta(1))W_1 + F(\theta(2))W_2]] - c \\ &= \frac{[pF(\theta(1) + (1-p)] \int_{\theta_2}^{\overline{\theta}} tf(t)dt}{1 - F(\theta(1))F(\theta(2))} \\ &+ \frac{(1 - F(\theta(1)))[p + (1-p)F(\theta(2))]W_1 - c}{1 - F(\theta(1))F(\theta(2))}. \end{aligned}$$

Assuming that c is sufficiently small, the Markov equilibrium is interior and can easily be characterized. Consider first the top agent. For a fixed W_1^*, the optimal choice θ_1^* is given by

$$\theta_1^* = W_1^*. \tag{5}$$

Replacing W_1^*, we compute the equilibrium threshold as the solution to the equation

$$\frac{\int_{\theta_1^*}^{\overline{\theta}}(t-\theta_1^*)f(t)dt}{1-F(\theta_1)} - \frac{c}{[p+(1-p)F(\theta_2)](1-F(\theta_1)} = 0. \tag{6}$$

Next consider the second agent. For fixed W_1^* and W_2^*, she selects the optimal θ_2^* as the solution to

$$\theta_2^* = \frac{(1-p)(1-F(\theta_1^*)}{pF(\theta_1^*+1-p}W_1^* + \frac{F(\theta_1^*)}{pF(\theta_1^*)+1-p}W_2^*. \tag{7}$$

Notice that, as opposed to the top agent, the second agent does not choose a threshold value θ_2^* equal to the value W_2^* because he expects to move to the top of the waiting list with some probability after a rejection. Replacing with the values of W_1^* and W_2^*, we compute the equilibrium threshold as the solution to the equation:

$$F(\theta_1^*)\int_{\theta_2^*}^{\overline{\theta}}(t-\theta_2^*)f(t)dt + (1-F(\theta_1^*))(\theta_1^*-\theta_2^*) - \frac{cF(\theta_1^*)}{1-p+pF(\theta_1^*)} = 0 \tag{8}$$

We now compare the equilibrium values and strategies under FCFS and other probabilistic queuing disciplines.

Proposition 5.5 *Let $n=2$. The equilibrium values of the two agents, W_1^* and W_2^* are strictly higher under the FCFS than under any other probabilistic queuing discipline. The threshold of the top agent θ_1^* is higher under FCFS than under any other probabilistic queuing discipline. The threshold of the second agent θ_2^* is higher under the FCFS than under the lottery.*

To understand Proposition 5.5 observe that the equilibrium threshold θ_1^* is always higher under the FCFS rule – when the top agent chooses his unconstrained optimal threshold. By Proposition 5.4, the equilibrium value of the second agent under FCFS is increasing in the threshold of the top agent. Hence the equilibrium value of the second agent is higher at the threshold pair

(θ_1^*, θ_2^*) than at any other threshold pair (θ_1, θ_2).[12] Using Proposition 5.3, we observe that the equilibrium value of the second agent at any arbitrary fixed threshold pair (θ_1, θ_2) is highest at $p = 1$, completing the proof. Because at the two extreme cases $p = 1$ and $p = \frac{1}{2}$, the equilibrium values of the agents are equal to their thresholds, the argument of Proposition 5.5 provides a comparison between the equilibrium thresholds θ_1^* and θ_2^*. For general values of p, the argument fails and we observe in the following example that the equilibrium threshold of the second agent θ_2^* is not a monotonic function of p.

Example 5.6 *Suppose that the distribution of values is uniform over* $[0, 1]$ *and let* $c = 0.1$*. The interior equilibrium is characterized by the two conditions*

$$\begin{aligned}
\theta_1^* &= \phi_1(\theta_2^*) \equiv 1 - \sqrt{\frac{0.2}{p + (1-p)\theta_2^*}}, \\
\theta_2^* &= \phi_2(\theta_1^*) \equiv 1 - \sqrt{1 - \theta_1^*(-2\theta_1^* + 3 - \frac{0.2}{p\theta_1^* + 1 - p}}.
\end{aligned}$$

Figure 2 displays the equilibrium threshold values θ_1^* and θ_2^* as a function of p. The threshold of the top agent is increasing in p but the threshold of the second agent is *not monotonic*. This non-monotonicity is due to the fact that, as in the discrete model, a change in p affects the second player's expectation that he may become the top agent next period. This probability decreases with p, creating a secondary effect which counters the primary effect that an increase in p increases the values of the two players. However, for the two extreme cases where $p = \frac{1}{2}$ and $p = 1$, the continuation value is exactly equal to the threshold, and this secondary effect disappears. It is then easy to see that the equilibrium threshold of the second agent is always higher in FCFS than in the lottery.

We now turn to the two other measures of efficiency.

The probability of misallocation measures the sum of the probability that the object is picked by the fist agent when the second agent draws a higher value and of the probability that the second agent picks the object when the first agent draws a higher value above θ_1^*. Hence, the probability of misallocation μ is given by

[12] This essential step in the proof cannot be extended to waiting lists of general length, preventing us from comparing equilibrium values in the general model.

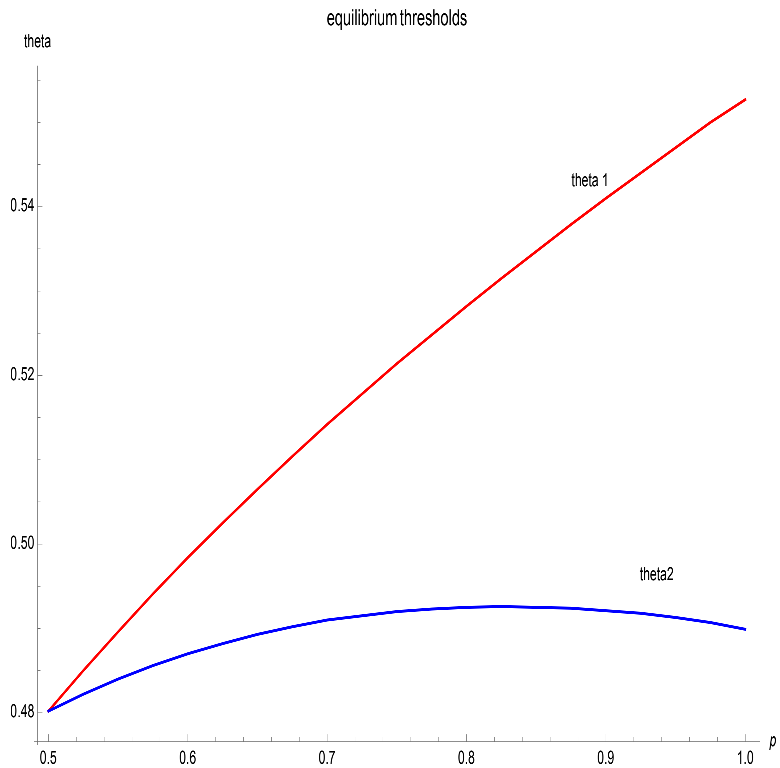

Figure 2: Equilibrium thresholds in the two-agent continuous model

$$
\begin{aligned}
\mu &= p\int_{\theta_1^*}^{\overline{\theta}}\int_{\theta}^{\overline{\theta}} f(t)dtf(\theta)d\theta + (1-p)\int_{\theta_2^*}^{\overline{\theta}}\int_{\max\{\theta,\theta_1^*\}}^{\overline{\theta}} f(t)dtf(\theta)d\theta, \\
&= p\int_{\theta_1^*}^{\overline{\theta}}\int_{\theta}^{\overline{\theta}} f(t)dtf(\theta)d\theta + (1-p)[\int_{\theta_2^*}^{\theta_1^*}\int_{\theta_1^*}^{\overline{\theta}} f(t)dtf(\theta)d\theta + \int_{\theta_1^*}^{\overline{\theta}}\int_{\theta}^{\overline{\theta}} f(t)dtf(\theta)d\theta, \\
&= \int_{\theta_1^*}^{\overline{\theta}}(1-F(\theta))d\theta + (1-p)\int_{\theta_2^*}^{\theta_1^*}(1-F(\theta_1^*)).
\end{aligned}
$$

Lemma 5.7 *When $n = 2$, the probability of misallocation is lowest at the FCFS rule.*

Finally, he probability of waste is given by

$$\nu = F(\theta_1^*)F(\theta_2^*),$$

a strictly increasing function of θ_1^* and θ_2^*. We observe that the probability of misallocation is always lower at the lottery than at the FCFS mechanism.

6 Variants and extensions

In this Section, we discuss extensions based on the two agent discrete model. We first discuss optimal mechanisms when agents have heterogeneous waiting costs. We then consider a model where agents apply for the object after they learn their value. We also study the effect of releasing information about the sequence chosen by the probabilistic queuing mechanism. Finally, we discuss the impact of a punishment scheme where agents are evicted from the queue with positive probability if they reject the object.

6.1 Heterogeneous waiting costs

We suppose that agents are divided into two categories: a fraction λ of agents with low waiting costs $\underline{c}$, and a fraction $1-\lambda$ of agents with high waiting costs $\overline{c}$. We suppose that the waiting cost is observable by the planner – for example, the planner can verify whether an agent currently lives in an apartment or not, or the health status of an agent waiting for a transplant. Agents are now characterized by two variables: their rank in the waiting list, and their waiting cost. In order to select an optimal mechanism, the designer faces a trade-off between these two characteristics, and must choose which weight to assign to seniority and waiting cost in offer sequence. More precisely, we assume that the designer, after observing the waiting costs of the two agents, (c, c) chooses the probability that the most senior agent is proposed the object first, $p(c, c)$. Because it may be optimal to let the second agent choose first when he has a high waiting cost, we do not put any restriction on $p(c, c)$.

Suppose that each agent knows the waiting cost of the other agent in the queue. The strategy of each agent assigns to each of the four possible vectors of waiting costs (c, c) a point in $\{0, 1\}$. As each agent in the queue chooses four actions, the total number of strategies makes it intractable to characterize admissible equilibrium configurations as a function of the parameters.[13] In order to understand the trade-off between waiting costs and seniority rank, we focus attention on one specific equilibrium configuration: one where the low waiting cost $\underline{c}$ is sufficiently low and the high waiting cost $\overline{c}$ sufficiently high so that *all agents with low waiting cost are selective, and all agents with high waiting costs accept both objects.*

[13]In principle, as each agent in the waiting list has $2^4 = 16$ choices, the total number of strategy vectors is $16 \times 16 = 256$. Characterizing equilibrium configurations with such a large strategy set becomes intractable.

In this equilibrium, we first compute the value of the first agent when his waiting cost is low.

$$\begin{aligned} V_1(\underline{c},\underline{c}) - \underline{c} &= \frac{p(\underline{c},\underline{c}) + (1-p(\underline{c},\underline{c}))(1-\pi) + p(\underline{c},\underline{c})(1-\pi)}{2-\pi} \\ &+ \frac{(1-p(\underline{c},\underline{c}))A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))}{2-\pi} - \frac{c}{\pi(2-\pi)}, \\ V_1(\underline{c},\overline{c}) - \overline{c} &= p(\underline{c},\overline{c})\pi + [p(\underline{c},\overline{c})(1-\pi) + 1 - p(\underline{c},\overline{c})]A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) - \underline{c}, \end{aligned}$$

where

$$\begin{aligned} A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) &= \lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c}) \\ &= 1 - \frac{(\lambda + (1-\lambda)\pi(2-\pi))\underline{c}}{\pi(\lambda[p(\underline{c},\underline{c}) + (1-p(\underline{c},\underline{c})(1-\pi))] + (1-\lambda)p(\underline{c},\overline{c})\pi^2(1-\pi))}. \end{aligned}$$

Notice that the expected continuation value $A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))$ is increasing in both probabilities $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$. As $A < 1$, the values $V_1(\underline{c},\overline{c})$ and $V_1(\underline{c},\overline{c})$ are also increasing in the probabilities $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$. We next compute the value of the first agent when his cost is high.

$$\begin{aligned} V_1(\overline{c},\underline{c}) &= p(\overline{c},\underline{c})\pi + (1-p(\overline{c},\underline{c})\pi(1-\pi) + \pi(1-p(\overline{c},\underline{c}))B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) - \overline{c}, \\ V_1(\overline{c},\overline{c}) &= p(\overline{c},\overline{c})\pi + (1-p(\overline{c},\overline{c}))B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) - \overline{c}, \end{aligned}$$

where

$$\begin{aligned} B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) &= \lambda V_1(\overline{c},\underline{c}) + (1-\lambda)V_1(\overline{c},\overline{c}) \\ &= \pi\frac{\lambda(p(\overline{c},\underline{c}) + (1-p(\overline{c},\underline{c}))(1-\pi) + (1-\lambda)p(\overline{c},\overline{c}) - c}{\lambda(1-\pi(1-p(\overline{c},\underline{c})) + (1-\lambda)p(\overline{c},\overline{c}))}. \end{aligned}$$

Notice that the expected continuation value $B(p(\overline{c},\underline{c}),p(\emptyset,\overline{c}))$ is increasing in $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$. As $B < \pi$, the values $V_1(\overline{c},\underline{c})$ and $V_1(\overline{c},\overline{c})$ are also increasing in the probabilities $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$. Turning to the second agent we compute his value when his cost is low as

$$\begin{aligned} V_2(\underline{c},\underline{c}) &= \frac{p(\underline{c},\underline{c})(1-\pi) + 1 - p(\underline{c},\underline{c}) + [p(\underline{c},\underline{c})}{2-\pi} \\ &+ \frac{+(1-p(\underline{c},\underline{c}))(1-\pi)A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))}{2-\pi} - \frac{\underline{c}}{\pi(2-\pi)}, \\ V_2(\overline{c},\underline{c}) &= p(\overline{c},\underline{c})A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) + (1-p(\overline{c},\underline{c}))[\pi + (1-\pi)A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))] - \underline{c} \end{aligned}$$

It is interesting to note that $V_2(\underline{c},\underline{c})$ is increasing in $p(\underline{c},\overline{c})$ but non monotonic in $p(\underline{c},\underline{c})$. The value $V(\overline{c},\underline{c})$ is increasing in $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$ but decreasing in $p(\overline{c},\underline{c})$. For the second agent with high costs

$$\begin{aligned} V_2(\underline{c},\overline{c}) &= p(\underline{c},\overline{c})[\pi B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c}))+(1-\pi)\pi]+(1-p(\underline{c},\overline{c}))\pi-\overline{c}, \\ V_2(\overline{c},\overline{c}) &= p(\overline{c},\overline{c})B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c}))+(1-p(\overline{c},\overline{c}))\pi-\overline{c}. \end{aligned}$$

We observe that $V_2(\overline{c},\underline{c})$ is increasing in $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$ but decreasing in $p(\underline{c},\overline{c})$ and $V_2(\overline{c},\overline{c})$ is increasing in $p(\overline{c},\underline{c})$ and non monotonic in $p(\overline{c},\overline{c})$.

Contrary to the case of homogenous waiting costs, the value of the second agent is not necessarily increasing in the probability that the first agent is proposed the object.[14]

In order to maximize EV_2 it is sufficient to maximize

$$\begin{aligned} E &= -\frac{7\underline{c}(5+2p(\underline{c},\underline{c})+p(\overline{c},\underline{c}))}{6(2+2p(\underline{c},\underline{c})+3p(\underline{c},\overline{c}))}-\frac{p(\underline{c},\overline{c})+2p(\overline{c},\overline{c})}{4} \\ &+ \frac{p(\underline{c},\overline{c})+2p(\overline{c},\overline{c})}{2}\frac{1+p(\overline{c},\underline{c})+2p(\overline{c},\overline{c})-8\overline{c}}{2+2p(\overline{c},\underline{c})+4p(\overline{c},\overline{c})}. \end{aligned}$$

We can check that $\frac{\partial E}{\partial p(\underline{c},\underline{c})} > 0$ and $\frac{\partial E}{\partial p(\overline{c},\overline{c})} < 0$. In addition, for sufficiently large values of $\overline{c}$ and sufficiently low values of $\underline{c}$, $\frac{\partial E}{\partial p(\underline{c},\overline{c})} < 0$ and $\frac{\partial E}{\partial p(\overline{c},\underline{c})} > 0$. Hence, in order to maximize the expected value of the second agent, the mechanism designer chooses $p(\underline{c},\underline{c}) = p(\overline{c},\underline{c}) = 1$ and $p(\underline{c},\overline{c}) = p(\overline{c},\overline{c}) = 0$. *The mechanism should always give the object to the second agent when he has a high waiting cost and to the first agent when he has a low waiting cost.*

6.2 Assignment with prior application

In the baseline model, we suppose that agents are given the opportunity to accept the object in sequence. This sequential assignment rule is time consuming as some agents choose to reject the object which is proposed to them. As an alternative, we consider an alternative assignment rule, where agents apply for the object after observing the value. The planner then chooses which applicant is assigned the object using a probabilistic priority mechanism p.

[14] In order to illustrate this fact, we consider the special case where $\lambda = \pi = \frac{1}{2}$ and compute the values of the probabilities which maximize the expected value of the second agent, $EV_2 = \frac{1}{4}V_2(\underline{c},\underline{c}) + V_2(\overline{c},\underline{c}) + V_2(\underline{c},\overline{c}) + V_2(\overline{c},\overline{c})$.

We consider a mechanism where, after learning his value, each agent announces $a(i) \in \{0,1\}$ where $a(i) = 1$ means that the agent applies to the object. A random order ρ is drawn by the mechanism designer and the first agent in the order ρ who chooses $a(i) = 1$ is assigned the object. Note that an agent with high value always applies to the mechanism and accepts the object in the sequential mechanism. An agent with low value applies in the mechanism with prior applications if and only if the expected value of participating is higher than the value of not participating.

The mechanism with prior application generates the same values for the two agents as the sequential mechanism but differs in the computation of the continuation values for the second agent. There are three possible equilibria: (i) one where neither of the two agents participates when the value of the object is low, and agents obtain values $V^2(1)$ and $V^2(2)$, (ii) one where agent 1 only applies when the value of the object is low and agent 2 always applies, and the values are $V^1(1)$ and $V^1(2)$, and (iii) one where both agents always apply and the values are $V^0(1)$ and $V^0(2)$.

The top agent with low value chooses to participate if and only if

$$(1-p)\pi(V(1)-c) \geq V(1)-c$$

if 2 is selective and

$$(1-p)(V(1)-c) \geq V(1)-c$$

if 2 is not selective. Both conditions amount to

$$V(1)-c \leq 0,$$

so that agent 1 participates when the value is low if and only if $V(1)-c \leq 0$, as in the sequential allocation model. The second agent with low value chooses to participate if and only if

$$p\pi(V(1)-c) \geq \pi[V(1)-c] + (1-\pi)[V(2)-c]$$

if 1 is selective and

$$p(V(1)-c) \geq \pi[V(1)-c] + (1-\pi)[V(2)-c]$$

if 1 is not selective. Hence, the equilibrium condition for agent 2 is not the same in the two models. The difference arises from differences in the computation of the continuation value in both cases. In the sequential mechanism, agent 2 updates his belief about the order ρ taking into the account that he is offered the object. In the mechanism with prior application, agent 2, if he does not participate, computes an expected continuation value taking expectations over all possible orders ρ chosen by the mechanism. More precisely, an equilibrium where agent 2 is selective exists if and only if

$$\pi(1-p)[V(1)-c)] + (1-\pi)[V(2)-c] \geq 0.$$

Comparing with the condition guaranteeing equilibrium in the sequential model,

$$\pi(1-p)[V(1)-c)] + (1-\pi+p)[V(2)-c] \geq 0.$$

we observe that, as $V(2) < V(1)$, there is no parameter region where agent 2 is selective in the sequential model but not in the model with prior application Hence, *agents are more selective in the mechanism with prior application.* This observation suggests that the assignment model with prior application dominates the sequential acceptance model in terms of the values of the agents inside the queue and the static probability of misallocation, but that the sequential model dominates the model with prior application with respect to the probability of waste.

6.3 Information about the sequence

In the analysis, we suppose that agents know their position in the waiting list, but ignore the sequence in which offers are made. We now consider an alternative model where agents are told the sequence of offers. In the two-agent queue, this information only affects the decision of the second agent. For this agent, we distinguish between two states at which decisions must be made: state $(2,1)$ when agent 2 knows that she is the first in the sequence of offers, and state $(2,2)$ where agent 2 knows that she is the second in the sequence of offers. Notice that we do not need to distinguish between different continuation values at the two states, and will instead only compute the expected continuation value $V(2)$ of the second agent before the offer sequence is drawn. Notice that the continuation value in state $(2,1)$ is $\pi V(1) + (1-\pi)V(2)$, whereas the continuation value in state $(2,2)$ is simply $V(2)$. As $V(1) \geq V(2)$, the continuation value of agent 2 is higher at state $(2,1)$ than at state $(2,2)$. The equilibrium in which agents 1 and 2 are selective in all circumstances results in continuation values

$$
\begin{aligned}
V^3(1) - c &= 1 - \frac{c}{\pi(1-\pi+p\pi)}, \\
V^3(2) - c &= 1 - \frac{2c}{\pi(2-\pi)}.
\end{aligned}
$$

As the behavior of the second agent in the two states are identical, the values are identical to the values in the baseline model when the sequence is not known. This equilibrium exists as long as $V^3(2) - c \geq 0$ or $c \leq \frac{\pi(2-\pi)}{2}$. Next consider an equilibrium where agent 2 is selective in state $(2,1)$ but not in state $(2,2)$. The values are given by

$$
\begin{aligned}
V^2(1) - c &= 1 - \frac{c}{\pi(1-\pi+p\pi)}, \\
V^2(2) - c &= p[\pi(V^2(1) - c) + \pi(1-\pi)] + (1-p)[\pi + \pi(1-\pi)(V^2(1) - c) \\
&+ (1-\pi)^2(V^2(2) - c)], \\
&= \frac{\pi(2-\pi) - 2c}{1 - (1-p)(1-\pi)^2}
\end{aligned}
$$

This equilibrium exists if $V^2(2) - c \leq 0$ and $\pi V^2(1) + (1-\pi)V^2(2) - c \geq 0$ or $\pi(2-\pi) < 2c$ and $\pi - \frac{c}{1-\pi+p\pi} + (1-\pi)\frac{\pi(2-\pi)-2c}{1-(1-p)(1-\pi)^2} \geq 0$. Observe that both $V^2(1)$ and $V^2(2)$ are increasing in p in the range of parameters for which the equilibrium exists. Next, consider an equilibrium where agent 2 is never selective. This results in the values

$$
\begin{aligned}
V^1(1) - c &= 1 - \frac{c}{p\pi}, \\
V^1(2) - c &= \pi(1 + p - p\pi) - 2c,
\end{aligned}
$$

as in the baseline case. This equilibrium exists if $\pi V^2(1) + (1-\pi)V^2(2) - c \leq 0$ and $V^1(1) - c \geq 0$ or $c \geq \frac{p\pi + p(1-\pi)\pi(1+p(1-\pi))}{1+2p(1-\pi)}$ and $c \leq p\pi$. Finally, in an equilibrium where no agent is selective, values are given by

$$
\begin{aligned}
V^0(1) - c &= \pi - \frac{c}{p}, \\
V^0(2) - c &= \pi - 2c.
\end{aligned}
$$

as in the baseline case, and the equilibrium exists if and only if $c \geq p\pi$. We now compare the two régimes where agents are informed and not informed about the sequence. We check that the utility of both agents are increasing in the degree of selectivity of the equilibrium, $V^3(1) - c = V^2(1) - c > V^1(1) - c > v^0(1) - c$ and $V^3(2) - c > V^2(2) - c > V^1(2) - c > V^0(2) - c$.

We also need to compare the parameter regions under which different equilibria exist. Notice that the parameter regions where the selective equilibrium (equilibrium 3) exists in the informed case is a subset of the parameter region under which the selective equilibrium (equilibrium 2) exists in the baseline case. However, the parameter region under which either equilibrium 2 or 3 exists is a superset of the region under which equilibrium 2 exists in the baseline case. Similarly, the parameter region under which equilibrium 1 exists in the informed case is a subset of the region under which the equilibrium exists in the baseline case, but the region under which either equilibrium 1 or 2 exists is a superset of the region under which equilibrium 1 exists in the baseline case. The parameter region where equilibrium 0 exists is identical in the two régimes. We illustrate these different regions when $\pi = \frac{1}{2}$ in Figure 3.

Giving information about the sequence makes agent 2 more selective when she is first in the sequence but less selective when she is second. For agent 1, this always results in a positive effect on value, as it increases the parameter region for which agent 1 is selective when choosing first, increasing the opportunity that agent 1 gets to pick the object. The balance between the two effects on agent 2's expected value is ambiguous. There are parameter regions for which the selective equilibrium exists in the baseline case but not in the informed régime, making agent 2 worse off, and parameter regions where equilibrium 2 exists in the informed régime but not in the baseline case, making agent 2 better off.

6.4 Eviction from the queue

We consider the effect of an eviction mechanism, where agents are taken away from the queue if they refuse an object with positive probability. Let $\beta(1)$ and $\beta(2)$ denote the probability that the first – respectively the second – agent remain in the queue if they refuse the object. In an equilibrium where both agents are selective, the equilibrium values are given by

$$\begin{aligned} V^2(1) - c &= \frac{p\pi + (1-p)(1-\pi)\pi - c}{p\pi + (1-p)(1-\pi)\pi + (1-\beta(1))[p(1-\pi) + (1-)(1-\pi)^2]}, \\ V^2(2) - c &= \frac{p\pi(1-\pi) + [p\pi + (1-pi)\pi\beta(2)][V^2(1) - c] - c}{1 - \beta(2)(1-\pi)^2}. \end{aligned}$$

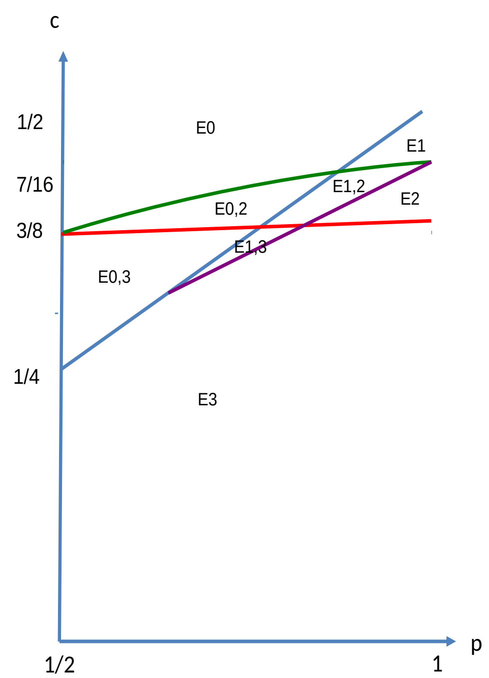

Figure 3: The two agent discrete model with information about the sequence

Clearly, $V^1(1)$ is increasing in $\beta(1)$ and $V^2(2)$ is increasing in $\beta(1)$ and $\beta(2)$. Evicting the agents from the queue decreases their expected continuation values, making them less likely to be selective. In the equilibrium where only agent 1 is selective, the values become

$$\begin{aligned} V^1(1) - c &= \frac{p\pi - c}{p\pi + (1-\beta(1))p(1-\pi)}, \\ V^1(2) - c &= p\pi(V1(1) - c) + p\pi(1-\pi) + (1-p)\pi - c. \end{aligned}$$

The values $V^1(1)$ and $V^1(2)$ are both increasing in $\beta(1)$. Finally, the values in the equilibrium where both agents accept both objects are clearly unaffected by the eviction probabilities. We thus observe that introducing eviction probabilities *reduces the values of the agents in the queue* and makes them less lily to be selective. As agents are less selective, the misallocation probability increases and the expected waste decreases. Hence, introducing eviction probabilities can only reduce the welfare of agents currently in the queue. It also accelerates the turnover in the queue, improving the well being of agents who are waiting to be included in the queue.

7 Conclusion

This paper analyzes the optimal assignment of objects which arrive sequentially to agents organized in a waiting list. Applications include the assignment of social housing and organs for transplants. We analyze the optimal design of probabilistic queuing disciplines, punishment schemes, the optimal timing of applications and information releases. We consider three efficiency criteria: the vector of values of agents in the queue, the probability of misallocation and the expected waste. Under private values, we show that the first-come first serve mechanism dominates a lottery according to the two first criteria, and that the lottery dominates first come first serve according to the third criterion. Punishment schemes accelerate turnover in the queue at the expense of agents currently in the waiting list, application schemes with commitment dominate sequential offers and information release always increases the value of agents at the top of the waiting list.

Our analysis thus gives support to the use of waiting time as a primary criterion in the priority order in order to maximize the value of agents inside the queue, and of the use of lotteries in order to minimize waste. It also shows that punishment schemes like eviction from the queue harm agents currently inside the queue but accelerate turnover in the queue. There remain a number of aspects of dynamic allocation that we have not yet studied. What happens if the waiting list is open and agents enter and exit the waiting list stochastically? What if agents have private information about their values and waiting costs and the planner designs a mechanism to elicit this information? How do our results extend to the case of common values? Do our results still hold when agents discount the value of future objects? We plan to tackle these issues in future research.

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A Proofs

Proof of Lemma 3.1: This is a special case of Lemma 4.3.

Proof of Lemma 4.1: To compute $\omega(i)$ and $\omega(j)$, consider a fixed realization of the order ρ and of the values θ and let $\omega(i, \rho, \theta)$ denote the probability that agent i is proposed the object. Fix one specific order ρ where $\rho(i) < \rho(j)$ and let ρ' be the order in which $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j)$ and $\rho(j) = \rho'(j)$. Clearly, the probability that the first of the two agents is proposed given ρ and θ is the same: $\omega(i, \rho, \theta) = 0 = \omega(j, \rho', \theta)$.
If $q(i) = q(j) = 1$, the probability that the second agent is proposed is equal to zero, so $\omega(i, \rho', \theta) = 0 = \omega(j, \rho, \theta)$.
If $q(i) = q(j) = 0$, if $\theta(i) = \theta(j) = 1$, the probability that the second agent is proposed is equal to zero ; if $\theta(i) = \theta(j) = 0$, the probability that the second agent is proposed is zero unless all agents are selective and pick low values, so that $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 0$ otherwise . If $\theta(i) = 1, \theta(j) = 0$, $\omega(j, \rho, \theta) = 0$ and $\omega(i, \rho', \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(i, \rho', \theta) = 0$ otherwise. Similarly if $\theta(i) = 0, \theta(j) = 1$, $\omega(i, \rho', \theta) = 0$ and $\omega(j, \rho, \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(j, \rho, \theta) = 0$. Hence, if $q(k) = 1$ for some $k \neq i, j$ such that $\rho(k) < \rho(j)$, $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 0$. If for all k, $\rho(k) < \rho(j)$, $q(k) = 0$, taking expectations over the realizations of θ,

$$\omega(i, \rho') = E_\theta \omega(i, \rho', \theta) \equiv \Pr[\theta(k) = 1 \forall k, \rho(k) < \rho(j)] = E_\theta \omega(j, \rho, \theta) \equiv \omega(j, \rho).$$

Finally, in order to compute $\omega(i)$, $\omega(j)$, we take expectations over all the orders. Given that the set of orders R can be decomposed into those orders for which $\rho(i) < \rho(j)$ and those for which $\rho(j) < \rho(i)$, we compute

$$\omega(i) = \sum_{\rho | \rho(i) < \rho(j)} p(\rho)\omega(i, \rho) + \sum_{\rho | \rho(j) < \rho(i)} p(\rho)\omega(i, \rho).$$

For any fixed order ρ such that $\rho(i) < \rho(j)$, consider the associated order ρ' such that $\rho'(j) = \rho(i), \rho'(i) = \rho(j)$.

$$\omega(i) = \sum_{\rho | \rho(i) < \rho(j)} (p(\rho)\omega(i, \rho) + p(\rho')\omega(i, \rho').$$

As noted above, $\omega(i, \rho) = \omega(j, \rho')$ and $\omega(i, \rho') = \omega(j, \rho)$. We also know that for a fixed order ρ, the expected probability of being proposed is higher for the first agent in the order. Hence, $\omega(i,\rho)-\omega(j,\rho) \geq \omega(j,\rho')-\omega(i,\rho') \geq 0$. Furthermore, by assumption 2.1, $p(\rho) \geq p(\rho')$. We thus have:

$$p(\rho)\omega(i,\rho) + p(\rho')\omega(i,\rho') \geq p(\rho)\omega(j,\rho) + p(\rho')\omega(j,\rho'). \tag{9}$$

concluding the proof of the Lemma.

Proof of Lemma 4.2; The proof is by induction on the rank of agents. Consider first agents 1 and 2. Let $q(1)$ and $q(2)$ be the equilibrium strategies of the two agents and suppose first that $q(1) = q(2)$. We compute

$$\begin{aligned} V(1) - c &= [1 - (1-\pi)q(1)) - \frac{c}{\gamma(1)}], \\ V(2) - c &= \frac{\gamma(1)}{\gamma(1)+\gamma(2)}(V(1)-c) + \frac{\gamma(2)}{\gamma(1)+\gamma(2)}[1-(1-\pi)q(2)) - \frac{c}{\gamma(2)}]. \end{aligned}$$

By Lemma 4.1, $\gamma(2) \leq \gamma(1)$ so that $[1-(1-\pi)q(1)) - \frac{c}{\gamma(2)}] \leq [1-(1-\pi)q(2)) - \frac{c}{\gamma(1)}]$, establishing that $V(1)-c \geq V(2)-c$. Next suppose that $q(1) \neq q(2)$ and let $\tilde{V}(1)$ denote the value of agent 1 if he plays strategy $q(2)$. By the preceding computation, $\tilde{V}(1) \geq V(2)$. But as agent 1 optimally chooses $q(1) \neq q(2)$, $V(1) \geq \tilde{V}(1) \geq V(2)$.

Suppose now that for all $j < i$, $V(j+1) \leq V(j)$. Assume first that $q(i) = q(i+1)$ and compute

$$\begin{aligned} V(i) - c &= \frac{\sum_{k=1}^{i-1}\gamma(k)}{\sum_{k=1}^{i}\gamma(k)}(V(i-1)-c) + \frac{\gamma(i)}{\sum_{k=1}^{i}\gamma(k)}[1-(1-\pi)q(i)) - \frac{c}{\gamma(i)}], \\ V(i+1) - c &= \frac{\sum_{k=1}^{i}\gamma(k)}{\sum_{k=1}^{i+1}\gamma(k)}(V(i)-c) + \frac{\gamma(i+1)}{\sum_{k=1}^{i+1}\gamma(k)}[1-(1-\pi)q(i+1)) - \frac{c}{\gamma(i+1)}]. \end{aligned}$$

By the induction hypothesis, $V(i) - c \leq V(i-1) - c$ and by Lemma 4.1, $\gamma(i+1) \leq \gamma(i)$ so that $[1-(1-\pi)q(i)) - \frac{c}{\gamma(i+1)}] \leq [1-(1-\pi)q(i)) - \frac{c}{\gamma(i)}]$. Hence $V(i+1) \leq V(i)$. If $q(i+1) \neq q(i)$, let $\tilde{V}(i)$ denote the value fo agent i when he chooses $q = q(i+1)$. By the same revealed preference argument as above, $V(i) \geq \tilde{V}(i) \geq V(i+1)$, completing the proof of the Lemma.

Proof of Lemma 4.3: Let $\mu(i)$ denote the probability that an agent $k < i$ accepts the object after is refusal so that the continuation value of agent i after a rejection is

$$W(i) \equiv \mu(i)V(i-1) + (1-\mu(i))V(i) - c.$$

Suppose by contradiction that there exists a (pure strategy) equilibrium where $q(i) = 1$ and $q(i+1) = 0$. This implies that $W(i) \leq 0 \leq W(i+1)$. By Lemma 4.2, $V(i+1) \leq V(i) \leq V(i-1)$. Furthermore, if p is not the uniform random order, an inspection of the proof of Lemma 4.1 shows that $\gamma(i) > \gamma(i+1)$ if $q(i) = q(i+1)$ so that the proof of Lemma 4.2 can be adapted to show that these inequalities are strict. Hence, if $W(i) \leq 0$ necessarily $V(i) < 0$ and if $W(i+1) > 0$, we must have $V(i) > 0$, yielding a contradiction.

Proof of Proposition 4.4: Consider first a selective agent $i \leq k$. Notice that, as all $t < i$ are selective in the k equilibrium, $\sum_{t=1}^{i} \omega(t) \leq (1 + (1-\pi) + ... + (1-\pi)^{k-1})$. The probability that the object is proposed to an agent $t = 1,..,i$ is maximized when p guarantees that the first agents who are proposed the object are agents $1,..,i$. In that case, irrespective of the order in which agents $1,..i$ are placed, the probabilities of being proposed are 1 for the first agent, $1-\pi$ for the second,...,$(1-\pi)^{-1}$ for the ith agent in the order ρ. Next notice that the strict seniority order is the only order which guarantees that for any $i \leq k$, $\sum_{t=1}^{i} \omega(t) \leq (1 + (1-\pi) + ... + (1-\pi)^{k-1})$. Now

$$V^k(i) = 1 - \frac{ic}{\pi \sum_{t=1}^{i} \omega(t)} \leq 1 - \frac{ic}{\pi(1 + (1-\pi) + ... + (1-\pi)^i)} = \hat{V}^k(i).$$

Next consider an agent $i > k$. Since the $k+1$th player in the order ρ will necessarily be nonselective and pick up the object, the sum of probabilities that an agent $t = 1,...i$ is proposed the object is bounded by $(1 + (1-\pi) + ... + (1-\pi)^k)$ for any $i > k$. Notice that the maximum is attained for any order ρ which places agents $1, 2..., k$ at the top. Furthermore, the probability that an agent $t = 1,...i$ picks up the object is always bounded by 1. In the strict seniority order, this probability is exactly equal to 1, as agent $k+1$ will always pick up the object when none of the selective agents has accepted it. Next, notice that

$$V^k(i) = \frac{\pi \sum_{t=1}^{i} \omega(t)}{\sum_{t=1}^{i} \gamma(t)} - \frac{ic}{\sum_{t=1}^{i} \gamma(t)}.$$

Given that in equilibrium $\pi \sum_{t=1}^{i} \omega(t) \leq ic$,

$$V^k(i) \leq \pi \sum_{t=1}^{i} \omega(t) - ic \leq \pi(1 + (1-\pi) + ... + (1-\pi)^k) - ic = \hat{V}^k(i).$$

Next suppose that $k = 0$, so that no agent is selective. Under the uniform random order $\omega(i) = \frac{1}{n}$ for all i (all agents have an equal probability to be chosen first) whereas, by Lemma 4.1 $\omega(i) \leq \omega(j)$ for $i < j$ in all mechanism $p \in \mathcal{P}$. Furthermore, as the first player picks the object, $\sum_{t=1}^{n} \omega(t) = 1$ and hence $\sum_{t=1}^{i} \omega(t) \geq \frac{i}{n} = \sum_{t=1}^{i} \tilde{\omega}(i)$. Now in a 0 equilibrium,

$$V^0(i) = \pi - \frac{ic}{\sum_{t=1}^{i} \omega(t)} \geq \pi - nc = \tilde{V}^0(i).$$

If $k = 1$, again under the uniform random order $\tilde{\omega}(i) = \tilde{\omega}(j) = \frac{1}{n}\sum_{t=0}^{n-1}(1-\pi)^t = \frac{1-(1-\pi)^n}{n\pi}$. For any other mechanism $p \in \mathcal{P}$, by Lemma 4.1, $\omega(i) \geq \omega(j)$ if $i < j$. Furthermore, $\sum_{i=1}^{n} \omega(i) = \frac{1-(1-\pi)^n}{\pi}$. Hence,

$$V^n(i) = 1 - \frac{ic}{\pi \sum_{t=1}^{i} \omega(t)} \geq 1 - \frac{nc}{1-(1-\pi)^n} = \tilde{V}^n(i),$$

completing the proof of the Proposition.

Proof of Proposition 4.5: Notice first that, along any realization of ρ where i precedes $k+1$, the probability that i is proposed the object remains the same under the k and the $k+1$ equilibrium, but along any realization of ρ where $k+1$ precedes i, the probability that i is proposed the object weakly increases. Hence, denoting by $\omega^k(i)$ the expected probability that i is proposed the object in the k equilibrium, $\omega^k(i) \leq \omega^{k+1}(i)$ for all $i \neq k+1$ and $\omega^k(k+1) = \omega^{k+1}(k+1)$. Now consider first $i \leq k$, as

$$W^{k+1}(i) = 1 - \frac{c}{\pi \sum_{t=1}^{i} \omega^{k+1}(t)},$$

and $\omega^{k+1}(t) \geq \omega^k(t)$ for all t, $W^{k+1}(i) \geq W^k(i)$. Next, consider $i \geq k+1$. We will prove the stronger statement:

$$[1 - \sum_{t=i+1}^{n} \omega^{k+1}(t)]W^{k+1}(i) \geq [1 - \sum_{t=i+1}^{n} \omega^k(t)]W^k(i).$$

Notice that for $i = k+1$ this statement is equivalent to:

$$\pi \sum_{t=1}^{k} \omega^{k+1}(t)W^{k+1}(k) + \pi\omega^{k+1}(k+1) - c \geq \pi \sum_{t=1}^{k} \omega^k(t)W^k(k) + \pi\omega^k(k+1) - c,$$

Using the fact that $\omega^{k+1}(t) \geq \omega^k(t)$ for all $t \neq k+1$, $\omega^{k+1}(k+1) = \omega^k(k+1)$ and $W^{k+1}(k) \geq W^k(k)$, , the inequality follows. Suppose now that the statement is true for all $t < i$ and consider i, we compute

$$[1-\sum_{t=i+1}^{n}\omega^{k+1}(t)W^{k+1}(i)=\sum_{t=1}^{i-1}\gamma^{k+1}(t)W^{k+1}(i-1)+\pi\omega^{k+1}(i)-c.$$

Now, as $i-1>k$, $\sum_{t=1}^{i-1}\gamma^{k+1}(t)=1-\sum_{t=i}^{n}\omega^{k+1}(t)$, and using the induction hypothesis and the fact that $\omega^{k+1}(i)\geq\omega^{k}(i)$,

$$\begin{aligned}
[1-\sum_{t=i+1}^{n}\omega^{k+1}(t)W^{k+1}(i) &= [1-\sum_{t=i}^{n}\omega^{k+1}(t)]W^{k+1}(i-1)+\pi\omega^{k+1}(i)-c\\
&\geq [1-\sum_{t=i}^{n}\omega^{k}(t)]W^{k}(i-1)\pi\omega^{k}(i)-c\\
&= [1-\sum_{t=i+1}^{n}\omega^{k}(t)]W^{k}(i),
\end{aligned}$$

completing the proof of the Proposition .

Proof of Proposition 4.6: Notice that the equilibrium under $\hat{p}$ is always more selective than the equilibrium under $\tilde{p}$. If both mechanisms admit a n equilibrium, the expected values of all agents are higher under $\hat{p}$ by Proposition 4.4. If the equilibrium under $\hat{p}$ is a k equilibrium and a 0 equilibrium under $\tilde{p}$, the expected values of all agents are higher under $\hat{p}$ by Propositions 4.5 and 4.4. The results on misallocation and expected waste derive from the fact that the equilibrium under $\hat{p}$ is always more selective.

Proof of Lemma 5.2 : The proof is by induction on the rank of agents. Consider first agents 1 and 2. Let $\theta(1)$ and $\theta(2)$ be the equilibrium strategies of the two agents and suppose first that $\theta(1)=\theta(2)=\theta$. We compute

$$\begin{aligned}
W(1) &= \frac{\int_{\theta}^{\overline{\theta}}tf(t)dt}{1-F(\theta)}-\frac{c}{\gamma(1)}\\
W(2) &= \frac{\gamma(1)}{\gamma(1)+\gamma(2)}W(1)+\frac{\gamma(2)}{\gamma(1)+\gamma(2)}[\frac{\int_{\theta}^{\overline{\theta}}tf(t)dt}{1-F(\theta)}-\frac{c}{\gamma(2)}].
\end{aligned}$$

By Lemma 5.1, $\gamma(2)\leq\gamma(1)$ so that $[\frac{\int_{\theta}^{\overline{\theta}}tf(t)dt}{1-F(\theta)}-\frac{c}{\gamma(2)}]\leq[\frac{\int_{\theta}^{\overline{\theta}}tf(t)dt}{1-F(\theta)}-\frac{c}{\gamma(1)}]$, establishing that $W(1)\geq W(2)$. Next suppose that $\theta(1)\neq\theta(2)$ and let $\tilde{W}(1)$ denote the value of agent 1 if he plays strategy $\theta(2)$. By the preceding computation, $\tilde{W}(1) \geq W(2)$. But as agent 1 optimally chooses $\theta(1)$, $W(1) \geq \tilde{W}(1) \geq W(2)$.

Suppose now that for all $j < i$, $W(j+1) \leq W(j)$. Assume first that $\theta(i) = \theta(i+1) = \theta$ and compute

$$\begin{aligned} W(1) &= \frac{\sum_{k=1}^{i-1} \gamma(k)}{\sum_{k=1}^{i} \gamma(k)} W(i-1) + \frac{\gamma(i)}{\sum_{k=1}^{i} \gamma(k)} [\frac{\int_\theta^{\overline{\theta}} t f(t) dt}{1 - F(\theta)} - \frac{c}{\gamma(i)}], \\ W(i+1) &= \frac{\sum_{k=1}^{i} \gamma(k)}{\sum_{k=1}^{i+1} \gamma(k)} W(i) + \frac{\gamma(i+1)}{\sum_{k=1}^{i+1} \gamma(k)} [\frac{\int_\theta^{\overline{\theta}} t f(t) dt}{1 - F(\theta)} - \frac{c}{\gamma(i+1)}]. \end{aligned}$$

By the induction hypothesis, $W(i) \leq W(i-1)$ and by Lemma 4.1, $\gamma(i+1) \leq \gamma(i)$ so that $[[\frac{\int_\theta^{\overline{\theta}} t f(t) dt}{1-F(\theta)} - \frac{c}{\gamma(i+1)}] \leq [\frac{\int_\theta^{\overline{\theta}} t f(t) dt}{1-F(\theta)} - \frac{c}{\gamma(i)}]$. Hence $W(i+1) \leq W(i)$. If $\theta(i+1) \neq \theta(i)$, let $\tilde{W}(i)$ denote the value fo agent i when he chooses $\theta(i+1)$. By the same revealed preference argument as above, $W(i) \geq \tilde{W}(i) \geq W(i+1)$.

Proof of Proposition 5.3: Rewrite the value of agent i as

$$V(i) = \frac{\sum_{j=1}^{i} \gamma_j [\frac{\int_{\theta(j)}^{\overline{\theta}} t f(t) dt}{1 - F(\theta(j))} - \frac{c}{\gamma_j}]}{\sum_{j=1}^{i} \gamma_j}.$$

Observe that because $\gamma(i) > \gamma(i+1)$, $-\frac{c}{\gamma_i} > -\frac{c}{\gamma_{i+1}}$ and because $\theta(i) \geq \theta(i+1)$, $\frac{\int_{\theta(i)}^{\overline{\theta}} t f(t) dt}{1-F(\theta(i))} \geq \frac{\int_{\theta(i+1)}^{\overline{\theta}} t f(t) dt}{1-F(\theta(i+1))}$. Hence $W(i)$ is a convex combination of the decreasing sequence of points $[\frac{\int_{\theta(j)}^{\overline{\theta}} t f(t) dt}{1-F(\theta(j))} - \frac{c}{\gamma_j}]$, $j = 1, .., i$. It is thus maximized when we place maximal weight on γ_1, then maximal weight on γ_2, etc. The probabilistic sequence in $\mathcal{P}$ which lexicographically maximizes the vector $(\gamma_1, ..., \gamma_i)$ for any strategy vector $(\theta(1), .., \theta(n))$ is the seniority order. The probabilistic sequence which minimizes the vector is the uniform lottery, establishing the result.

Proof of Proposition 5.4: For any i and any $(\theta(1), .., \theta(i-1))$ let $\Theta_i(\theta(1), ..., \theta(i-1))$ denote the optimal threshold of agent i given the thresholds of the preceding agents. As a first step of the proof, we show the following claim:

Claim A.1 *For any i, any $(\theta(1), ..., \theta(i-2))$ and any $\theta(i_1) \leq \Theta_{i-1}(\theta(1), ..., \theta(i-2))$, $g_i(\theta(1), .., \theta(i-1)) \equiv -\theta(i-1) + \int_{\Theta_i(\theta(1), .., \theta(i-1)}^{\overline{\theta}} t f(t) dt + F(\Theta_i(\theta(1), .., \theta(i-1))\phi_i(\theta(1), ..., \theta(i-1)) \geq 0$.*

Proof of Claim A.1: Recall that the threshold of agent i under the priority rule is computed as the unique solution to

$$\begin{aligned}\theta(i)[1-F(\theta(1))..F(\theta(i))] &= \int_{\theta(1)}^{\overline{\theta}} tf(t)dt + F(\theta(1))\int_{\theta(2)}^{\overline{\theta}} tf(t)dt + .. \\ &+ .. + F(\theta(1))..F(\theta(i-1))\int_{\theta(i)}^{\overline{\theta}} tf(t)dt - ic = 0.\end{aligned}$$

We compute

$$\text{sign}\frac{\partial\Theta_i}{\partial\theta(i-1)} = \text{sign}[-\theta(i-1) + \int_{\Theta_i}^{\overline{\theta}} tf(t)dt + F(\Theta_i)\Theta_i] \equiv g_i$$

Our objective is to sign this last expression g_i. Wet compute the derivative of g_i with respect to $\theta(i-1)$:

$$\frac{g_i}{\theta(i-1)} = -1 + F(\Theta_i)\frac{\partial\Theta_i}{\partial\theta(i-1)}.$$

Because $g(i) = 0$ if and only if $\frac{\partial\Theta_i}{\partial\theta(i-1)} = 0$, we conclude that $\frac{\partial g_i}{\partial\theta(i-1)} < 0$ when $g_i = 0$. As $g(i) > 0$ for $\theta(i-1) = 0$, this establishes that there exists a $\overline{\theta}(i-1)$ such that, whenever $\theta(i-1) < \overline{\theta}(i-1)$, $g_i(\theta(i-1)) > 0$. In order to prove the claim, we will show that $g_i(\theta(1),..,\theta(i-2),\Theta_{i-1}(\theta_1),..\theta(i-2)) > 0$.

In fact, we use the characterization of the optimal threshold θ_{i-1} to compute

$$\begin{aligned}\Theta_{i-1}(1-F(\theta(1))..F(\theta(i-1)) &= \int_{\theta(1)}^{\overline{\theta}} tf(t)dt + ... \\ &+ F(\theta(1))..F(\theta(i-2))\int_{\Theta_{i-1}}^{\overline{\theta}} tf(t)dt - (i-1)c.\end{aligned}$$

Replacing in g_i and multiplying by $(1-F(\theta(1),...F(\Theta_{i-1})$, we obtain

$$
\begin{aligned}
B_i &\equiv g_i(\theta(1),..,\theta(i-2),\Theta_{i-1})(1-F(\theta(1),...F(\Theta_{i-1}) \\
&= -\int_{\theta(1)}^{\overline{\theta}} tf(t)dt - ... - F(\theta(1))..F(\theta(i-2))\int_{\Theta_{i-1}}^{\overline{\theta}} tf(t)dt \\
&+ (i-1)c + (1-F(\theta(1),...F(\Theta_{i-1})[\int_{\Theta_i}^{\overline{\theta}} tf(t)dt + F(\Theta_i)\Theta_i, \\
&= -\int_{\theta(1)}^{\overline{\theta}} tf(t)dt - ... - F(\theta(1))..F(\Theta(i-1))\int_{\Theta_i}^{\overline{\theta}} tf(t)dt \\
&+ (i-1)c + \int_{\Theta_i}^{\overline{\theta}} tf(t)dt + [1-F(\theta(1))...F(\Theta_{i-1})F(\Theta_i)]\Theta_i - (1-F(\Theta_i))\Theta_i.
\end{aligned}
$$

Now using the characterization of the optimal threshold Θ_i at $(\theta(1),..,\theta(i-2),\Theta_{i-1},$

$$
[1-F(\theta(1))...F(\Theta_{i-1})F(\Theta_i)]\Theta_i = \int_{\theta(1)}^{\overline{\theta}} tf(t)dt + ... + F(\theta(1))..F(\Theta(i-1))\int_{\Theta_i}^{\overline{\theta}} tf(t)dt - ic,
$$

so that

$$
B_i = \int_{\Theta_i}^{\overline{\theta}} (t-\Theta_i)f(t)dt - c.
$$

Now observe that $\int_{\hat{\theta}(1)}^{\overline{\theta}}(t-\hat{\theta}(1))f(t) - c = 0$, that we must necessarily have $\Theta_i < \hat{\theta}(1)$ because $\hat{\theta}(1)$ is the value of an agent at an unconstrained maximum and that $\int_{\theta}^{\overline{\theta}}(t-\theta)f(t)dt$ is a decreasing function of θ to conclude that $g_i(\theta(1),..,\theta(i-2),\Theta_{i-1}) \geq 0$, establishing the claim.

Using Claim A.1, we immediately observe that $-\theta(1) + \int_{\phi_2}^{\overline{\theta}} tf(t)dt + F(\phi_2)\phi_2 \geq 0$ so that $\phi_2(\theta)$ is an increasing function for $\theta \leq \hat{\theta}_1$. Observe also that by Claim A.1, for any $\theta(j) < \Theta_j$, $-\theta(j) + \int_{\Theta_{j+1}}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1})\Theta_{j+1} \geq 0$. The next step of the proof uses an induction argument. Consider an agent at rank i. Suppose that for all $j < i$, $\phi_j(\theta)$ is increasing in θ for $\theta \leq \hat{\theta}_1$. We compute

$$
\frac{\partial \phi_i}{\partial \theta} = \sum_{j=1}^{i-1} \frac{\partial \phi_j}{\partial \theta}\frac{\partial \theta_i}{\partial \theta_j} + \frac{\partial \theta_i}{\partial \theta}.
$$

By the inductive step, $\frac{\partial \phi_j}{\partial \theta} \geq 0$ for all j. Now compute for any $j \leq i$,

$$\begin{aligned}
\text{sign}\frac{\partial \theta_i}{\partial \theta_j} &= \text{sign} - \theta(j) + \int_{\Theta_{j+1}}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1}) \int_{\Theta_{j+2}}^{\overline{\theta}} tf(t)dt \\
&+ \;....F(\Theta_{j+1})...F(\theta_{i-1}) \int_{\Theta_i}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1})...F(\Theta_i)\Theta_i.
\end{aligned}$$

Next we compute

$$\begin{aligned}
D &= -\theta(j) + \int_{\Theta_{j+1}}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1}) \int_{\Theta_{j+2}}^{\overline{\theta}} tf(t)dt \\
&+ \;....F(\Theta_{j+1})...F(\theta_{i-1}) \int_{\Theta_i}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1})...F(\Theta_i)\Theta_i, \\
&= [-\theta(j) + \int_{\Theta_{j+1}}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+1})\Theta_{j+1}] \\
&+ \;F(\Theta_{j+1})[-\Theta(j+1) + \int_{\Theta_{j+2}}^{\overline{\theta}} tf(t)dt + F(\Theta_{j+2})\Theta_{j+2}] \\
&+ \;...F(\Theta_{j+1})...F(\Theta_{i-1})[-\Theta_{i-1} + \int_{\Theta_i}^{\overline{\theta}} tf(t)dt + F(\Theta_i)\Theta_i]
\end{aligned}$$

By the inductive step, all summands in D are positive so that $D \geq 0$, completing the proof of the Lemma.

Proof of Proposition 5.5: As a first step in the proof, we show that $\theta_1^* = W_1^*$ is maximized at $p = 1$. Fix a value θ_2, and consider the mappings $h(\theta, p) = W_1(p, \theta, \theta_2)$. By Proposition **??**, $h(\theta, p)$ is increasing in p. Hence, for all θ and all $p < 1$, $h(\theta, 1) > h(\theta, p)$. Now, by equation 5, θ_1^* is a fixed point of the mapping $h(p, \theta)$. By Theorem 1 in Milgrom and Roberts (1994), because $h(p, \theta)$ is monotonically increasing in p, the lowest and highest fixed points of $h(p, \theta)$ are increasing in p. In addition, at $p = 1$, there exists a unique fixed point θ_1^*, which is independent of θ_2. Hence, for any θ_2, the unique fixed point $\theta_1^*(1) > \theta^(1)(p, \theta_2) \geq \theta_1^*(p)$, establishing the result.

To show that W_2^* is maximized at $p = 1$,, consider any $p < 1$ and pick equilibrium thresholds $\theta_1^*(p)$ and $\theta_2^*(p)$. By Lemma 5.2, $\theta_1^*(p) \geq \theta_2^*(p)$ (with strict inequality when $p > \frac{1}{2}$). By Proposition 5.3, W_2 evaluated at $(\theta_1^*(p)\theta_2^*(p))$ is increasing in p, so that

$$W_2(1, \theta_1^*(p), \theta_2^*(p)) > W_2(p, \theta_1^*(p), \theta_2^*(p)) = W_2^*(p).$$

In addition, because agent 2 can optimally choose θ_2^* as a best response to $\theta_1^*(p)$ when $p = 1$,

$$\tilde{W_2}(\theta_1^*(p)) \geq W_2(1, \theta_1^*(p), \theta_2^*(p)).$$

Finally, notice that we have shown that $\theta_1^*(1) > \theta_1^*(p)$ and, by Proposition 5.4, for any $\theta_1 < \theta_1^*(1)$, $W_2^*(1)$ is strictly increasing in θ_1.. Hence,

$$W_2^*(1) = \tilde{W_2}(\theta^*(1)) > W\tilde{-}2(\theta_1^*(p),$$

establishing that $W_2^*(1) > W_2^*(p)$ for all $p < 1$.

Proof of Lemma 5.7:

Consider the probability of misallocation μ and notice that, as $\theta_1^*(1) > \theta_1^*(p)$ for all p,

$$\begin{aligned}
\mu(1) &= \int_{\theta_1^*(1)}^{\bar{\theta}} (1 - F(\theta))d\theta \\
&< \int_{\theta_1^*(p)}^{\bar{\theta}} (1 - F(\theta))d\theta \\
&< \int_{\theta_1^*(p)}^{\bar{\theta}} (1 - F(\theta))d\theta + (1-p)\int_{\theta_2^*(p)}^{\theta_1^*(p)} (1 - F(\theta_1^*(p)) \\
&= \mu(p).
\end{aligned}$$