# Aggregative Games with Entry[1]

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This version: May 2010

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[1]We thank Daniel Halbheer, and seminar participants at the Federal Trade Commission and Johns Hopkins University for their comments.

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**Abstract**

Aggregative games are used to model strategic interaction in many fields of economics, including industrial organization, political economy, international trade, and public finance. In such games, each player's payoff depends on his/her own actions and an aggregate of all player's actions. Examples in industrial organization are the Cournot oligopoly model, logit and CES differentiated products Bertrand models (and linear demand models in the short run), and R&D games. We suppose a change affects some of the players, such as cost shocks (subsidies, tariffs), privatization, and a merger or RJV. In a unifying framework, we determine the impact of the change on the aggregate variable, producer surplus and consumer surplus under free entry. We also show that the IIA property of demands implies that consumer surplus depends on the aggregate alone, and the corresponding Bertrand pricing game is aggregative.

**JEL Classifications:** D43, L13

**Keywords:** Aggregative games; entry; strategic substitutes and complements; IIA property.

# 1 Introduction

Consider a mixed oligopoly industry of a public firm and private firms producing differentiated products and demand characterized by the CES model and a free entry zero profit condition. The public firm is then privatized, and there is entry or exit so that private firms again make zero profit. At the new equilibrium, total surplus is higher if and only if the privatized firm was making a loss before the change. Or consider an industry with logit demands and suppose there is a merger, with a zero profit condition for marginal firms both before and after. Even though the merged firm's price rises and there is new entry, consumer surplus is unaltered. Or consider an R&D race with free entry. Some firms participate in the race as members of a cooperative R&D arrangement. R&D cooperation has no impact on the aggregate rate of innovation in the industry (hence need not be encouraged or discouraged) even though the number of participants in the race increases with R&D cooperation. Or indeed compare a homogenous Cournot industry with free entry with one where a firm acts as Stackelberg leader. The size of fringe firms is the same in these industrial structures, although there are fewer of them. Moving outside IO, consider a rent-seeking (lobbying) game (Tullock) where efforts determine success probabilities via a contest success function. Equilibrium efforts of unaffected firms are independent of efficiency gains (to lobbying) to infra-marginal firms.

What ties together all these examples is that they are all aggregative games, and consumer surplus is a monotonic function of the aggregator. Many noncooperative games studied in the literature are aggregative games, where each player's payoff depends on his/her own actions and an aggregate of all player's actions. The goal of this paper is to consider the impact of entry in such games. We consider scenarios where an exogenous change affects some of the players in the market. We detail the comparative statics and welfare properties of efficiency or behavioral changes to infra-marginal agents. Examples of the exogenous changes include (i) cost shocks (subsidies, tariffs), (ii) privatization, and (iii) merger or formation of a RJV. We develop a unifying framework to analyze the impact of the change on the aggregate variable, producer surplus and consumer surplus when the change is followed by entry or exit. The analysis makes extensive use of the cumulative best reply function introduced by Selten (1970), for which we derive the corresponding maximal profit function.

Acemoglu and Jensen (2009) analyze the short-run comparative statics properties of aggregative games. Our contribution is to consider entry, and impact on consumer welfare and total surplus. A further difference from Acemoglu and Jensen (2009) is that they only consider comparative static changes which affect all of the firms in the industry (so they fall in type (i) above) whereas we consider a broader set of changes, but the changes affect only a subset of the firms.

# 2 Preliminaries

## 2.1 Game

We consider two-stage games where in the first stage, firms simultaneously make entry decisions. Entry involves a sunk cost. In the second stage, after observing how many firms have sunk the entry cost, the firms simultaneously choose their actions. We will also consider sequential choices of actions in the applications.

We consider a class of aggregative games whereby payoffs can be written as the sum of the actions (or strategic variables) of all agents as well as one's own action. These strategic variables may be monotonic transformations of the "direct" choices: in some central examples of Bertrand games with differentiated products, we will write the strategic variables as monotonically decreasing in prices. For what follows, it may be easiest for the reader to think of Cournot games, where the aggregator is simply the sum of all firms' output choices. Other applications will be brought in where pertinent.

For (homogeneous product) Cournot games, payoffs are written simply as a function of own output, $q_i$, and the sum of all other firms' outputs, $Q_{-i}$. That is, $\pi_i = p\left(Q_{-i} + q_i\right) q_i - C_i\left(q_i\right)$, or indeed $\pi_i = p\left(Q\right) q_i - C_i\left(q_i\right)$, where $Q$ is the value of the aggregator, here total output. Note that here the consumer surplus depends only on the price, $p\left(Q\right)$, and so tracking what happens to the aggregator is a sufficient statistic for tracking what happens to consumer welfare. The benchmark case to which we shall sometimes refer is the Cournot model with log-concave demand, $p\left(Q\right)$, and constant marginal cost, $C_i\left(q_i\right) = c_i q_i + K$, where $K$ is the sunk entry cost.

Another example of an aggregator is in the Bertrand differentiated products model with logit demands. Here we have $\pi_i = (p_i - c_i) \frac{\exp[(s_i - p_i)/\mu]}{\sum_{j=0}^{n} \exp[(s_j - p_j)/\mu]}$ where the $s_j$ represent vertical "quality" parameters and $\mu > 0$ represents the degree of preference heterogeneity across products. The "outside" option has price 0. The denominator of this expression is the aggregator. It can be written as the sum of individual choices by defining $q_j = \exp[(s_j - p_j)/\mu]$ so that we can think of firms as choosing the values $q_j$. Then we write $\pi_i = (s_i - \mu \ln q_i - c_i) \frac{q_i}{Q}$. Note here that $q_j$ varies inversely with price, $p_j$, so that the strategy space $p_j \in [c_j, \infty]$ under the transformation becomes $q_j \in [0, \exp[(s_j - c_j)/\mu]]$. Strategic complementarity of prices implies strategic complementarity of the $q$'s.

## 2.2 Payoffs

Consider the sub-games in the second (post-entry) stage of the game.

Let $A$ be the set of active firms: an active firm is one which has sunk the entry cost. Let $Q$ be the value of the aggregator (total output in the Cournot case), and let $q_i \geq 0$ be each active firm's contribution, so $Q = \sum_{i \in A} q_i$. Let $Q_{-i} = Q - q_i$ be the aggregate choices of all firms other than $i$. Now let $\pi_i(Q_{-i} + q_i, q_i)$ be Firm $i$'s profit, and let $\pi_i^*(Q_{-i})$ be $i$'s maximized profit function, so $\pi_i^*(Q_{-i}) = \pi_i(Q_{-i} + r_i(Q_{-i}), r_i(Q_{-i}))$ where $r_i(Q_{-i})$ denotes the best-reply (or reaction) function, i.e., $r_i(Q_{-i}) = \arg\max_{q_i} \pi_i(Q_{-i} + q_i, q_i)$.

**Assumption 1** (Competitive Games) $\pi_i(Q_{-i} + q_i, q_i)$ is strictly decreasing in $Q_{-i}$.

This competitiveness assumption means that agents are hurt by larger rival actions. Note that the aggregator we use for Bertrand differentiated product games will vary inversely with price, so competitiveness applies in that context too. Assumption 1 implies that $\pi_i^*(Q_{-i})$ is strictly decreasing for $Q_{-i} < \bar{Q}_{-i}$, where $\bar{Q}_{-i}$ is defined as the smallest value of $Q_{-i}$ such that $r_i(\bar{Q}_{-i}) = 0$. This will imply below that there is a unique free entry equilibrium. Note that $\bar{Q}_{-i}$ is allowed to differ across firms. For example, when marginal cost is constant, in the central case of homogenous products Cournot oligopoly with constant marginal costs, this means $p(\bar{Q}_{-i}) = c_i$.

**Assumption 2** (Payoff functions) For any $Q_{-i} \in \mathbb{R}^+$, $\pi_i(Q_{-i} + q_i, q_i)$ is continuous, twice differentiable, and strictly quasi-concave in $q_i$ for $Q_{-i} < \bar{Q}_{-i}$, with a strictly negative second derivative with respect to $q_i$ at the maximum.

Assumption 2 implies a continuous and differentiable best response function, $r_i(Q_{-i})$, $i \in A$. These are characterized by $\frac{d\pi_i}{dq_i} = 0$. We will concentrate on two cases for these reaction functions.

The case of strict strategic substitutes arises for $\frac{\partial^2 \pi_i}{\partial q_i \partial Q_{-i}} < 0$. This means that the derivative $\frac{d\pi_i}{dq_i}$ decreases in $Q_{-i}$ so that the value of $q_i$ where this is zero, which characterizes the best response, is smaller when $Q_{-i}$ is larger. Then, $r_i(Q_{-i})$ is a strictly decreasing function for $Q_{-i} < \bar{Q}_{-i}$ and is equal to zero otherwise. This holds in the benchmark Cournot model if demand is log-concave.

Conversely, the case of strict strategic complements applies when $\frac{\partial^2 \pi_i}{\partial q_i \partial Q_{-i}} > 0$. Then $r_i(Q_{-i})$ is strictly increasing because marginal profits rise with rivals' strategic choices.

The next assumption ensures that the value of the aggregator does not fall when rivals increase their actions.

**Assumption 3** (Stability) $\frac{\partial^2 \pi_i}{\partial q_i^2} < \frac{\partial^2 \pi_i}{\partial q_i \partial Q_{-i}}$.

This is a stability condition.[1] The condition is readily verified in the benchmark Cournot model.[2,3] The next result shows that this condition implies there will be no over-reaction: if all others have collectively increased their actions, the reaction of $i$ should not cause the aggregator to fall lower than where it started.

**Lemma 1** *The relation $Q_{-i}+r_i(Q_{-i})$ is monotonically increasing in $Q_{-i}$ under the stability condition $\frac{\partial^2 \pi_i}{\partial q_i^2} < \frac{\partial^2 \pi_i}{\partial q_i \partial Q_{-i}}$.*

**Proof.** First note that $Q_{-i} + r_i(Q_{-i})$ is necessarily increasing in $Q_{-i}$ if $r_i(Q_{-i})$ is non-decreasing (strategic complementarity). So consider $r_i(Q_{-i})$ decreasing (strategic substitutability). Since $r_i(Q_{-i})$ is defined by the first-order condition $\frac{\partial \pi_i}{\partial q_i} = 0$, then $r_i'(Q_{-i}) = \frac{-\partial^2 \pi_i}{\partial q_i \partial Q_{-i}} / \frac{\partial^2 \pi_i}{\partial q_i^2}$. Since the denominator on the LHS is negative by the second-order condition (see Assumption 2), then the stability condition implies that $r_i'(Q_{-i}) > -1$. Hence $Q_{-i} + r_i(Q_{-i})$ is monotonically increasing. ■

[1] See Vives, pp. 47 and 97.

[2] Consider the Cournot profit function $p(Q)q_i - C_i(q_i)$, with first derivative $p'(Q)q_i + p(Q) - C_i'(q_i)$. The stability condition in Assumption 3 is $p''(Q)q_i + 2p'(Q) - C_i''(q_i) < p''(Q)q_i + p'(Q)$ or $p'(Q) < C_i''(q_i)$, which readily holds for $C_i''(q_i) \geq 0$.

[3] For a general treatment of aggregative games, can assume this property. Give example from R&D models.

Given the monotonicity relation established in Lemma 1, then we can specify pertinent relations as functions of $Q$ instead of $Q_{-i}$, and this will be used extensively in the analysis that follows. In particular, since $Q = Q_{-i} + r_i(Q_{-i})$, we can invert this relation (given that the RHS is monotonic, by Lemma 1) to write $Q_{-i} = f_i(Q)$. This is used in the next sub-section. The construction of $Q_{-i} = f_i(Q)$ is illustrated in Figure 1 for the case of strategic substitutes and Figure 2 for strategic complements. In each case, we construct $Q$ for given $Q_{-i}$, and $f_i(Q)$ is then given by flipping the axes (inverting the relation).

### 2.3 Cumulative best reply function

The cumulative best reply of Firm $i$ gives the optimal action of firm $i$ which is consistent with a given value of the aggregator, $Q$. The concept was first introduced by Selten (1970). See Acemoglu and Jensen (2009) for some historical details, and Vives (1999, p. 42) for a discussion.[4]

Let $\tilde{r}_i(Q)$ stand for this cumulative best reply function, and below we will work with $\tilde{r}_i(Q)$ instead of $r_i(Q_{-i})$. Hence, $\tilde{r}_i(Q)$ stands for the portion of $Q$ optimally produced by firm $i$ (i.e., $Q - Q_{-i} = r_i(Q_{-i}) = \tilde{r}_i(Q)$). A continuous and differentiable $r_i(Q_{-i})$ gives us a continuous and differentiable $\tilde{r}_i(Q)$ function (by construction).

Geometrically, $\tilde{r}_i(Q)$ can be constructed in the following way. For strategic substitutes, the stability condition gives us $q_i = r_i(Q_{-i})$ decreasing as a function of $Q_{-i}$, with slope greater than $-1$. At any point on the reaction function, draw down an isoquant (slope $-1$) to reach the $Q_{-i}$ axis, which it attains before the reaction function reaches the axis. The $x-$intercept is the $Q$ corresponding to the $Q_{-i}$ once augmented by $i$'s reaction function contribution. This also relates $q_i = \tilde{r}_i(Q)$. Clearly, $Q$ and $q_i$ are negatively related. This construction is illustrated in Figure 3.

The case of strategic complements gives $Q$ and $q_i$ positively related. Hence, strategic substitutability or complementarity is preserved in the cumulative best replies: for strategic substitutes a higher total means a lower own contribution, and for strategic complements a higher total means a higher own contribution. This construction is illustrated in Figure 4.

[4] Cornes and Hartley (2009) call it the replacement function.

**Lemma 2** *If Assumption 3 (the stability condition) holds,* $\tilde{r}_i(Q)$ *is strictly decreasing for* $Q < \bar{Q}_{-i}$ *for strict strategic substitutes. For strict strategic complements,* $\tilde{r}_i(Q)$ *is strictly increasing. In both cases,*

$$\frac{d\tilde{r}_i}{dQ} = \frac{r_i'}{1 + r_i'}.$$

**Proof.** By definition, $\tilde{r}_i(Q) = r_i(f_i(Q))$. Differentiating yields

$$\frac{d\tilde{r}_i(Q)}{dQ} = \frac{dr_i(Q_{-i})}{dQ_{-i}} \frac{df_i(Q)}{dQ}.$$

Since $Q_{-i} = f_i(Q)$ from the relation $Q = Q_{-i} + r_i(Q_{-i})$, applying the implicit function theorem gives us

$$\frac{df_i}{dQ} = \frac{1}{1 + r_i'}$$

and hence

$$\frac{d\tilde{r}_i}{dQ} = \frac{r_i'}{1 + r_i'}.$$

Note that $\tilde{r}_i' \to 0$ as $r_i' \to 0$ and $\tilde{r}_i' \to \infty$ as $r_i' \to -1$. For the case of strategic substitutes, since $-1 < r_i' < 0$ by Lemma 1, then $\tilde{r}_i' < 0$. For the case of strategic complements, if $0 < r_i' < 1$, then $0 < \tilde{r}_i' < 1/2$. ■

Therefore, the slope of the cumulative best reply has the same sign as the slope of the best reply itself. Lemma 2 implies that $\sum_i \tilde{r}_i(Q)$ will be decreasing for strategic substitutes. This is useful because the cumulative best reply function can be used to establish the existence of an equilibrium: this is why the concept emerged in the literature (Selten, 1970) in the first place. An equilibrium exists if and only if $\sum_i \tilde{r}_i(Q)$ has a fixed point.[5] Uniqueness follows from Assumption 3 (Vives, 1999, p. 43) for the strategic substitutes case, for strategic complements, it suffices to assume that $\sum \tilde{r}_i'(Q) < 1$, where the sum is over all active firms.

Define now $\tilde{\pi}_i^*(Q)$ as the value of profit given a total output of $Q$ and given that Firm $i$ is maximizing its profit given the output of the others, and doing so renders the total as $Q$. This optimized profit level is therefore given by $\tilde{\pi}_i^*(Q) = \tilde{\pi}_i^*(Q, \tilde{r}_i(Q)) = \pi_i^*(Q_{-i} + r_i(Q_{-i}), r_i(Q_{-i})) = \pi_i^*(Q_{-i})$.

[5] Note that flat parts in $\sum_i \tilde{r}_i$ would not cause a problem. Hence, we do not need $\sum_i \tilde{r}_i$ to be strictly decreasing (at least for the present purpose). If flat portions are coming from flat portions in $r_i(Q_{-i})$, then the strict strategic substitutes assumption is violated. Note that if $r_i(Q_{-i})$ has flat parts, then $\tilde{r}_i(Q)$ will have flat parts. More on this below.

**Lemma 3** *Under Assumptions 1-3, the function* $\tilde{\pi}_i^*(Q)$ *is strictly decreasing for* $Q < \bar{Q}_{-i}$ *and is constant (zero) otherwise.*

**Proof.** Recall that $\pi_i^*(Q_{-i})$ is strictly decreasing in $Q_{-i}$ for $Q_{-i} < \bar{Q}_{-i}$ and is constant otherwise. Hence, since $Q_{-i} = f_i(Q)$ is a strictly increasing function for $Q_{-i} < \bar{Q}_{-i}$ (by Lemma 1), then the function $\tilde{\pi}_i^*(Q)$ defined by $\tilde{\pi}_i^*(Q) = \pi_i^*(f_i(Q))$ is strictly decreasing in $Q$ for $Q < \bar{Q}_{-i}$ and is constant otherwise. ■

## 2.4 Free Entry Equilibrium (FEE)

We are now in a position to define an equilibrium with entry. First define by $Q_A$ the equilibrium value of the aggregator when the set $A$ of agents has sunk the entry cost. That is, agents enter and the sub-game ensues with outcome $Q_A$. We already know from above that $Q_A$ is unique given Lemma 3. Notice there could be some zero actions in there: this depends on whether $\tilde{r}_i(Q)$ is symmetric across firms. If it is, then we will have a symmetric equilibrium. Of course, following an exogenous change, which is what we are interested in, $\tilde{r}_i(Q)$ will no longer necessarily be symmetric.

**Definition 1** *A Free Entry Equilibrium (FEE) is defined by a set of entrants, A, such that*

$$\tilde{\pi}_i^*(Q_A) \geq K \quad \textit{for all } i \in A$$

*and*

$$\tilde{\pi}_i^*\left(Q_{\{A+i\}}\right) < K \quad \textit{for all } i \notin A.$$

The first of these conditions means the firms which are in the market do not prefer to exit (and earn zero), while the second condition means the converse, where the notation $\{A+i\}$ denotes the set of agents $A$ plus $i$. Generally, an equilibrium set of agents will not be unique. This is especially true when there are many homogenous agents.

We will further suppose, for the sake of comparisons in the equilibria we consider, that some set of agents is different from others, either before and/or after some exogenous change which will be the subject of our investigation. We think of these as having some advantage (like a lower cost of production, superior product quality) or indeed being subject to different behavior, like Stackelberg leadership, or a different objective function, like with a merger or nationalization. This means they are in $A$ both before and after the change. Denote the set of such firms by $A_I$ and so we will be interested in equilibria where $A_I \subset A$. We shall also be primarily concerned with equilibria at which there is at least one member (but not all the members) of the Fringe, which is the set of firms, $F$, which all have the same profit functions. This will mean that effectively the fringe firms determine the equilibrium characteristics. The assumption of symmetry among the Fringe is a crucial assumption. We relax it in Section 5.

A substantial part of the literature on free entry equilibrium assumes that entry occurs until the marginal fringe firm makes zero profit (and effectively ignores any possible integer issue). To the end of illuminating the role of the zero profit condition in the equilibrium configuration, we introduce the following definition.

**Definition 2** *At a Zero Profit Fringe Equilibrium (ZPFE):*

$$\begin{aligned} \tilde{\pi}_i^* (Q_A) &\geq K \quad \text{for all } i \in A_I \subset A \\ \tilde{\pi}_i^* (Q_A) &= K \quad \text{for all } i \in F \cap A, \end{aligned}$$

*where $F \cap A$ is non-empty.*

The ZPFE is used by many papers in the literature, but it can be criticized because it does not account for integer constraints. In a later section, we specifically account for integers, using a bounds approach.

Suppose now that there is some change (affecting the firms $i \in A_I$) that shifts the equilibrium set of agents from $A$ to $A'$, both of which contain $A_I$, and both $A$ and $A'$ constitute Zero Profit Fringe Equilibria. We wish to characterize the positive and normative characteristics of the comparison of these equilibria.

# 3 Core propositions

We are now in a position to prove the central positive and normative results.

## 3.1 Aggregator

**Proposition 1** *Suppose that there is some change affecting the firms $i \in A_I$, which shifts the ZPFE set of agents from $A$ to $A'$, both of which contain $A_I$ and at least one Fringe firm.*

*Then* $Q_A = Q_{A'}$*.*

**Proof.** By Lemma 3, $\tilde{\pi}_i^*(Q)$ is strictly decreasing in $Q$, which implies that there is a unique solution for the aggregator at any ZPFE. As long as $\tilde{\pi}_i^*(0) < K$, It is given by $Q = \tilde{\pi}_i^{*-1}(K)$, for $i \in F$. Hence, we must have $Q_A = Q_{A'} = \tilde{\pi}_i^{*-1}(K)$. ■

Proposition 1, while simple, is quite a powerful result. The composition of $Q_A$ and $Q_{A'}$ may be quite different from each other due to the change experienced by the infra-marginal firms. For example, in the context of privatization with the CES model (Anderson, de Palma, and Thisse, 1997), there can be more or fewer firms present in the market. Moreover, the result applies irrespective of whether the actions of the firms are strategic substitutes or complements. In most models without entry, which analyze firm behavior in the short run, whether the actions of the firms are strategic substitutes or complements has a significant impact on the equilibrium predictions.

The crucial assumption behind Proposition 1 is that the marginal entrant has the same type before and after the change. In Section 5, we discuss how the results are modified if this is not the case.

### 3.2 Individual actions of fringe and unaffected firms

Proposition 1 implies that the equilibrium actions of the fringe firms will be the same before and after the change.

**Proposition 2** *Suppose that there is some change directly affecting the firms* $i \in A_I$*, which shifts the ZPFE set of agents from* $A$ *to* $A'$*, both of which contain* $A_I$*. Then* $q_{i,A} = q_{i,A'}$ *for all* $i \in F$*, and* $q_{i,A} = q_{i,A'}$ *for all unaffected firms* $i \in A_I$*.*

**Proof.** Since $Q_A = Q_{A'}$ and the cumulative best reply function $\tilde{r}_i(Q)$ is the same for all $i \in F$, we have $\tilde{r}_i(Q_{A'}) = \tilde{r}_i(Q_A)$. Similarly, for an unaffected firm (that is, one whose profit function remains unchanged) we have $\tilde{r}_i(Q_{A'}) = \tilde{r}_i(Q_A)$. ■

For the Cournot model, this means that *per firm* fringe output stays the same at a long-run equilibrium (although the number of fringe firms will typically change) following any change to some infra-marginal firms. Likewise, any infra-marginal firms not directly affected will also see no change in their output. For the privatization model noted above

(Anderson, de Palma, and Thisse, 1997), the equilibrium prices of private firms must be the same independent of the presence of a public firm.

It is important to note that although the equilibrium action of each active fringe firm does not change, the number of fringe firms may change substantially due to exit or entry after the change. We explore the conditions for this in Section 3.3.

### 3.3 Producer surplus and the number of fringe firms

Here we address the question whether insider equilibrium rents increase when some change occurs to the insiders. For example, there may be a privatization or a merger, or their costs may decrease everywhere. For simplicity, we start by analyzing cost or quality improvements. We first (Proposition 3) analyze the effect if a single insider is affected, supposing that it is the only one affected.

We are first interested in how the change affects the affected firm's rents, and we draw out the crucial distinction between the total profit and the marginal profit affect of the change. Then we show that the change in producer surplus is measured as the change in the affected firm's rent.

In what follows, we denote firm $i$'s type parameter by $\theta_i$, and we are interested in how an improvement to $\theta_i$ translates to changes in producer surplus. In what follows we therefore introduce parameters $\theta_i$ into the profit functions of insiders.

**Proposition 3** *Suppose* $\frac{d\pi_i(Q_{-i}+q_i,q_i;\theta_i)}{d\theta_i} > 0$ *for some* $i \in A_I$. *Then higher* $\theta_i$ *raises firm* $i$*'s equilibrium rents at a ZPFE if* $\frac{\partial^2\pi_i(Q_{-i}+q_i,q_i;\theta_i)}{\partial\theta_i \partial q_i} \geq 0$.

**Proof.** Proposition 1 implies that at a ZPFE with homogeneous fringe firms, $Q$ remains unchanged as $\theta_i$ rises. Hence, we wish to show that $\frac{d\tilde{\pi}_i^*(Q;\theta_i)}{d\theta_i} > 0$ with $Q$ fixed.

Let $f_i(Q,\theta_i)$ denote the $Q_{-i}$ locally defined by $Q - Q_{-i} - r_i(Q_{-i},\theta_i) = 0$. Then, by the implicit function theorem,

$$\frac{df_i(Q,\theta_i)}{d\theta_i} = \frac{-\partial r_i/\partial\theta_i}{1+\partial r_i/\partial Q_{-i}}.$$

Invoking Lemma 1, the denominator is positive. Hence, this expression is negative if $\partial r_i/\partial\theta_i$ is positive. Applying the implicit function theorem to the reaction function, which is given in implicit form as $\frac{d\pi_i(Q_{-i}+q_i,q_i;\theta_i)}{dq_i} = 0$, shows that $\partial r_i/\partial\theta_i \geq 0$ if and only if $\frac{\partial^2\pi_i(Q_{-i}+q_i,q_i;\theta_i)}{\partial\theta_i\partial q_i} \geq 0$. Hence, $\frac{df_i(Q,\theta_i)}{d\theta_i} \leq 0$ under this condition.

Now, since by definition $\tilde{\pi}_i^*(Q;\theta_i) = \pi_i^*(f_i(Q,\theta_i);\theta_i)$, we have

$$\frac{d\tilde{\pi}_i^*(Q;\theta_i)}{d\theta_i} = \frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial Q_{-i}}\frac{df_i(Q,\theta_i)}{d\theta_i} + \frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial\theta_i} \tag{1}$$

The term $\frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial Q_{-i}}$ on the RHS is negative by Assumption 1. Hence, $\frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial Q_{-i}}\frac{df_i(Q,\theta_i)}{d\theta_i}$ is positive under the condition stated in the Proposition.

Consider now the last term on the RHS (1). Since $\pi_i(Q_{-i}+q_i,q_i;\theta_i)$ increases with $\theta_i$ by supposition, so does the maximized value represented by $\pi_i^*(Q_{-i};\theta_i)$: hence, $\frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial\theta_i} > 0$.

Hence, $\frac{d\tilde{\pi}_i^*(Q;\theta_i)}{d\theta_i} > \frac{\partial\pi_i^*(Q_{-i};\theta_i)}{\partial\theta_i} > 0$, and $i$'s rents necessarily rise at a ZPFE. ■

The qualification $\frac{\partial^2\pi_i(Q_{-i}+q_i,q_i;\theta_i)}{\partial\theta_i\partial q_i} \geq 0$ in Proposition 3 represents an increasing *marginal* profitability. If marginal profits instead fell with $\theta_i$, then there is a tension between the direct effect of the improvement to $i$'s situation, and the induced effect through a lower action. This tension is illustrated in the following example, where we show that a cost improvement that has a "direct" effect of raising profits may nonetheless end up decreasing them once we duly account for the new free entry equilibrium reaction.

**Example 1** *Consider a Cournot model with demand $p(Q) = 1-Q$. Suppose then that firm costs are $C(q) = cq + K$ for all Fringe firms, and $C_1(q) = (c+\theta)q_1 - \beta\theta + K$ for Firm 1. The Fringe firms' first order conditions are $1 - Q - c = q$, and their zero profit condition is $q = \sqrt{K}$. For Firm 1, the first order conditions are $1 - Q - c - \theta = q_1$, where we see the impact of higher marginal cost reducing output. Hence, $q_1 = q - \theta = \sqrt{K} - \theta$. Since profit is $q_1^2 + \beta\theta - K$, then, using the zero profit condition of the fringe and the output relation just derived this profit is $\left(\sqrt{K}-\theta\right)^2 + \beta\theta - K$ at the zero-profit free entry equilibrium solution. The derivative of this profit expression with respect to $\theta$ is $-2\left(\sqrt{K}-\theta\right) + \beta$, which we can write as $-2q_1 + \beta$. Now notice that the "direct" effect of a marginal change in $\theta$ is $-q_1 + \beta$, meaning that this is the change in profit if all outputs were held constant (of course, Firm 1's output would adjust, but, by the envelope theorem, this expression measures the profit change for given output of rivals). Clearly, depending on the size of $\beta$, a positive direct effect can nonetheless mean a negative final effect, once we factor in the output restriction of the affected firm, and the entry response of the fringe.*

The example shows that the direct profit effect and the marginal profit effect can work in different directions. There are, of course, other examples in the literature where the response of rivals can overwhelm the direct effect.[6] Bulow, Geanakoplos, and Klemperer (1985) provide several examples where a purported benefit turns into a liability once reactions are factored in, in the context of multi-market contact. The original Cournot Merger Paradox of Salant, Switzer, and Reynolds (1983) is a case where merging firms end up worse off after coordinating their output. This latter example can easily be tailored to fit the case at hand, with free entry. Note first that two merging firms in an industry of symmetric firms earn zero profits both before and after merging, so there is no long-run benefit (or cost) to them merging. Suppose instead then that two firms in a (Cournot) industry had lower marginal production costs than their rivals, and the industry is in free entry equilibrium. Then when these two low-cost firms merge, they will restrict output and more firms will enter. The final state will have them worse off than before merging, at least if there are no synergies from the merger.

We now return to the question of total producer surplus following a change to some (or all) of the non-fringe active firms.

**Proposition 4** *Suppose some firms' profit functions change. Then all other firms' rents remain unchanged at a ZPFE, so the change in producer surplus equals the change in rents to the affected firms.*

**Proof.** This follows directly from Propositions 1 and 2: the aggregator remains the same, the best replies remain the same, and, since the profit functions are the same, the rents remain the same for the unaffected firms. Hence, the total change to producer surplus from the change is measured as the change in the affected firms' rents. ■

Thus, all insiders not directly affected will not change their equilibrium behavior, and nor do their rents change. Nor do the rents to fringe firms change, even though their numbers may change, since all $i \in F$ make zero profit whether in the industry or not. Changes in total producer surplus therefore follow what happens to the affected insiders.

If the changes affect all firms in the same way, then we can simply determine what

[6] Although we do not know of examples using free entry as the mechanism?

happens to the equilibrium number of fringe firms. The next result says how that number changes following a change to a single firm.

**Proposition 5** *Suppose* $\frac{d^2\pi_i(Q_{-i}+q_i;\theta_i)}{d\theta_i dq_i} > 0$. *Then, a higher* $\theta_i$ *for firm* $i$ *increases its equilibrium output and decreases the number of fringe firms (and hence the number of active firms) at a ZPFE.*

**Proof.** We know from Proposition 1 that $Q$ is unchanged when $\theta_i$ changes for an inframarginal firm $i$. We also know from Proposition 2 that each active fringe firm's output remains constant and each unaffected insider's output remains constant. Hence, the number of fringe firms must fall in a ZPFE if $i$'s action goes up, so this is what we want to show, namely that $\frac{d\tilde{r}_i(Q,\theta_i)}{d\theta_i} > 0$: which means that Firm $i$ produces more as a result of the change in $\theta_i$.

As we argued in Proposition 3, $\frac{d^2\pi_i(Q_{-i}+q_i;\theta_i)}{d\theta_i dq_i} > 0$ if and only if $\frac{dr_i(Q_{-i},\theta_i)}{d\theta_i} > 0$.

Now, by definition $\tilde{r}_i(Q;\theta_i) = r_i(f_i(Q,\theta_i);\theta_i)$, where we recall that $f_i(.)$ denotes the $Q_{-i}$ locally defined by the relation $Q - Q_{-i} - r_i(Q_{-i};\theta_i) = 0$. Hence,

$$\frac{d\tilde{r}_i(Q;\theta_i)}{d\theta_i} = \frac{\partial r_i(Q_{-i};\theta_i)}{\partial Q_{-i}}\frac{df_i(Q)}{d\theta_i} + \frac{\partial r_i(Q_{-i};\theta_i)}{\partial \theta_i}.$$

As we showed in Proposition 3,

$$\frac{df_i(Q)}{d\theta} = \frac{-\partial r_i/\partial\theta_i}{1+\partial r_i/\partial Q_{-i}}.$$

Hence,

$$\frac{d\tilde{r}_i(Q;\theta_i)}{d\theta_i} = \frac{\partial r_i/\partial\theta_i}{1+\partial r_i/\partial Q_{-i}}, \tag{2}$$

which is positive since the denominator is positive for strategic substitutes or strategic complements with the stability condition (see Lemma 1). ■

The proof relies on showing how the individual cumulative reaction function changes. If there are several affected firms, and they all change the same way (in the sense that $\frac{d\pi_i(Q_{-i}+q_i;\theta_i)}{dq_i}$ is moved in the same direction by the individual changes), then the number of fringe firms goes down if the individual marginal profitabilities move up, etc.

The result at the end of the Proposition (eq., (2)) can be illustrated graphically. This is done in Figure 5 for strategic substitutes, and Figure 6 for strategic complements. In both cases, the cumulative reaction functions shift up when the reaction functions do. Indeed, given $Q = Q_{-i} + r_i(Q_{-i}; \theta)$, a higher $\theta$ leads to a higher $r_i(Q_{-i})$ and hence to a higher $Q$. Hence, inverting, there is a higher $Q$ for a given $Q_{-i}$ and so, since $Q_{-i}$ is increasing in $Q$, we have now a *lower* $Q_{-i}$ for a given $Q$, which means a *higher* $\tilde{r}(Q)$. That is, $q_i$ is now a larger part of any total; as indeed $i$ reacts with a higher action to any $Q_{-i}$.

The construction used in Figures 3 and 4 also neatly illustrates how a higher $\theta$ raises $\tilde{r}_i(Q)$. This is done for the two cases in Figures 7 and 8, respectively. Since a higher $\theta$ raises the reaction function, the same $Q$ line needs to get to a higher $q$. For the case of strategic complements the reaction function slopes up; stability means it is flatter than the diagonal. From the reaction function, draw a line of slope -1 back to the $Q$ axis to give the $Q_{-i}$ associated to $Q$ and $q$; $Q_{-i}$ is increasing in both. Hence $\tilde{r}$ increases when $r$ increases. Raising $\theta$ raises the reaction function. The same $Q_{-i}$ (extending the 45-degree line) associates to a higher $q$ and $Q$. Alternatively, hold $Q$ constant by going vertically above the reaction function to the new one, and observe that the associated $q$ rises while $Q_{-i}$ falls. This indicates the property that if higher a $\theta$ means a higher $\tilde{r}$, then it means a higher $r$, and conversely.

### 3.4 Consumer welfare

In oligopoly examples, the other agents who can be affected by changes are consumers. Since it is the consumers who generate the demand system, their behavior is integral to the firms' problem. Their welfare is an important or even decisive (under a consumer welfare standard) criterion for evaluating the desirability of changes in the industry.

In the benchmark case of Cournot competition with homogeneous goods, the aggregator $Q$ is total output, and consumer welfare depends directly on the aggregator via the market price, $p(Q)$. Therefore, an increase in the aggregator is a sufficient statistic for consumer welfare to rise. This is the case whenever the consumer welfare depends just on the aggregator.

**Proposition 6** *Suppose that consumer surplus depends solely on the value of the aggregator. Suppose that there is some change affecting the firms $i \in A_I$, which shifts the ZPFE set of agents from $A$ to $A'$, both of which contain $A_I$ and at least one Fringe firm. Then* *consumer surplus remains unchanged.*

**Proof.** By Proposition 1, $Q_A = Q_{A'} = \tilde{\pi}^{*-1}(K)$ at any ZPFE, and the result follows immediately. ■

The result does not hold if the composition of $Q$ matters to consumers. This may be the case for example when there is an externality, like pollution, which varies across firms. Then a shift in output composition to less polluting firms raises welfare.

There are important other cases when consumer welfare depends solely on the value of the aggregator (and not its composition). Consider Bertrand (pricing) games, with differentiated products, and suppose the profit function takes the form $\pi_i = (p_i - c_i) D_i(\vec{p})$ where $\vec{p}$ is the vector of prices set by firms (and prices are infinite for inactive firms, reflecting the unavailability of their goods) and $D_i(\vec{p})$ is Firm $i$'s demand function. We are interested in the conditions – which are therefore the properties of the demand function, $D_i(\vec{p})$ – under which the game is an aggregative one and in which also the consumer welfare depends only on the value of the aggregator.

To pursue these questions, consider a quasi-linear consumer welfare (indirect utility) function, $V(\vec{p}, Y) = \phi(\vec{p}) + Y$. Demands are generated from the price derivatives of $\phi(.)$. Suppose first that we can write $\phi(\vec{p})$ as an increasing function of the sum of decreasing functions of $p_i$, so $\phi(\vec{p}) = \tilde{\phi}\left(\sum_i g_i(p_i)\right)$ where $\tilde{\phi}' > 0$ and $g_i'(p_i) < 0$. Then $D_i(\vec{p}) = -\tilde{\phi}'\left(\sum_i g_i(p_i)\right) g_i'(p_i) > 0$, which therefore depends only on the summation and the derivative of $g_i(.)$. Assume further that $D_i(\vec{p})$ is decreasing in own price ($\frac{dD_i(\vec{p})}{dp_i} = -\tilde{\phi}''\left(\sum_i g_i(p_i)\right)[g_i'(p_i)]^2 - \tilde{\phi}'\left(\sum_i g_i(p_i)\right) g_i''(p_i) < 0$).[7] Since $g_i(p_i)$ is decreasing, its value uniquely determines $p_i$ and hence the term $g_i'(p_i)$ in the demand expression. Therefore, demand can be written as a function solely of the summation and $g_i(p_i)$. This means that the game is aggregative, by choosing $g_i(p_i) = q_i$ and $Q = \sum_i q_i$. Hence too the consumer welfare depends only on $Q$ (and not its composition). This structure has another important property, namely that the demand functions satisfy the IIA property: the ratio of any two demands depends only on their own prices and is independent of the price of any other

[7] For the logsum formula which generates the logit model, we have $g_i(p_i) = \exp[(s_i - p_i)/\mu]$ and so $g_i''(p_i) > 0$. However, $\tilde{\phi}$ is concave in its argument, the Sum.

option in the choice set. That is, $\frac{D_i(\vec{p})}{D_j(\vec{p})} = \frac{g_i'(p_i)}{g_j'(p_j)}$. In summary:

**Proposition 7** *Let $\pi_i = (p_i - c_i) D_i(\vec{p})$ and $D_i(\vec{p})$ be generated by an indirect utility function $V(\vec{p}, Y) = \tilde{\phi}\left(\sum_i g_i(p_i)\right) + Y$ where $\tilde{\phi}$ is twice differentiable, strictly convex in $p_i$, and $g_i(p_i)$ is twice differentiable and decreasing. Then demands exhibit the IIA property, the Bertrand pricing game is aggregative, and consumer welfare depends only on the aggregator, $Q = \sum_i q_i$.*

Important examples include the logit and CES models. For the logit model, $V = \mu \ln\left(\sum_i \exp[(s_i - p_i)/\mu]\right) + Y$, and the action variables are $q_i = \exp[(s_i - p_i)/\mu]$.

For the CES model, it is more common to write the direct utility function first, which in quasi-linear form is $U = \frac{1}{\rho}\ln\left(\sum_i x_i^{\rho}\right) + X_0$, where $X_0$ denotes numeraire consumption and $x_i$ is consumption of the numeraire commodity. This implies that demands are $x_i = \frac{p_i^{-\lambda-1}}{\sum_i p_i^{-\lambda}}$ where $\lambda = \frac{\rho}{1-\rho}$. Then $X_0 = Y - 1$ and $V = \frac{1}{\rho}\ln\left(\sum_i \left(\frac{p_i^{-\lambda-1}}{\sum_j p_j^{-\lambda}}\right)^{\rho}\right) + Y - 1$, or $V = \frac{1}{\lambda}\ln\left(\sum_j p_j^{-\lambda}\right) + Y - 1$.[8]

There is also a converse to Proposition 7. Suppose that demands exhibit the IIA property. Then, following Theorem 1 (p.389) in Goldman and Uzawa (1964), and imposing quasi-linearity, $V$ must have the form $\tilde{\phi}\left(\sum_i g_i(p_i)\right) + Y$ where for them $\tilde{\phi}(.)$ is increasing and $g_i(p_i)$ is any function of $p_i$. If we further stipulate that demands must be differentiable, then the differentiability assumptions made in Proposition 7 must hold, and assuming that demands are strictly downward sloping implies that $\tilde{\phi}\left(\sum_i g_i(p_i)\right)$ must be strictly convex in $p_i$. In summary:

**Proposition 8** *Let $\pi_i = (p_i - c_i) D_i(\vec{p})$ and $D_i(\vec{p})$ be twice continuously differentiable and strictly decreasing in own price; furthermore suppose that the demand functions satisfy the IIA property. Then the demands $D_i(\vec{p})$ can be generated by an indirect utility function $V(\vec{p}, Y) = \tilde{\phi}\left(\sum_i g_i(p_i)\right) + Y$ where $\tilde{\phi}$ is twice differentiable, strictly convex in $p_i$, and $g_i(p_i)$ is twice differentiable and decreasing. Then the Bertrand game is aggregative, and consumer welfare depends only on the aggregator, $Q = \sum_i q_i$.*

[8] A useful task is to find a general $V$ which nests these two cases.

However, the fact that a game is aggregative does not imply the IIA property on $V(Q)$. For example, the linear differentiated products demand system of Ottaviano and Thisse (2004) is an aggregative game for a fixed number of firms (i.e., in the short-run) with Bertrand competition since demand can be written as a sum of all prices and own price. However, it does not satisfy the IIA property.[9]

### 3.5 Total welfare

The results in Sections 3.3 and 3.4 give us total welfare results.

**Proposition 9** *Suppose that consumer surplus depends solely on the value of the aggregator. Suppose that there is some change affecting the firms $i \in A_I$, which shifts the ZPFE set of agents from $A$ to $A'$, both of which contain $A_I$ and at least one Fringe firm. Then the change in welfare is measured solely by the change in the affected firms' rents.*

**Proof.** This result follows immediately from Propositions 6 and 4. ■

That is, since the aggregator remains unchanged at any ZPFE, then consumer welfare is unaffected and so is the rent earned by all infra-marginal firms not directly affected by the change.

## 4 Integer constraints

So far, we have considered changes that lead to long-run profits of zero for fringe firms both before and after any changes. In this section, we will take the integer constraint into account, and thus determine bounds on $Q_A$ and $Q_{A'}$ - equilibrium output levels before and after the change - and the associated welfare consequences.

### 4.1 Strategic substitutes

The following is about bounds on $Q_A$, but the same arguments apply to $Q_{A'}$.

Let $Q_L$ and $Q_U$ stand for the lower and the upper bound on the equilibrium aggregate output level with a discrete number of firms and at least one fringe entrant. That is, we

[9] As shown in Erkal and Piccinin (2010), the same demand system is an aggregative game both in the short run and the long run under Cournot competition, but again it does not satisfy the IIA property.

are looking for the lowest value of $Q_A$ such that there is no entry. If $Q_A < Q_L$, there will be entry. Similarly, for the upper bound, we are looking for the highest amount of $Q_A$ such that there is no exit: $\tilde{\pi}_i^* (Q_U) \geq K$ for all $i \in A$. If $Q_A > Q_U$, there will be exit.

The first thing to note is that the aggregate output level cannot be larger than the equilibrium output level in the case when the number of fringe firms is treated as a continuous variable. Hence, $Q_U$ is defined by $\pi_i^* (Q_U) = K$.

As $Q_A$ decreases, the potential entrants outside of the market have increasing incentives to enter the market: however, they need to rationally anticipate that the post-entry equilibrium, having observed the pre-entry statistic, will not exceed $Q_U$. Hence, we can determine the lower bound by considering their incentives to enter. If the potential entrant expects the aggregate reaction of the rivals to be $Q_{-i} < Q_U - \widetilde{r}_i (Q_U)$, it will enter the market. This is because by definition $\pi_i^* (Q_U) = K$ and so for all $Q_{-i} < Q_U - \widetilde{r}_i (Q_U)$, the potential entrant will expect to make positive profits upon entry. If it expects $Q_{-i} \geqslant Q_U - \widetilde{r}_i (Q_U)$, it will not enter the market.

Hence, we can treat $Q_U - \widetilde{r}_i (Q_U)$ as a lower bound on $Q_A$. Actually, under strategic substitutability, $Q_A$ will be higher than this because the rival firms will choose higher actions in the absence of entry: entry tends to be accommodated in the sense that incumbents reduce their equilibrium.

We can now show the following result.

**Proposition 10** *Assume actions are strategic substitutes. Any free-entry equilibrium with at least one fringe entrant must have $Q_A \in [Q_L, Q_U]$, where $Q_U$ is defined by $\pi_f^* (Q_U) = K$ and $Q_L = Q_U - \widetilde{r}_f (Q_U)$.*

**Proof.** First, note that $Q_U = \pi_f^{*-1} (K)$ is the upper bound on the aggregator with at least one fringe firm active, since $\pi_i^* (Q)$ is decreasing in $Q$ by Lemma 3 and $Q_U$ is the largest aggregator value at which an active fringe firm can make a non-negative profit.

Second, we wish to show that any $Q_A < Q_L$ cannot be an equilibrium because it must attract profitable entry by a fringe firm. This is equivalent to showing that if a single fringe entrant joins the set $A$ of active firms, then the ensuing equilibrium aggregator value will not be above $Q_U = \pi_f^{*-1} (K)$. Suppose this is not the case and suppose $Q_{\{A+i\}} > Q_U$ with

$i \in F$. Since, by Lemma 2, $\tilde{r}_i(Q)$ is decreasing under strategic substitutes, each firm in $A$ must be choosing a lower action than before (i.e., at the equilibrium with the fringe firm excluded). This means their actions sum to less than $Q_A$. Likewise, for the incremental fringe entrant, $\tilde{r}_f\left(Q_{\{A+i\}}\right) < \tilde{r}_f(Q_U) = Q_U - Q_L$ under strategic substitutability (also by Lemma 2). Hence, its action must be less than its action at $Q_U$. But then the sum of the actions cannot exceed $Q_U$, a contradiction. ■

Hence, before and after any change, the equilibrium value of the aggregator lies in $[Q_L, Q_U]$. The maximum consumer welfare increase from the change is given by $CW(Q_U) - CW(Q_L)$. The maximum consumer welfare decrease from the change is given by $CW(Q_L) - CW(Q_U)$.[10]

**Example 2** *Consider Cournot competition with symmetric firms and linear demand. Suppose firms are symmetric after the change. Suppose demand is given by $P = 1 - Q$, and marginal costs are zero. Then, $Q_U = 1 - \sqrt{K}$, because each firm would just make zero profit by producing its equilibrium output of $q_i = \sqrt{K}$. The best response function is given by $r_i(Q_{-i}) = \frac{1-Q}{2}$ and the cumulative best response function is given by $\widetilde{r}_i(Q) = 1 - Q$. Hence, the lower bound is given as $Q_U - \widetilde{r}_i(Q_U) = Q_L = 1 - 2\sqrt{K}$. Note that there are $n = \frac{1-\sqrt{K}}{\sqrt{K}}$ firms at $Q_U$ if this is an integer. Suppose then we took out one (indifferent) firm, and check what the new equilibrium total quantity is. For this benchmark Cournot model, $n-1$ firms produce a total output of $\frac{n-1}{n}$. Substituting the value of $n$ given above, we have a total output of $\frac{1-2\sqrt{K}}{1-\sqrt{K}} > Q_L$, which represents the actual lower bound in this case. The ratio of $Q_L$ to the actual one gets small as $K$ gets small, as indeed does the ratio $\frac{Q_L}{Q_U}$.*

## 4.2 Strategic complements

Dealing with the bounds for the strategic complements case is more tricky because $\sum \widetilde{r}_i'(Q)$ can be arbitrarily close to 1. The equilibrium output with one less firm may be very far away from $Q_U$. At the other extreme, if $\sum \widetilde{r}_i'(Q)$ is close to zero, then $Q_L$ is close to the bound found above for the case of strategic substitutes.

[10] Graphing $q_i$ against $Q_{-i}$ and $Q$ reveals that as $Q_{-i} = Q_L$ increases (which happens when $K$ decreases), $q_i$ decreases.

# 5 Heterogeneous fringe

We have assumed until now that the Fringe consists of firms with the same profit functions. The simplest generalization is when firms differ in a manner that can be unambiguously ranked, such as costs, qualities, or entry costs.

## 5.1 Heterogeneous entry costs

We first analyze the case of differing entry costs, by supposing that all fringe firms have the same profit functions, except that they differ by idiosyncratic $K$. Note that Lemmas 1, 2 and 3 of Section 2 of the paper are still pertinent to the current case since they apply to the post-entry sub-games.

Let $K(n)$ denote the entry cost of the $n$th lowest cost fringe entrant (rather similar to a supply curve). Assume the marginal firm earns zero profit at a zero profit equilibrium. Then the equilibrium solution for any set of active firms, $A$, is given by

$$\sum_{i \in A} \tilde{r}_i(Q) = Q, \tag{3}$$

which is a fixed point. We assume the LHS has slope less than 1 (see Figures 11 and 12 below). This is guaranteed for strategic substitutes since the LHS is decreasing because all its components are. For strategic complements, matters are more delicate. Assuming a slope less than 1 at any fixed point guarantees a unique equilibrium. Alternatively, following a device frequently deployed in analyzing comparative statics with strategic complements, we can describe the comparative static properties of the largest and smallest equilibria. The slope condition must hold for both.

Suppose now that one of the insiders' marginal profit rises (for example, due to a cost reduction), and this causes the equilibrium set of firms to move from $A$ to $A'$. From Proposition 5, we know that if all fringe firms have the same entry cost, such a change increases the affected firm's action while leaving the value of the aggregator unchanged. In the following proposition, we state how the equilibrium changes in the case of heterogeneous fringe firms.

**Proposition 11** *Let entry costs be strictly increasing across fringe firms. Suppose one of the insiders ($j \in A$) experiences a change that increases its marginal profit. Let $A$ and $A'$*

*stand for the set of firms in the two ZPFE before and after the change, respectively. Then: (i) $Q_A < Q_{A'}$, (ii) the change causes some fringe firms to exit (i.e., the number of fringe firms is higher at A than at $A'$), (iii) each fringe firm chooses a higher (lower) action if and only if actions are strategic substitutes (complements), and (iv) insider Firm $j$ (the firm which has been exposed to the change) chooses a higher action.*

**Proof.** (i) To show that $Q_A < Q_{A'}$, we can break down the fixed-point condition (3) above to write

$$n\tilde{r}_f(Q) + \sum_{i \in A_I} \tilde{r}_i(Q) = Q. \tag{4}$$

The marginal fringe entrant makes zero profit, or

$$\tilde{\pi}_f(Q) = K(n). \tag{5}$$

Together, these two equations determine the equilibrium outcome.

Totally differentiating (4) gives

$$\tilde{r}_f(Q)\,dn + \left(n\tilde{r}'_f(Q) + \sum_{i \in A_I} \tilde{r}'_i(Q)\right) dQ + d\tilde{r}_j(Q) = dQ \tag{6}$$

where we take into account the fact that insider Firm $j$ experiences a change that increases its marginal profit (a positive $d\tilde{r}_j(Q)$ means it is more aggressive). Totally differentiating (5), solving for $dn$, and substituting in (6) gives

$$\left(\tilde{r}_f(Q)\frac{\partial\tilde{\pi}_f(Q)/\partial Q}{K'(n)} + \left(n\tilde{r}'_f(Q) + \sum_{i \in A_I} \tilde{r}'_i(Q) - 1\right)\right) dQ = -d\tilde{r}_j(Q) \tag{7}$$

Note from (5) that $Q$ and $n$ vary inversely, since $\tilde{\pi}_f(Q)$ is a decreasing function (for both strategic substitutes and complements). Hence, the first term on the LHS of (7) is always negative. The second term, in parentheses, is negative immediately for strategic substitutes, and by invoking the slope condition noted above for strategic complements. Hence, if insider Firm $j$ experiences a change that renders it more aggressive, it must be the case that $Q_A < Q_{A'}$.

(ii) From Lemma 3, recall that $\tilde{\pi}_i^*(Q)$ is strictly decreasing. Hence, if $Q_A < Q_{A'}$, then $\tilde{\pi}_i^*(Q_A) > \tilde{\pi}_i^*(Q_{A'})$. The result then follows from (5), which implies that there must be fewer fringe firms in $A'$ than in $A$ since the marginal firm has a lower gross profit and hence a lower entry cost, too.

(iii) Whether fringe firms' actions are larger or smaller depends on the sign of $\tilde{r}'_f(Q)$. Lemma 2 implies that if $Q_A < Q_{A'}$, the fringe firms in $A'$ choose a lower (higher) action iff actions are strategic substitutes (complements).

(iv) A change that increases the marginal profit of insider Firm $j$ causes its reaction function to shift out. Since $Q_A < Q_{A'}$, the firm clearly chooses a higher action if actions are strategic complements. We prove by contradiction that the same result holds in the case of strategic substitutes. Suppose not and suppose that the insider Firm $j$ chooses a lower action. But then the value of the aggregator must be larger in order to overturn the marginal profit effect of increasing the action level. From (ii) and (iii), there must be fewer fringe firms and each fringe firm must choose a lower action under strategic substitutes. Since every action level is smaller, this is inconsistent with the purported higher level of the aggregator. Hence, the insider Firm $j$ must produce more. ■

Proposition 11 states that a change which increases the marginal profit of one of the insiders increases the equilibrium value of the aggregator irrespective of whether the actions of the firms are strategic substitutes or complements. For example, for the Cournot model, this would imply a higher total output level and for the Logit model, this would imply a lower price level (which implies a higher total output level).[11] In cases where the aggregator value increases consumer welfare, consumers must be better off in both cases.

Proposition 11 also states that such a change will always result in exit by the fringe firms, which implies that the fringe firms which remain active in the market must be earning lower rents after the change. (This also follows from the fact that the value of the aggregator is smaller in the new equilibrium and $\tilde{\pi}_i^*(Q)$ is strictly decreasing by Lemma 3.) Although the firm which experiences the change reacts positively to it by increasing its own action, whether the fringe firms' actions increase or decrease depends on the sign of $\tilde{r}'_f(Q)$. The equilibrium with only fringe firms is illustrated in Figure 9, while Figure 10 shows the case where other firms are also present.

[11] The Cournot case concurs well with intuition for a dominant firm-fringe scenario.

### 5.2 The Logit model with differentiated quality-costs

The analysis above readily adapts to the case of a fringe with different quality-costs and the same entry cost, $K$. Anderson and de Palma (2001) considered this model, but did not look at the comparative static properties of the equilibrium (instead, they considered the pricing, profit, and output comparison of firms within an equilibrium: higher quality cost firms have higher mark-ups and sell more, while entry is excessive.)

Suppose then that $\pi_i = (p_i - c_i) \frac{\exp(s_i - p_i)/\mu}{\sum\limits_{j=0,..,n} \exp(s_j - p_j)/\mu}$ where the $s_j$ represent vertical "quality" parameters and $\mu > 0$ represents the degree of preference heterogeneity across products. The "outside" option has price 0 and "quality" $s_0$. The denominator of this expression is the aggregator, $Q = \sum\limits_{j=0,..,n} \exp(s_j - p_j)/\mu$. This can be written as the sum of individual choices by defining $q_j = \exp(s_j - p_j)/\mu$ so that we can think of firms as choosing the values $q_j$. Then we write $\pi_i = (s_i - \mu \ln q_i - c_i) \frac{q_i}{Q}$. Note here that $q_j$ varies inversely with price, $p_j$, so that the strategy space $p_j \in [c_j, \infty]$ under the transformation becomes $q_j \in [0, \exp(s_j - c_j)/\mu]$. Strategic complementarity of prices implies strategic complementarity of the $q$'s.

Label firms by decreasing quality cost, so that $s_1 - c_1 \geq s_2 - c_2 \geq ... \geq s_n - c_n$, etc. The free-entry equilibrium is described by zero profits for the marginal firm, Firm $n$, so that

$$\tilde{\pi}(Q; n) = K.$$

where $\tilde{\pi}(Q; n)$ denotes the profit value of the $n$th firm when the aggregator has value $Q$, and is decreasing in $Q$ and $n$. Let $A$ be the set of active firms, i.e., the first $n$ firms. The fixed point condition becomes

$$\sum_{i \in A} \tilde{r}(Q; n) = Q.$$

Then total differentiation of the latter condition gives

$$\tilde{r}(Q; n)\, dn + \left( \sum_{i \in A} \tilde{r}'(Q; i) - 1 \right) dQ = -d\tilde{r}_j(Q; j)$$

where we assume that insider Firm $j$ experiences a change that causes its reaction function to change. Since the zero profit condition implies $(\partial \tilde{\pi}(Q; n)/\partial Q)\, dQ + (\partial \tilde{\pi}(Q; n)/\partial n)\, dn = 0$, we can write

$$\left(\tilde{r}\left(Q;n\right)\frac{-\partial\tilde{\pi}\left(Q;n\right)/\partial Q}{\partial\tilde{\pi}\left(Q;n\right)/\partial n}+\left(\sum_{i\in A}\tilde{r}'\left(Q;i\right)-1\right)\right)dQ=-d\tilde{r}_j\left(Q;j\right).$$

Since $\frac{\partial\tilde{\pi}(Q;n)/\partial Q}{\partial\tilde{\pi}(Q;n)/\partial n}>0$ and $\sum_{i\in A}\tilde{r}'\left(Q;i\right)<1$ by the slope condition, the LHS bracketed term is negative, and so a higher aggregator results from a change that makes Firm $j$ more aggressive. Then fewer firms are active at the new equilibrium, and each unaffected one has a higher $q_i$, meaning a lower mark-up. Consumers are better off because the aggregator has risen.

# 6 Monopolistic competition

Models of monopolistic competition downplay strategic interaction by assuming firms are so small that they do not factor in the effect of their own actions on the aggregator. Such models, most prominently the CES model, are widely applied to a variety of economic problems, and are particularly common in international trade theory, and the new empirics of trade. To relate outcomes from such an assumption to the "oligopoly" models considered thus far, it is useful to first derive the cumulative best-reply from another direction.

We have derived the function $\tilde{\pi}_i^*\left(Q\right)$ from a maximized value function $\pi_i^*\left(Q_{-i}\right)=\pi_i\left(Q_{-i}+r\left(Q_{-i}\right),r\left(Q_{-i}\right)\right)$, where we wrote the basic profit function as $\pi_i\left(Q_{-i}+q_i,q_i\right)$. This latter profit can be written directly as $\tilde{\pi}_i\left(Q,q_i\right)$, and the agent's choice of $q_i$ can be written directly from this profit function as

$$\frac{\partial\tilde{\pi}_i\left(Q,q_i\right)}{\partial Q}\frac{dQ}{dq_i}+\frac{\partial\tilde{\pi}_i\left(Q,q_i\right)}{\partial q_i}=0, \tag{8}$$

where the term $\frac{dQ}{dq_i}$ depends on the context, as explained momentarily:[12]

1. For the standard simultaneous move Nash "oligopoly" equilibrium, the term is 1. For example, in Cournot games, this represents how much an agent increases the aggregator. For Bertrand CES and Logit games, this is

2. Under the "monopolistic competition" assumption, the term is ZERO. Monopolistically competitive firms in the CES context are assumed not to affect the "price index" for example.

[12] This device can also be used to compare Nash to Stackelberg equilibrium outcomes in a similar manner, as we discuss in the Leaders section below.

Note now that A1 is the assumption that $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q} < 0$. This implies immediately that *at the Nash equilibrium*, the other term must be positive, i.e., $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial q_i} > 0$. (For example, in Cournot competition, $\tilde{\pi}_i(Q, q_i) = p(Q) q_i - C(q_i)$, so $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q} = p'(Q) q_i < 0$ and $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial q_i} = p(Q) - C'(q_i)$ which then must be positive in equilibrium.)

An implication is that, comparing Nash equilibrium ("oligopoly") to monopolistic competition for any given number of firms, the aggregator will be higher under monopolistic competition. The argument is a local derivative one: at the Nash equilibrium, $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial q_i} > 0$, as just stated. Therefore, since this is the only term at play in the monopolistically competitive equilibrium, firms want to have higher $q_i$ than at Nash. This is because firms do not factor into their calculus the impact of their own actions on the aggregate.

Writing $\tilde{\pi}_i(Q, q_i)$, and deriving the cumulative reaction function (8) as above gives us a $q_i$ as a function of $Q$; that is, this finds $\tilde{r}_i(Q)$. Notice that the corresponding "maximum value function" is $\tilde{\pi}_i^*(Q) = \pi_i(Q, \tilde{r}_i(Q))$ which was derived before from the traditional reaction function approach. Differentiation of this version yields

$$\frac{d\tilde{\pi}_i^*(Q)}{dQ} = \frac{\partial \tilde{\pi}_i(Q, q_i)}{\partial Q} + \frac{\partial \tilde{\pi}_i(Q, q_i)}{\partial q_i} \tilde{r}_i'(Q).$$

Together with the first order condition above (8) (with $\frac{dQ}{dq_i} = 1$, for the Nash equilibrium "oligopoly" case) means that we can write equivalently[13]

$$\begin{aligned} \frac{d\tilde{\pi}_i^*(Q)}{dQ} &= \frac{\partial \pi_i(Q, q_i)}{\partial q_i} \left( \tilde{r}_i'(Q) - 1 \right) \\ &= \frac{\partial \pi_i(Q, q_i)}{\partial Q} \left( 1 - \tilde{r}_i'(Q) \right). \end{aligned}$$

These expressions, and variants, are used in the next section.

# 7 Optimal numbers of firms

There is a long literature on whether the market provides optimal variety. It was the central question of Dixit-Stiglitz (1977), and Spence (1976), and goes back to Chamberlin

[13] Note also the cumulative best reply function slope from this is

$$\tilde{r}_i'(Q) = \frac{-\left( \frac{\partial^2 \pi_i(Q,q_i)}{\partial Q^2} + \frac{\partial \pi^2(Q,q_i)}{\partial Q \partial q_i} \right)}{\frac{\partial^2 \pi_i(Q,q_i)}{\partial q_i^2} + \frac{\partial \pi^2(Q,q_i)}{\partial Q \partial q_i}}.$$

(who suggested the market outcome "might be a sort of ideal", though apparently on the premise that equilibrium and optimum provide the same comparative statics regarding entry costs and taste for variety). Mankiw and Whinston (1986) provide a Cournot analysis.

Various welfare benchmarks have been proposed, primarily first-best, and second best with either a zero-profit constraint or subject to firm pricing. Here we consider the last of these.

Suppose then that there is free entry of a symmetric fringe. Write welfare as a function of the number of entrants, and suppose this number can be varied, though the subsequent level of the aggregator is determined according to a Nash equilibrium of the ensuing aggregative game. Then $Q$ depends on $n$ as per the short-run analysis, and $Q$ increases with $n$. We can write

$$W(n) = \tilde{\phi}(Q) + n\tilde{\pi}^*(Q)$$

with the subscripts suppressed under the current symmetry assumption. Let us now evaluate the welfare derivative at the free entry equilibrium, $n^e$, where the profit of the fringe firms is zero:

$$W'(n^e) = \left[\tilde{\phi}'(Q) + n^e\tilde{\pi}^{*\prime}(Q)\right]\frac{dQ}{dn}.$$

The two terms in brackets are simply the two externalities identified by Spence (1976), namely the non-appropriability of consumer surplus and the business stealing effect on other firms' profits. The factor $\frac{dQ}{dn}$ is positive since adding another firm's cumulative best reply function must raises the equilibrium value of the aggregator.

Now, let us sign the bracketed term: if positive, the second best optimum has more firms than the equilibrium.

First treat Cournot competition. Then $\tilde{\phi}(Q)$ is NET consumer surplus, and so $\tilde{\phi}'(Q) = -Qp'(Q)$. We noted above that $\frac{d\tilde{\pi}_i^*(Q)}{dQ} = \frac{\partial \pi_i(Q,q_i)}{\partial Q}\left(1 - \tilde{r}_i'(Q)\right)$, and, since $\frac{\partial \pi_i(Q,q_i)}{\partial Q} = p'(Q)\,q$ (under symmetry, $q_i = q$), then the desired term is

$$\begin{aligned} \tilde{\phi}'(Q) + n^e\tilde{\pi}^{*\prime}(Q) &= -Qp'(Q) + np'(Q)\,q\left(1 - \tilde{r}_i'(Q)\right), \\ &= -p'(Q)\,Q\tilde{r}_i'(Q). \end{aligned}$$

Hence the derivative has the sign of $\tilde{r}_i'(Q)$. The result of Mankiw-Whinston (1986) of excessive entry for Cournot competition then follows immediately since they assume strategic substitutes (this they do effectively by assuming that $q$ falls with $n$). However, the opposite result holds under strategic complementarity because then $\tilde{r}_i'(Q) > 0$. This property holds if demand is "convex enough" – for example, for log-convex demand functions if marginal costs are zero.

## 7.1 Bertrand application and monopolistic competition

Recall from above that the sign of $W'(n^e) = \left[\tilde{\phi}'(Q) + n^e \tilde{\pi}^{*\prime}(Q)\right] \frac{dQ}{dn}$ determines whether there should be more firms. Results yield grudgingly (to quote Spence, 1976), but some special cases are of interest. In particular, it can be shown that the logit model has exactly the right number of products by this welfare criterion (which result has not been established elsewhere), while the CES model has insufficient variety.

We now wish to apply the formula above to the Bertrand pricing model. Recall the profit function (written in terms of prices) is $\pi_i = (p_i - c_i) D_i(p_i, Q)$ where $Q$ is the value of the aggregator, and recall that $D_i(p_i, Q) = -\tilde{\phi}(Q) g_i'(p_i)$ Suppose a symmetric situation, so all marginal costs are the same, as are the demand functions, so we can drop to the corresponding $i$ subscripts. We therefore seek the expression for $\tilde{\pi}^{*\prime}(Q)$. The simplest context is monopolistic competition: recall from above that $\frac{d\tilde{\pi}_i^*(Q)}{dQ} = \frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q} + \frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial q_i} \tilde{r}_i'(Q)$, but under monopolistic competition, firms set $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial q_i} = 0$, so that here $\frac{d\tilde{\pi}_i^*(Q)}{dQ} = \frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q}$. Furthermore, $\frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q} = (p_i - c) \frac{\partial D(p_i,Q)}{\partial Q} = -(p_i - c)\tilde{\phi}"(Q) g'(p_i)$. This can be further simplified by taking the first order condition to the problem $\max_{p_i} (p_i - c) D(p_i, Q)$ (where $D(p_i, Q) = -\tilde{\phi}(Q) g'(p_i)$), which yields $-\tilde{\phi}(Q) g'(p_i) + (p_i - c)\left(-\tilde{\phi}(Q) g"(p_i)\right) = 0$. Substituting for the mark-up, we get

$$\begin{aligned} \frac{\partial \tilde{\pi}_i(Q, q_i)}{\partial Q} &= -\frac{\tilde{\phi}(Q) g'(p_i)}{\left(-\tilde{\phi}(Q) g"(p_i)\right)} \tilde{\phi}"(Q) g'(p_i) \\ &= \frac{g'(p_i)}{g"(p_i)} \tilde{\phi}"(Q) g'(p_i). \end{aligned}$$

Now, we seek the sign of $\tilde{\phi}'(Q) + n^e \frac{\partial \tilde{\pi}_i(Q,q_i)}{\partial Q}$, which therefore has the sign of

$$\tilde{\phi}'(Q) + n^e \frac{\left[g'(p_i)\right]^2}{g"(p_i)} \tilde{\phi}"(Q).$$

This turns out to be remarkably simple for the logit case. Indeed, in that case $\tilde{\phi}(Q) = \mu \ln Q$ and $g(p_i) = \exp(-p_i/\mu)$, so the expression becomes:

$$\frac{\mu}{Q}\left(1 + n^e \exp(-p_i/\mu)\frac{-1}{Q}\right).$$

However, since $n^e \exp(-p_i/\mu) = Q$, the expression in parentheses is zero. This means that the (second-best pricing) optimum and the equilibrium coincide.

We can run through the same exercise for the CES monopolistic competition model, and recall we are trying to determine the sign of $\tilde{\phi}'(Q) + n^e \frac{[g'(p_i)]^2}{g"(p_i)}\tilde{\phi}"(Q)$. For the CES, we have $\tilde{\phi}(Q) = \frac{1}{\lambda}\ln Q$ and $g(p_i) = p_i^{-\lambda}$.[14] Substituting then yields

$$\frac{1}{\lambda Q}\left(1 + n^e \frac{\lambda p_i^{-\lambda}}{\lambda + 1}\frac{-1}{Q}\right)$$

Now, since $n^e p_i^{-\lambda} = Q$, the expression in parentheses equals $\left(\frac{1}{\lambda+1}\right) > 0$, and so there is always underentry for this model compared to the (second-best pricing) optimum.

One way to write the condition $\tilde{\phi}'(Q) + n^e \frac{[g'(p_i)]^2}{g"(p_i)}\tilde{\phi}"(Q)$ is as an elasticity product. Recalling that $n^e g(p_i)$, this gives us

$$\tilde{\phi}'(Q)\left(1 + \frac{g'(p_i)}{g"(p_i)p_i}\frac{g'(p_i)p_i}{g(p_i)}\frac{Q\tilde{\phi}"(Q)}{\tilde{\phi}'(Q)}\right),$$

where we recall that $\tilde{\phi}'(Q) > 0$, and so the direction of the solution depends on these elasticities.

# 8 Applications

In this section, we consider several implications of the results stated above. It is readily confirmed in these examples that Assumption 1 which states that $\pi_i(Q, q_i)$ is decreasing in $Q$ holds quite generally.

## 8.1 Mergers

We next consider the short-run and long-run equilibrium effects of a merger. As illustrated below, this objective is made simple with the device of the cumulative reaction function.

[14] Here $\lambda = \frac{\rho}{1-\rho} > 0$.

A merger implies that the merged firms can maximize joint profits. In addition, a merger can result in cost savings and/or synergies. As explained in Farrell and Shapiro (1990), synergies allow the merged entity to improve their joint production capabilities (through learning, for example) while cost savings result from the opportunity to better allocate production among the different facilities (rationalization). In order to understand the impact of synergies on the performance of a merger, we first consider the case where mergers do not result in any synergies.

### 8.1.1 No synergies

In this section we consider mergers with or without cost savings, but no synergies. A merger may result in fixed cost savings (perhaps due to economies of scope) or a more optimal allocation of production within the merged entity by shifting production from high-cost facilities to low-cost ones. Note that the second kind of cost saving is more likely to happen if the firms are producing homogeneous products. If the firms are producing differentiated products, then it may not be feasible to shift production between firms.

We first consider the impact of the merger on the actions of the merged firms. Suppose that two firms, $j$ and $k$, merge. Before the merger, their cumulative reaction functions are given by $\tilde{r}_j(Q)$ and $\tilde{r}_k(Q)$. $\tilde{r}_j(Q)$ is defined by the solution to the first order condition

$$\frac{\partial \widetilde{\pi}_j(Q, q_j)}{\partial Q} + \frac{\partial \widetilde{\pi}_j(Q, q_j)}{\partial q_j} = 0.$$

Similarly for $\tilde{r}_k(Q)$. After the merger, we can treat the merged entity as a multi-plant (in case of homogeneous products) or a multi-product firm (in case of differentiated products). Let $\widetilde{\pi}_m(Q, q_j, q_k)$ stand for the profit function of the merged entity. It solves

$$\max_{q_j, q_k} \widetilde{\pi}_m(Q, q_j, q_k) = \widetilde{\pi}_j(Q, q_j) + \widetilde{\pi}_k(Q, q_k).$$

There are two first order conditions which determine $q_j$ and $q_k$. The first of these is

$$\frac{\partial \widetilde{\pi}_j(Q, q_j)}{\partial Q} + \frac{\partial \widetilde{\pi}_j(Q, q_j)}{\partial q_j} + \frac{\partial \widetilde{\pi}_k(Q, q_k)}{\partial Q} = 0, \tag{9}$$

which differs from the pre-merger case by the last term, which is the internalization of the effect of the aggregator on the sibling profit. From this first order condition, we can solve for optimal choice of $q_j$ as a function of $Q$ and $q_k$. Similarly, the second first order condition defines the optimal choice of $q_k$ as a function of $Q$ and $q_j$. These two first order conditions can be solved simultaneously to determine the optimal choices of $q_j$ and $q_k$ as functions of the aggregator $Q$. Let $\tilde{r}_j^m(Q)$ and $\tilde{r}_k^m(Q)$ denote these individual cumulative best reply functions for the members of the merged entity. Adding them up gives us the cumulative best reply for the merged entity, $\tilde{R}^m(Q)$.

For illustration purposes, consider the derivation of $\tilde{r}_j^m(Q)$ and $\tilde{r}_k^m(Q)$ in the next example, where there is a merger between two producers of homogeneous products in a market with Cournot competition. Note that since the merging firms have different cost functions in this example, it allows for cost savings.

**Example 3** *Consider a merger between firms $j$ and $k$ in a market where firms engage in Cournot competition and demand is $P = 1 - Q$. The cost function of firm $j$ is $C_j(q_j) = q_j^2$ and the cost function of firm $k$ is $C_k(q_k) = q_k^2/2$. The merger allows the firms to shift production between themselves so that each unit is produced by the firm with the lower marginal cost for that unit. In equilibrium, the last unit produced at both firms should have the same marginal cost.*

*The firms maximize $(1-Q)q_j - q_j^2 + (1-Q)q_k - q_k^2/2$. The FOCs are*

$$\begin{aligned} 1 - Q_{-j} - 2q_j - 2q_j - q_k &= 1 - Q_{-j-k} - 4q_j - 2q_k = 1 - Q - 3q_j - q_k = 0 \\ 1 - Q_{-k} - 2q_k - q_k - q_j &= 1 - Q_{-j-k} - 3q_k - 2q_j = 1 - Q - 2q_k - q_j = 0 \end{aligned}$$

*The first expression gives us $q_j$ as a function of $Q$ and $q_k$. Similarly for the second expression. We can solve these two expressions simultaneously to get the optimal $q_j$ and $q_k$ as functions of $Q$. We get*

$$\widetilde{r}_j^m(Q) = \frac{1-Q}{5} \text{ and } \widetilde{r}_k^m(Q) = \frac{2(1-Q)}{5}$$

*where we use the notation $\widetilde{r}_j^m(Q)$ and $\widetilde{r}_k^m(Q)$ to denote the quantities produced by firms $j$ and $k$ within the merged entity, respectively.*

The following lemma states that the merger causes the cumulative best reply function of the firms which are part of the merger to shift down.

**Lemma 4** *Let $A$ be the set of active firms in the market. Suppose that two of the firms in $A$, $j$ and $k$, merge. Then, $\tilde{r}_j^m(Q) \leq \tilde{r}_j(Q)$, $\tilde{r}_k^m(Q) \leq \tilde{r}_k(Q)$, and hence $\tilde{R}^m(Q) < \tilde{r}_j(Q) + \tilde{r}_k(Q)$.*

**Proof.** First consider the case when both $j$ and $k$ are active in production after the merger. Assumption 1 states that $\pi_k(Q_{-k} + q_k, q_k)$ is strictly decreasing in $Q_{-k}$. Writing $\pi_k(Q_{-k} + q_k, q_k)$ directly as $\widetilde{\pi}_k(Q, q_k)$, this implies that $\widetilde{\pi}_k(.)$ is decreasing in its first argument. Hence, the third term in (9) is negative by Assumption 1 so that, for any $q_k$, the choice of $q_j$ must be lower at any given $Q$. That is, firms $j$ and $k$ will choose lower actions (lower quantity in the Cournot benchmark, higher price in the differentiated products Bertrand oligopoly) as a result because they internalize the negative effect of their action on each other. The same logic applies for the first order condition for $q_k$. This implies that $\tilde{R}^m(Q) < \tilde{r}_j(Q) + \tilde{r}_k(Q)$.

Note that in the special case where the firms produce homogeneous products at constant but heterogeneous marginal costs, the lower-cost firm will do all of the production after the merger. In this case, we have $\tilde{r}_j^m(Q) \leq \tilde{r}_j(Q)$, $\tilde{r}_k^m(Q) \leq \tilde{r}_k(Q)$, with one inequality holding as a strict equality, and $\tilde{R}^m(Q) < \tilde{r}_j(Q) + \tilde{r}_k(Q)$. ■

That the merged entity would decrease its total production after the members internalize the externalities they impose on each other was stressed by Salant et al. (1983) in the case of Cournot competition with homogeneous products. In Lemma 4, we illustrate this result more generally (for any aggregative game) using the concept of the cumulative best reply function for the merger.

We next explore the impact of the merger on the value of the aggregator in the short run. Recall that the equilibrium condition for the aggregator, for a given set of firms, $A$, is

$$\sum_{i \in A} \tilde{r}_i(Q) = Q.$$

In an aggregative game, a merger does not affect the reaction functions or the cumulative reaction functions of the firms which are not part of the merger. Therefore, we only need to keep track of what happens to the cumulative reaction functions of those merging, which we have just argued goes down. Hence, the sum $\sum_{i \in A} \tilde{r}_i(Q)$ (i.e., $\sum_{i \neq j,k} \tilde{r}_i(Q) + \tilde{R}^m(Q)$) goes down once there is a merger. Since it will intersect the 45° line at a lower $Q$, the equilibrium value of $Q$ necessarily falls in the short run for the strategic substitutes case and, invoking the slope condition, also for the strategic complements case.

The new equilibrium values for the remaining firms follow directly from the reaction function slope ($\tilde{r}_i'(Q)$, which has the same sign as $r_i'(Q_{-i})$). They therefore rise under strategic substitutability; for example, in the Cournot case, other firms expand output, and hence the merged firm's total output must contract by a higher amount to deliver the lower total $Q$. For the strategic complements case, others' actions fall (higher prices under Bertrand competition), and the merged firm's actions fall for the twin reasons of the direct lowering of the reaction functions and their positive slope. In both cases though, since $Q$ has fallen, profits of non-merging firms have risen (since $\tilde{\pi}_i^*(Q)$ is decreasing by Lemma 3). Also, consumer welfare decreases as long as consumer surplus decreases in the value of the aggregator. In summary:

**Proposition 12** *Let $A$ be the set of active firms in the market. Suppose a subset $M$ of the firms in $A$ merge. If there are no merger-specific synergies, the value of the aggregator decreases in the short run. Hence, the non-merged firms' profits go up, and consumer welfare goes down.*

In the short run, in the absence of cost savings or synergies, the merged firms may or may not benefit from the merger. If actions are strategic substitutes, since the non-merged firms' response does not reinforce the merged firms' actions, mergers are not profitable unless they include a sufficiently large percentage of the firms in the market. This effect is exemplified in the Cournot merger paradox of Salant, Switzer, and Reynolds (1983). Without synergies, non-merged firms benefit from a merger while merging firms can lose. If actions are strategic complements (as in the case of Bertrand competition), Deneckere and Davidson (1985) show that mergers are always profitable. However, non-merged firms still benefit more from a merger than the merged firms. This is because each merged firm cannot choose the action that maximizes its individual profits while each non-merged firm does.

These counter-intuitive results have played an important role in the mergers literature over the last few decades. The following proposition summarizes the results in the long run case. In the statement of the proposition, we assume that

**Proposition 13** *Let $A$ be the set of active firms in the market. Suppose a subset $M$ of the firms in $A$ merge, the merger does not result in any synergies, and a ZPFE prevails after the merger as well as before. Then:*

*(i) The value of the aggregator and, as a result, the non-merged firms' profits and consumer welfare remain unchanged.*

*(ii) If there are no fixed cost savings, the merged firms' individual profits weakly decrease. If there are fixed cost savings, the merged firms' individual profits may increase depending on the level of the fixed cost savings.*

**Proof.** (i) By Proposition 1, the aggregator does not change if the fringe is homogeneous. That the non-merged firms' profits and consumer welfare remain unchanged follows from Lemma 3 and Proposition 6.

(ii) Let $\tilde{\pi}_j^*(Q) = \tilde{\pi}_j^*(Q, \tilde{r}_j(Q))$ stand for the maximized profit of firm $j$ before the merger. By definition, $\tilde{r}_j(Q)$ stands for the portion of $Q$ that is optimally produced by firm $j$. Let $\tilde{\pi}_j^m(Q) = \tilde{\pi}_j^m\left(Q, \tilde{r}_j^m(Q)\right)$ stand for the profit of firm $j$ after the merger, assuming the firm takes the action that maximizes the merged firms' joint profits. We know from Lemma 4 that each merged firm produces weakly less after the merger: $\tilde{r}_j^m(Q) \leq \tilde{r}_j(Q)$. Hence, it must be the case that $\tilde{\pi}_j^*(Q) \geq \tilde{\pi}_j^m(Q)$. ■

Part (i) of Proposition 13 states that with entry, whether actions are strategic substitutes or strategic complements, the value of the aggregator returns back to its pre-merger level. This result has been shown in Davidson and Mukherjee (2007) for the case of homogeneous goods Cournot competition. Proposition 13 extends it to show that in any set-up where the game can be described as an aggregative game in the long run, the aggregator is unchanged by the merger. This includes differentiated goods Cournot competition (Ottaviano and Thisse, 2004), CES, and logit demand systems. Part (i) of Proposition 13 also implies that entry in the long run counteracts the negative impact of mergers on consumer welfare in the short run. Given our results in Section 3.4, this holds for any demand system where consumer welfare can be written as a function of the aggregator only. As we show in Section 3.4, this includes demand systems which satisfy the IIA property.

Propositions 12 and 13 together imply that while the non-merged firms benefit from the merger in the short run, their profits go back to the pre-merger level (zero) in the long run. Importantly, part (ii) of Proposition 13 implies that in the absence of merger-specific synergies and/or fixed cost savings, the merger affects the profits of its member firms adversely.[15] Hence, the profitability of mergers under Bertrand competition no longer holds in the long run. Merger-specific synergies and/or fixed cost savings are required for the profitability of mergers in the long run.

Proposition 13 applies generally in the case of symmetric or asymmetric firms as long as the type of the marginal firm does not change. If the firms are symmetric to start with, we have the following corollary.

**Corollary 1** *Let $A$ be the set of active firms in the market. Suppose all of the firms in $A$ are symmetric, a subset $M$ of the firms in $A$ merge, and a ZPFE prevails after the merger as well as before. Then, if there are no fixed cost savings, each merged firm earns less than each non-merged firm after the merger and makes negative profits.*

In a market with symmetric firms, the firms must be making zero profits before the merger. After the merger (and following entry), the non-merged firms continue to make zero profits while the merged firms start to make negative profits if the merger results in no fixed cost savings.

### 8.1.2 Synergies

We next consider how synergies affect the performance of a merger. Using our notation from before, let $\theta$ stand for the level of synergies and let $c(\theta)$ stand for the marginal cost function of each merged firm, where $c' < 0$. We know from Lemma 4 that without synergies, each merged firm will reduce its action after the merger. If there are synergies created by the merger, Lemma 4 may no longer hold.

We are mainly interested in synergies which have a positive direct and strategic effect. Hence, synergies are assumed to shift the marginal and/or fixed cost functions of each

[15] In the special case where the firms produce homogeneous products at constant but heterogeneous marginal costs, it is profitable for the lower-cost firm to do all of the production after the merger. In this case, the lower-cost firm's profits will not be affected by the merger while the higher-cost firm will make strictly less as a result of the merger since it will shut down. In such cases, there may be side payments between the firms, which are not taken into account in the statement of Proposition 13. If there are side payments between the firms, their post-merger payoffs will clearly be different from what they individually make as a result of the merger.

merged firm. A positive strategic effect implies $\frac{\partial \tilde{R}^m(Q;\theta)}{\partial \theta} > 0$.

**Proposition 14** *Let $A$ be the set of active firms in the market. Suppose a subset $M$ of the firms in $A$ merge. For sufficiently low (high) synergies, a merger is followed by entry (exit) in the long run.*

**Proof.** By Proposition 1, the value of the aggregator does not change with the merger. Let $\bar{Q}$ stand for this value. Let $\bar{\theta}$ stand for the level of synergies such that $\tilde{R}^m\left(\bar{Q};\bar{\theta}\right) = \sum_{i \in M} \tilde{r}_i\left(\bar{Q}\right)$. Since $\frac{\partial \tilde{R}^m(Q;\theta)}{\partial \theta} > 0$, for all $\theta < \bar{\theta}$, the merged entity's action is lower than it was before the merger. By Proposition 2, each non-merged firm's optimal action is the same with and without the merger. Hence, it must be the case that there is entry after the merger. Similarly, for all $\theta > \bar{\theta}$, the merged entity's action is higher after the merger and, hence, it must be the case that the merger is followed by exit. ∎

We next show that if the level of synergies is such that the members of the merger are just indifferent between merging and not merging, then it must be the case that the merger is followed by entry.

**Proposition 15** *Let $A$ be the set of active firms in the market. Suppose a subset $M$ of the firms in $A$ merge. If the merged firms' profit remain unchanged as a result of the merger, it must be the case that the merger results in entry.*

**Proof.** For the merged firms, consider their maximized per-firm profits before and after the merger. Before the merger, each earn $\tilde{\pi}_i^*\left(\bar{Q}\right)$. Let $\tilde{\pi}_{i\in M}^*\left(\bar{Q};\theta\right)$ denote the maximized per-firm profits after the merger. Consider the level of synergies such that $\tilde{\pi}_{i\in M}^*\left(\bar{Q};\theta\right) = K = \tilde{\pi}_i^*\left(\bar{Q}\right)$. Let $\hat{\theta}$ stand for this value of synergies. Proposition 13 implies that without synergies, $\tilde{\pi}_{i\in M}^*\left(\bar{Q}\right) < \tilde{\pi}_i^*\left(\bar{Q}\right)$. Since synergies increase the profits of the merged firms, we can find such a $\theta$.

To prove the proposition, we need to show that $\hat{\theta} < \bar{\theta}$. However, this must be the case because by definition at $\bar{\theta}$, each merged firm's post-merger action is equal to its pre-merger action. However, at this level, the merged firm must be making positive profits post-merger because its marginal cost is lower. Hence, the level of synergies which brings it back to its pre-merger profit level (which is zero) must be lower. ∎

It is important to comment on how general this result is. The results in Propositions 14 and 15 hold for all games which can be classified as aggregative games in the long run. Davidson and Mukherjee (2007) shows these results in the case of homogeneous goods Cournot competition. Our contribution is to extend it to multi-product firms and to firms participating in differentiated goods markets with Bertrand and Cournot competition.

In terms of welfare, Proposition 13 implies that mergers are socially desirable from a total welfare standpoint if and only if they are profitable. Put another way, the models we consider here build in that property. The result is tempered by integer issues, by a rising fringe supply, and if the long-run takes a long time to attain. Most important, perhaps, is the global competition property built into the demand functions satisfying the IIA property.

## 8.2 Contests

Another example of an aggregative game can be found in the literature on contests, where players spend effort to win a prize.[16] One specific context in which contests take place is in research and development (R&D).

### 8.2.1 Competition and cooperation in R&D environments

Starting with Loury (1979) and Lee and Wilde (1980), the standard approach to R&D competition assumes that the size of the innovation is exogenously given, but its timing stochastically depends on the R&D investments chosen by the firms through a Poisson process (see Reinganum (1989) for a survey). This mainstream model of R&D competition has been used to analyze issues ranging from the impact of market structure on innovation to the optimal design of patent policy.

Each firm which participates in the R&D race operates an independent research facility. Time is continuous, and the firms share a common discount rate $r$. Firms choose an investment $x$ at the beginning of the race, which provides a stochastic time of success that is exponentially distributed with hazard rate $h(x)$. A higher value of $h(x)$ corresponds to a shorter expected time to discovery. Suppose that $h'(x) > 0$, $h''(x) < 0$, and $h(0) = 0$.

[16] Starting with the seminal contributions of Tullock (1967 and 1989), Krueger (1974), Posner (1975) and Bhagwati (1982), a large literature has emerged to analyze the various aspects of contests seen in social, political and economic life.

$\lim_{x \to 0} h'(x)$ is sufficiently large to guarantee an interior equilibrium and $\lim_{x \to \infty} h'(x) = 0$.

The main difference between Loury (1979) and Lee and Wilde (1980) is in the formulation of the investment costs. Loury (1979) assumes that a firm which chooses an investment $x$ pays $x$ at $t = 0$. Lee and Wilde (1980) assumes that the same firm pays a fixed cost $F$ at $t = 0$ and a flow cost $x$ as long as it stays active in the R&D race.

Following Lee and Wilde (1980), we can express the payoff function of participant $i$ in the R&D race as

$$\frac{h(x_i) V - x_i}{r + h(x_i) + \alpha_i} - F,$$

where $V$ stands for the value of the innovation and $\alpha_i = \sum_{j \neq i} h(x_j)$ stands for the aggregate hazard rate of the rival firms. Using our notation from above, we can think of each firm as choosing $q_i = h(x_i)$. Hence, letting $Q_{-i} = \sum_{j \neq i} h(x_j)$ and $Q = \sum_i h(x_i)$, we can the firm's payoff function as

$$\frac{q_i V - h^{-1}(q_i)}{r + Q} - F.$$

Clearly, we have an aggregative game, where each firm's payoff depends on its own action and the aggregator, $Q$.

Let $r(Q_{-i}, V, r)$ and $\widetilde{r}(Q, V, r)$ stand for the reaction function and the cumulative reaction function of firm $i$, respectively. Using the implicit function theorem, it is straightforward to show that both functions are upward sloping. Hence, investment choices are strategic complements.[17]

Although the discussion above treats the number of firms as fixed, several papers have assumed that the number of participants in the R&D race is determined by free entry.[18] Using the envelope theorem, we can verify that Lemma 3 holds within this framework.[19] This powerful result allows one to pin down the aggregate investment level in a long-run equilibrium.

[17] Following Loury's (1979) formulation of the investment costs yields a payoff function of the form $\frac{q_i V}{r+Q} - -h^{-1}(q_i)$. In this case, $\frac{\partial r}{\partial Q_{-i}} \gtrless 0$ as $q_i \gtrless \frac{Q+r}{2} = Q_{-i} + r$. Hence, actions are strategic substitutes if $q_i < Q_{-i} + r$ and strategic complements if $q_i > Q_{-i} + r$ However, in a symmetric equilibrium with $N \geqslant 2$, actions are necessarily strategic substitutes since $Q^* = Nh(x^*)$.

[18] Loury (1979) and Lee and Wilde (1980) show that with free entry, there will be too much entry into the R&D race. For some more recent contributions, see Denicolo (1999 and 2000), Etro (2004), Erkal (2005) and Erkal and Piccinin (2010).

[19] See Lemma 1 in Erkal and Piccinin (2010).

One obvious exogenous change to consider in such a framework is the impact of R&D cooperation on market structure and performance. Erkal and Piccinin (2010) study the impact of R&D cooperation in a long run equilibrium. They compare free entry equilibria with R&D competition to free entry equilibria with R&D cooperation, where an exogenously given number of firms participate in the R&D race as members of a cooperative R&D arrangement. Following Kamien, Muller and Zang (1992), two common ways in which cooperative R&D has been defined in the literature are R&D cartels and RJV cartels. In an R&D cartel, the partner firms choose their investment levels to maximize their joint profit function, but they do not share the outcomes of their research efforts. In an RJV cartel, the partner firms both choose their investment levels to maximize their joint profit function and share the outcomes of their research efforts. In both cases, the firms compete in the product market following the R&D stage. Hence, if the firms form an R&D cartel and if one of the cartel members is the first to develop the innovation, there is only one firm with the new technology in the product market whereas if the firms form an RJV cartel and if one of the cartel members is the first to develop the innovation, all of the cartel members compete in the product market using the new technology.

Proposition 1 above implies that regardless of the type of cooperation, as long as there are participants in the R&D race which are not members of the cooperative R&D arrangement, the aggregate rate of innovation ($Q = \sum_i h(x_i)$) remains the same with and without a cooperative R&D arrangement. This is despite the fact that the number of participants in the R&D race is different under R&D competition and R&D cooperation. This surprising indifference result implies that any welfare gain from R&D cooperation cannot be driven by its impact on the aggregate rate of innovation.[20]

Regarding the impact of R&D cooperation on firm profitability, Erkal and Piccinin (2010) show that R&D cartels are unprofitable in the long run. The logic behind this result is similar to the one presented in Lemma 4 and Proposition 13 in Section 8.1. That is, the members of the R&D cartel, because they maximize joint profits, reduce their investment choices. This causes them to make less profits than they would make in an industry without an R&D cartel. If firms are symmetric, then it implies that they would be making negative

[20] For the short-run effects of RJVs in a Poisson environment, see Miyagiwa and Ohno (2002).

profits instead of zero profits.

Although Erkal and Piccinin (2010) do not consider synergies, the results from Section **??** can be used to state that R&D cartels with synergies may be profitable depending on the size of the synergies.[21]

### 8.2.2 R&D subsidies

Another relevant policy question which can be explored within our framework is the impact of R&D subsidies in a long run equilibrium. R&D subsidies are used in many countries throughout the world. Depending on the program, the goal may be to support specific types of industries, technological problems, or early vs. late stage research.

Consider a subsidy program that affects only a subset of the firms in an industry.[22] Suppose that, as in the Lee and Wilde (1980), investment in R&D entails the payment of a fixed cost $F$ at $t = 0$ and a flow cost, and that the subsidy decreases the recipient's marginal cost of R&D. Based on our results in section 3, we can state the following.

**Proposition 16** *Let $A$ be the set of active firms in the market. Suppose the government adopts an R&D subsidy policy which affects the firms $i \in A_I \subset A$. Then:*

*(i) The rate of innovation in the short run is higher with the policy than without.*

*(ii) The long run rate of innovation is unaffected by the policy.*

*(iii) The number of participants in the R&D race is lower with the policy than without.*

*(iv) With the policy, the investment choices of the affected firms increase and the investment choices of the unaffected firms do not change in the long run.*

*(v) With the policy, the expected profits of the affected firms go up and the expected profits of the fringe firms remain unchanged in the long run.*

**Proof.** (i) The R&D subsidy causes the recipients' reaction functions to shift up. Since actions are strategic complements in Lee and Wilde (1980), this causes the rate of innovation in the short run, $Q$, to increase.

[21] In the context of R&D cooperation, synergies can emerge if, for example, the cooperating firms can learn from each others' patents or research teams. Such learning may cause either the marginal cost of investment to decrease or the marginal benefit of investment to increase.

[22] Alternatively, we can think about R&D subsidies in an international setting, where the firms in a single country or a group of countries can benefit from R&D subsidies.

(ii) Follows from Proposition 1.

(iii) Follows from Proposition 5.

(iv) Follows from Propositions 5 and 2.

(v) That the affected firms earn higher with the policy follows from Proposition 3. The fringe firms make zero by definition in a long run equilibrium. ■

The important result in Proposition 16 is that although the government can increase the rate of innovation in the short run by adopting an R&D subsidy policy, it cannot affect the rate of innovation in the long run with this kind of a policy.

### 8.3 Privatization of public firms

Anderson, de Palma and Thisse (1997) use a CES model to compare free-entry equilibria (ignoring integer constraints) in two scenarios. The first is when some firms are run as public companies, with the objective of maximizing their contribution to social surplus. There remains a competitive fringe. Note first that the public firms do not price at marginal cost, they price above it. This is because they are assumed to care about producer surplus of other private firms – raising price just above marginal cost has a second-order effect on the sum of public firm profit plus consumer surplus, but has a first-order effect of increasing private firms' profits. They also price below private firms because they care about consumers too. Paradoxically perhaps, (paradoxically because the private firms are the assumed profit maximizers) it can also happen that the public firms make a profit at a ZPFE, even though the private firms (with the same cost and demand structure) do not. This is because public firms price lower and produce more.[23]

Since the CES model has the IIA property, Proposition 8 applies: the game is an aggregative one, and consumer surplus depends only on the value of the aggregator. Moreover, then our core propositions apply. Namely from Proposition 9, the affect of privatization is to change total welfare by the change in the rents earned by the nationalized firms. Hence, if they earned profits as nationalized firms, total surplus falls by the size of the profits now

[23] Recall that price leaders in strategic complements Bertrand differentiated products games will price above the Nash equilibrium level in order to induce a favorable price rise from the price followers. All earn more than at the Nash price equilibrium, but the leader earns less than the follower(s). The nationalized firm's incentives run the other way: it prices below the private firms, and this brings down their prices. This per se lowers profits for all, but can lower them more for the private firms. Of course, if the public firm prices close enough to marginal cost, its profits must be lower than the private firms'.

no longer earned. In this case, consumers experience a price rise, but this is exactly offset by the change in product variety as new entrants are attracted to the industry by reduced price competition from the erstwhile public firms.[24] Note that this suggests that profitable public firms ought not be privatized if there is free-entry, and if the market demands are well characterized by the IIA property.

## 8.4 Leaders and followers

Etro (2006, 2007, 2008) considers the effect of introducing a Stackelberg leader into the free-entry model. His main results can be derived succinctly and extended using the framework above.

We consider two variations on the Stackelberg theme. In the first, the leader simply chooses its value, $q_l$, and in doing so rationally anticipates the subsequent entry level and follower firm action levels. In the second, following Brander and Spencer (1983) and Spencer and Brander (1983), the leader firm, as first mover, commits to an investment level which allows it to present a reaction function to the followers which may differ from the cost-minimizing reaction function level.

We first note how the leader's action differs from the action it would choose in a simultaneous-move game. The cumulative reaction function $\tilde{r}_i(Q)$ is implicitly defined by

$$\frac{d\widetilde{\pi}_i(Q,q_i)}{dq_i} = \frac{\partial\widetilde{\pi}_i(Q,q_i)}{\partial Q}\frac{dQ}{dq_i} + \frac{\partial\widetilde{\pi}_i(Q,q_i)}{\partial q_i} = 0.$$

Assumption 1 states that $\pi_i(Q_{-i}+q_i,q_i)$ is strictly decreasing in $Q_{-i}$. Writing $\pi_i(Q_{-i}+q_i,q_i)$ directly as $\widetilde{\pi}_i(Q,q_i)$, this implies that $\widetilde{\pi}_i(.)$ is decreasing in its first argument. This implies that in a Nash equilibrium (with a fixed number of firms), the second term must be positive.

The equilibrium behavior of a first-mover with a fixed number of firms depends on whether actions are strategic substitutes or complements. If one of the firms has the privilege to choose its action before the others, it will take into account the impact of its action on the behavior of the followers. This implies that if actions are strategic complements,

[24] Lemma 3 in Anderson, de Palma and Thisse (1997) - that all private firms charge the same price in a long-run equilibrium - follows from Proposition 2 above. Proposition 2 in Anderson et al. (1997) -that the social gain (loss) from privatization equals the loss (profit) made by the public firms before they are privatized - is Proposition 9 above.

$dQ/dq_i > 1$. Since in a simultaneous-move Nash equilibrium $dQ/dq_i = 1$, the leader acts less aggressively than it would in a simultaneous-move game. On the other hand, if actions are strategic substitutes (i.e., $dQ/dq_i < 1$), the leader acts more aggressively than it would in a simultaneous-move game.

Consider now how the leader behaves in the long run, if its action is followed by entry. Proposition 1 implies that the first term is zero. Hence, the action taken by the leader in the long run must be larger than the action it takes in a simultaneous-move game. This result does not depend on whether we have a game with strategic complements or strategic substitutes.

A first result on welfare is quite immediate:

**Proposition 17** *Assume that one firm acts before the others (Stackelberg leader), and the subsequent equilibrium is a ZPFE. Then welfare is higher than under the Nash equilibrium, although consumer surplus stays the same.*

**Proof.** The consumer surplus result follows because $Q$ is the same, given the outcome is a ZPFE. Welfare is higher because the leader's rents must rise: it always has the option of choosing the Nash action level, and can generally do strictly better. ■

As discussed in Section 3.4, examples of cases where this results hold include markets with homogenous goods Cournot competition and Bertrand competition with demands satisfying the IIA property (such as logit/CES).

A second result follows from the competitiveness assumption, Assumption 1. Indeed, Assumption 1 implies that if we write $\pi_i (Q_{-i} + q_i, q_i)$ directly as $\pi_i (Q, q_i)$, then $\pi_i (.)$ is decreasing in its first argument. Consider now the equilibrium choice of action by Firm $i$ in a Nash equilibrium. It is characterized by the first-order condition which begets the reaction function, which is usefully rewritten in the following form:

$$\frac{\partial \pi_i (Q, q_i)}{\partial Q} \frac{dQ}{dq_i} + \frac{\partial \pi_i (Q, q_i)}{\partial q_i} = 0. \tag{10}$$

Since the first term is negative by Assumption 1, then the second must be positive at any Nash equilibrium. This means that if $Q$ is held fixed, as it is at a ZPFE, then the profit derivative with respect to $q_i$ must be positive at $i$'s Nash equilibrium action level. Given profit quasi-concavity (Assumption 2), then $q_l$ must be higher.

**Proposition 18** *(Replacement effect) Assume that one firm acts before the others (Stackelberg leader), and the subsequent equilibrium is a ZPFE. Then its action level is higher and there are fewer fringe firms, although each fringe firm retains the same action level.*

The contrast with equilibria with fixed numbers of firms is quite striking. The larger leader result is familiar from the classic strategic substitutes case of Cournot competition, though there the followers react by reducing output, and the leader endogenizes this effect. In the canonical strategic complements model of differentiated products, the leader sets a higher price, inducing a higher price from the followers (and this induced higher price is the objective since the action level $Q$ is reduced).[25] At the ZPFE, by contrast, the leader sets a lower price (higher $q_i$) and all fringe firms have the same price, regardless of the presence of a leader.

We term the above result the Replacement Effect because, with a fixed $Q$, the Leader would rather do more of it itself, knowing that it crowds out one-for-one the following fringe. In some cases, the Leader wants to fully crowd out the fringe. For example, in the Cournot model, with $\pi_i(Q, q_i) = p(Q) q_i - C_i(q_i)$ we have $\frac{\partial \pi_i(Q,q_i)}{\partial q_i} = p(Q) - C_i'(q_i)$, so the leader will fully crowd out the fringe if $p(Q) \geq C_i'(Q)$. This condition necessarily holds if marginal cost is constant (since $p(Q) > c$ at a ZPFE).

#### 8.4.1 Credible commitment

We assumed above the Leader firm is able to credibly commit to its action. However, this may not be feasible without some commitment device, and so the Leader may be constrained in the extent to which it can crowd out the fringe. Nevertheless, the tenor of the above results goes through: the leader will be more aggressive in order to credibly commit to crowd out the fringe, by presenting a tougher reaction function.

Following Brander and Spencer (1983) and Spencer and Brander(1983), suppose that the leader can irrevocably choose its cost technology, and by doing so present a more or less aggressive stance to the subsequent Fringe movers.

More formally, let us append a cost function $\Gamma(\theta_i)$ to the Leader's profit function, with the idea that higher $\theta_i$ shifts the net profit function, which we wrote before as

[25] These results can be quite readily derived within our framework.

$\pi_i(Q_{-i}+q_i, q_i;\theta_i)$, and so we now write the firm's profit as $\pi_i(Q_{-i}+q_i, q_i;\theta_i) - \Gamma(\theta_i)$. Naturally then, $\Gamma'(\theta_i) > 0$. For example, a higher capital investment translates into a lower marginal cost of production.

We first describe the standard ZPFE when the firm in question acts in standard Nash fashion, but we add to its choice the level of $\theta_i$. We then compare to the case in which the firm acts as Leader.

As we know from Section 3.3, the reaction function for given $\theta_i$, $r_i(Q_{-i}, \theta_i)$, is determined by

$$\frac{d\pi_i(Q_{-i}+q_i, q_i;\theta_i)}{dq_i} = 0.$$

As we noted above (see (10)), this can be written as

$$\frac{\partial\pi_i(Q, q_i;\theta_i)}{\partial Q} + \frac{\partial\pi_i(Q, q_i;\theta_i)}{\partial q_i} = 0, \tag{11}$$

where the first term is negative by Assumption 1 and so the second term must be positive at the solution. We now also have a new first order condition:

$$\frac{d\pi_i(Q_{-i}+q_i, q_i;\theta_i)}{d\theta_i} = \Gamma'(\theta_i), \tag{12}$$

which equalizes marginal benefits and marginal costs.

Consider now the case of the Leader firm, which chooses $\theta_l$ in advance of the fringe's actions. Regardless of what it does, it knows that the total market action, $Q_{ZPFE}$, is independent of its own actions. It also rationally anticipates that it will choose $q_l = \tilde{r}_l(Q_{ZPFE}, \theta_l)$ in the sub-game in which actions and entry decisions are determined. Therefore, its choice of $\theta_l$ is determined by the solution to the problem

$$\max_{\theta_l} \pi_l(Q_{ZPFE}, q_l;\theta_l) - \Gamma(\theta_l)$$

where $q_l = \tilde{r}_l(Q_{ZPFE}, \theta_l)$, and the Leader treats $Q_{ZPFE}$ as fixed. The derivative with respect to $\theta_l$ is given by

$$\frac{\partial\pi_l(Q_{ZPFE}, q_l;\theta_l)}{\partial q_l}\frac{d\tilde{r}_l(Q_{ZPFE};\theta_l)}{d\theta_l} + \frac{\partial\pi_l(Q_{ZPFE}, q_l;\theta_l)}{\partial\theta_l} - \Gamma'(\theta_l) \tag{13}$$

where $\frac{d\tilde{r}_l(Q_{ZPFE};\theta_l)}{\partial\theta_l}$ has the sign of $\frac{\partial^2\pi_i(Q_{-i}+q_i, q_i;\theta_i)}{\partial\theta_i\partial q_i}$ whether or not actions are strategic substitutes or strategic complements.[26]

[26] Recall from (2) that $\frac{d\tilde{r}_i(Q,\theta_i)}{d\theta_i} = \frac{\partial r_i/\partial\theta_i}{1+\partial r_i/\partial Q_{-i}}$.

Now, since $Q_{ZPFE}$ is the same whether or not the firm acts as a leader, the first term $\frac{\partial \pi_l(Q_{ZPFE}, q_l; \theta_l)}{\partial q_l}$ in (13) is positive when evaluated where $\theta_l$ is given by the non-strategic (simultaneous move) choice, i.e., (12), at which point the last two terms in (13) are also zero. Therefore the derivative (13) at that choice has the sign of $\frac{\partial^2 \pi_i(Q_{-i}+q_i, q_i; \theta_i)}{\partial \theta_i \partial q_i}$. This means that the choice of $\theta_l$ is then higher under pre-commitment (assuming the problem is sufficiently well-behaved that it is quasi-concave) if marginal profit is increasing in $\theta_i$ and it is lower if marginal profit is decreasing. In either case, $\tilde{r}_l(Q)$ increases from the simultaneous game level: this is the Aggressive Leaders result of Etro (2006).[27]

Short-run incentives with a fixed number of firms can also readily be compared. The Leader wants to do at the margin whatever causes the other firms to cut back, so this is clearly to be more aggressive if actions are strategic substitutes (see Figure 11), and to be less aggressive if actions are strategic complements (Figure 12).[28]

# 9 Further applications and comments

Aggregative games are common in other applications, such as public finance, where a selectively-applied exogenous tax or subsidy may affect the marginal or fixed costs of firms (see e.g. Anderson et al. (2001a and 2001b) and Besley (1989)), and in public good games (see Cornes and Hartley (2007)). They are also extensively applied in international trade models, along with the free entry condition, in order to analyze unilateral changes in trade policy (see for example Etro (2010), Markusen and Stahler (NBER), Markusen and Venables, Venables (1985), Horstmann and Markusen (1986, 1992). Most recently, CES models are at the heart of the empirical renaissance in trade (see Melitz, Melitz and Ottaviano, and Helpman and Itskhoki, and the survey article in JEP (2007)). Further applications of CES based models arise in the new economic geography, while common property resource problems and congestion problems (including consumer information overload) have a strong aggregative game presence too.

One of the shortcomings of the applications (and results) we have considered so far is that they are all based on models of global competition. Some special cases of localized

[27] Results for the case of heterogeneous followers can be readily derived.

[28] With a lower leader action, rivals reply with lower actions, which is the response the leader wants to induce, under A1.

competition which are aggregative games include the Hotelling model with two firms and the circular city model with 3 firms. The short run results presented above would apply in these special cases.

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