# The Birth and Burst of Asset Price Bubbles*

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**Abstract**

We develop a model of rational bubbles, based on the assumptions of an unknown potential market size (liquidity) and delegation of investment decisions. In a bubble, the price of an asset rises above its steady-state value, which must be justified by rational expectations about possible future price developments. The higher the expected future price increase, the more likely is the market potential reached, in which case the bubble will burst. Depending on the interaction of uncertainty about the market potential, fundamental riskiness of the asset, the compensation scheme of the fonds manager, and the risk-free interest rate, we give a condition for whether rational bubbles are possible. Based on this analysis, several widely-discussed policy measures are investigated with respect to their effectiveness to prevent bubbles. A modified Taylor rule, long-term compensation, and capital requirements can have the desired effect. Caps on bonuses and a Tobin tax can create or destroy the possibility of bubbles, depending on their implementation.

**Keywords:** Bubbles, Rational Expectations, Bonuses, Compensation Schemes, Financial Crises, Financial Policy.

**JEL-Codes:** G01, G12, G18.

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# 1 Introduction

What causes asset price bubbles? In the light of recent economic experience, this question seems important and topical. The last 15 years have seen at least two important market developments that are considered as bubbles by now. Both, the so-called dot-com bubble in the late '90s and the recent housing bubble in the United States and elsewhere have produced large reallocations of wealth during their buildup, and especially after their respective crashes. These bubbles have not only affected the parties directly participating in the bubble markets. Also outsiders were impacted heavily, e. g., by mass layoffs that took place as a result of the crashes. Although bubbles are a phenomenon known (at least) since the tulip mania in 1637, economic policy has apparently not been able to prevent their repeated occurrence. Neither does a commonly accepted theoretical model of bubbles exist, which could be used to derive policy implications. Our paper contributes to the development of such an understanding, which might eventually help in guiding policymakers.

We construct a simple workhorse model of a bubble, based on an overlapping generations model and the crucial assumption that the potential amount of liquidity in the market is not precisely known. In the model, the number of potential investors is a random variable. Consequently, information about the market size is noisy except for the limiting case of a finite market of fixed size. We think that, as financial markets become more complex and opaque, the assumption of imprecise information about the market size seems very natural. Within a bubble, managers are only willing to invest if they believe that there might be another investor in the next period to whom they can sell the asset at an even higher price. As already observed by Tirole (1982), if the number of investors were known, the highest possible price of the concerned asset could be calculated, and by backward induction no bubble could emerge from the beginning.[1]

The second important feature of our model is delegation. In particular, we consider investors who delegate investment to fonds managers. The model applies, however, directly to more general intermediated finance such as through banks, investment banks, insurance companies, private equity firms, and the like. In the absence of a bubble, we find that the risk appetite induced by the limited liability of fonds managers pushes asset prices above their fundamental values (as already noted by Allen and Gale, 2000). Because of the limited liability in case of a low or zero return, the manager can increase her expected payoff by engaging in riskier assets. This effect drives asset prices above fundamentals, but in a static way. These price deviations are not induced by expectations, and there are no sudden corrections (bursts).

[1] Tirole (1985) extends the model of Tirole (1982) to an overlapping-generations model with perfect foresight, showing that under certain conditions bubbles can occur. However, these bubbles do not grow faster than the real interest rate. Also Santos and Woodford (1997) show that the conditions for the existence of bubbles are very restrictive, if one is to assume a fixed number of households that participate in the asset market and own a finite aggregate endowment.

Adding the assumption of unknown market liquidity extends the space of possible price paths drastically. Combined with a high-powered incentive, an expectations-induced bubble with a dynamic price path may emerge. Higher prices increase the probability that the current asset holders do not find future buyers for even higher prices. Given this increased risk, today's buyers demand a higher expected gain from the asset.[2] This mechanism drives prices up over time, until the bubble collapses because either the previously unknown ceiling is hit or the underlying fundamental breaks down (e .g., a bankruptcy of the issuing firm).[3] Importantly, the described mechanism can, but does not have to allow for bubbles. Depending on the interaction of limited liability, uncertainty about the market size, riskiness of the asset and the interest rate on an alternative safe asset the prerequisites for bubbles can be fulfilled or not. This stands in contrast to previous models, in which bubbles always exist if the ceiling in the market is unknown, or are always ruled out if this ceiling is known (Brunnermeier, 2008). In these kinds of models, no comparative statics and policy implications can be derived.

Since the model allows us to derive conditions under which bubbles can exist, we can also test several policies that could prevent bubbles. This is particularly important, since bubbles harm the welfare of market participants. One of the widely-discussed possible policy measures is a cap on bonuses. We find that a system that reduces the bonus payments but keeps their proportionality to investment success could actually backfire and make bubbles possible. A maximum cap on bonuses, on the other hand, can effectively prevent the emergence of bubbles. Similarly, a financial transaction ('Tobin-') tax can create the possibility of bubbles if levied on all forms of financial assets. On the other hand, if it is imposed on the risky asset only, it can effectively prevent the emergence of bubbles. Also a monetary policy rule that takes asset price inflation into account, as discussed in Bernanke and Gertler (2001), can render bubbles impossible. Finally, mandatory long-term compensation and/or capital requirements fulfill the same purpose.

[2]Note that this mechanism does not require asymmetric expectations. This differentiates the models from Allen, Morris, and Postlewaite (1993), in which private information can drive a price above its fundamental value, and those of Scheinkman and Xiong (2003) and Bolton, Scheinkman, and Xiong (2006), who assume that buyers of an asset hope to sell it to overoptimistic agents in the next period. This is only possible in case of heterogenous beliefs. Note that different to our model, the latter paper is concerned with executive compensation, as Calcagno and Heider (2007). Allen and Gorton (1993) show that asymmetry of information between investors and heterogenous managers can lead to deviations of prices from fundamentals. The model of Brunnermeier and Abreu (2003) relies on dispersed opinions. Together with coordination failure, they can trigger bubbles. In this context, Froot, Scharfstein, and Stein (1992) analyze which information can influence trading, potentially leading to herding equilibria. Allen, Morris, and Shin (2006) analyze the role of higher-order expectations if traders have asymmetric information.

[3]Referring to the dot-com bubble, Brunnermeier and Nagel (2004) provide evidence that hedge funds were riding the bubble, a result similar to a previous finding by Wermers (1999). They relate this to, among others, a short-term horizon of the managers. This is in line with our model. Here, riskiness and herding are no opposites, such that the argument of Dass, Massa, and Patgiri (2008)—high-powered incentive schemes will induce managers to break out from herding—does not apply.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 2.2 constructs a steady-state (rational expectations) equilibrium price process. Section 3 constructs a simple example of a non steady-state (rational expectations) equilibrium price process, which we call a bubble. We give a necessary and sufficient condition for the existence of such example-bubbles. In section 4, we show that this very condition is necessary and sufficient for the existence of bubbles in general. This condition lends itself to basic policy analysis, done in section 5 by discussing several policy measures. Some measures require a slight generalization of the model. While all other sections take the managers' compensation scheme as given, we consider one (of possibly many) ways to endogenize bonus payments in section 6. Section 7 concludes. All proofs are in the appendix.

# 2 The Model

## 2.1 Setup

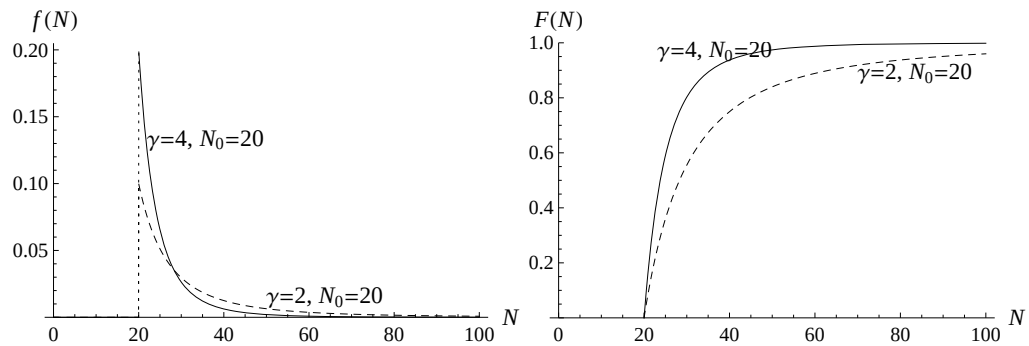
Consider an infinite horizon economy with overlapping generations of two types of agents, investors and fonds managers. In each period, a continuum of measure $N$ *investors* is born, each with an endowment of 1 dollar. Investors die in the next period. They consume only in the period they die. Investors cannot participate in the financial market. There is a continuum of fonds managers (short: managers), and in order to invest in bonds or stocks, each investor needs to employ one of these manager. Since the number of managers is assumed to be unlimited, an investor will always find a manager to handle her wealth. Each manager can handle the funds of one investor only. The manager is compensated by a linear scheme with limited liability. Her compensation can consist of a success-depending bonus and a base salary $S$. Earning a yield $y$, she receives $\max\{\alpha(y-\beta);\, 0\} + S$, with $\alpha \in [0;1]$ and $\beta, S \geq 0$. So if a manager invests 1 dollar into an asset at price $p_t$ and the price rises to $p_{t+1}$, she receives $\max\{\alpha(p_{t+1}/p_t - \beta);\, 0\} + S$. The contract will be treated as exogenous within this section and will be endogenized in section 6.

There are two assets, safe assets (bonds) of unlimited supply and a single risky asset. The safe bond bears a net interest of $r$. The risky asset can be interpreted as the shares of a firm. This firm pays total dividends of $d$ each period.[4] However, in each period, there is a probability $1-q$ that the firm will go bankrupt and cease to pay dividends forever. Hence, the time of bankruptcy is determined by a Poisson process. The total amount of shares of the firm is normalized to 1. The risky asset is traded in each period. Its price follows a time-discrete stochastic process $\{p_t\}_{t\geq 0}$.

[4] One may also interpret the asset as real estate. If the house is let, then $d$ is the rent per period.

The number of investors $N$ is unknown ex-ante. It follows a Pareto Distribution, with the density $f(N) = \gamma\, N_0^\gamma / N^{1+\gamma}$ and the distribution function $F(N) = 1 - N_0^\gamma / N^\gamma$ (both for $N > N_0$). Here, $N_0$ is some lower bound on the number of investors, and $\gamma$ is a shape parameter. The smaller $\gamma$, the more uncertainty exists about the number of investors, the thicker is the tail of the distribution. In fact, the mean of the distribution is $\mu = N_0 \frac{\gamma}{\gamma-1}$ for $\gamma > 1$, and $\mu = \infty$ for $\gamma \leq 1$. The standard deviation is given by $\sigma = N_0 \frac{1}{\gamma-1} \left(\frac{\gamma}{\gamma-2}\right)^{1/2}$ for $\gamma > 2$, and $\sigma = \infty$ for $\gamma \leq 2$. The following figure 1 shows the distributions and density functions for $N_0 = 20$ and shape parameters $\gamma = 2$ (dashed) and $\gamma = 4$ (solid).[5] For $\gamma \to \infty$, we get the limiting case of a known number of investors.

Figure 1: Pareto Density and Distribution Functions




## 2.2 The Steady-State Price

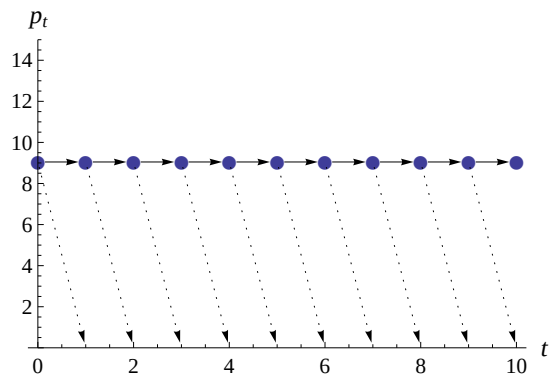
Consider the following simple stochastic process. The price of the asset is a constant, $\tilde{p}_t = \bar{p}$. Only if the underlying firm goes bankrupt (with probability $1 - q$) and cash ceases to flow, the price drops to $\tilde{p}_t = 0$. Hence, the price follows a very simple binomial process with $\Pr_t\{\tilde{p}_{t+1} = \bar{p} | p_t = \bar{p}\} = q$. The zero is an absorbing state. Let us derive the price $\bar{p}$ for which this process can be a rational expectations equilibrium.

In a market equilibrium, prices must be such that the managers' compensation is the same for storage and for the risky asset. If the manager stores, the compensation is

[5] In fact, we would only need the assumption that the upper tail of the distribution of potential market participants $F(N)$ can be approximated by a Pareto distribution. The theorem of Pickands, Balkema and de Haan states that this assumption holds for a large class of distributions (see Embrechts, Klüppelberg, and Mikosch, 2008). For our purpose, the Pareto distribution has the important feature that the probability to exceed a threshold $N$, conditional on that we exceed $N - \Delta N$, does not approach zero as $N \to \infty$. In fact, $(1 - F(N))/(1 - F(N - \Delta N)) = ((N - \Delta N)/N)^\gamma \to 1$ as $N \to \infty$.

$\max\{0;\, \alpha\,(1+r-\beta)\} + S = \alpha\,(1+r-\beta) + S$, assuming for now that $\beta \leq 1+r$.[6] If the manager buys shares of the firm at a price $p_t = \bar{p}$, she benefits from the dividend with probability $q$. She thus earns $d/p_t$ with probability $q$. If the firm does not pay a dividend, the price drops to zero. Otherwise, the price remains at $\tilde{p}_{t+1} = \bar{p}$, and the manager gets additionally $\tilde{p}_{t+1}/p_t = \bar{p}/\bar{p} = 1$ from selling the asset. This stochastic process is depicted in figure 2 (with parameters $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 10\%$).

Figure 2: A Binomial Price Process




Given this price process, a date-$t$-manager's expected compensation is

$$E_t \max\left\{0;\, q\,\alpha\left(\frac{\tilde{p}_{t+1}}{p_t} + \frac{d}{p_t} - \beta\right)\right\} + S \tag{1}$$

In the market equilibrium, managers must be indifferent between the asset and storage, hence

$$\alpha\,(1+r-\beta) + S = E_t \max\left\{0;\, q\,\alpha\left(\frac{\tilde{p}_{t+1}}{p_t} + \frac{d}{p_t} - \beta\right)\right\} + S. \tag{2}$$

Since the left-hand side is positive, we get

$$\alpha\,(1+r-\beta) + S = q\,\alpha\left(\frac{\bar{p}+d}{\bar{p}} - \beta\right) + S. \tag{3}$$

The *steady-state price* $\bar{p}$ is above the fundamental value of the asset that would obtain if investment were not delegated to managers, denoted by $\underline{p} := d\,q/(1-q+r)$.

$$\bar{p} = \frac{d\,q}{(1-\beta)\,(1-q)+r}. \tag{4}$$

[6] This assumption is confirmed for an endogenized contract in section 6.

If $\beta = 0$ or if $q = 1$, the two prices are equal, $\underline{p} = \bar{p}$. The effect that managers with limited liability push up prices of risky assets above their fundamental value has been analyzed before by Allen and Gale (2000). Like Allen and Gale, we find that an increase in uncertainty (i.e. a higher $q$), keeping the fundamental value constant, drives the steady-state price $\bar{p}$ up.

**Remark 1** *Keeping the fundamental value constant, the steady-state price of a riskier asset is higher above its fundamental value.*[7]

Let us make one important clarification. In the above numerical example, the fundamental value is $\underline{p} = 6.33$, but the steady-state price is $\bar{p} = 9.05$. This price deviation is due to the limited liability of managers. However, it is a *static* deviation, which is driven by fundamentals ($q$, $d$, and $r$) and the managers compensation package ($\beta$ and $\alpha$, where $\alpha$ is irrelevant). The price deviations is hence driven by fonds managers' expectations about future risk ($q$) and dividends ($d$), but *not* on their expectations about future price developments. The deviation is constant over time and cannot burst, such that its existence is less interesting from a financial stability perspective. Nevertheless, this deviation can magnify price movements. By contrast, the bubble described in the following is dynamic by nature. It can be sustained only if the price is expected to keep on increasing in the future. Price deviations will be fueled by the expectation that in the future, other managers will buy at an even higher price (if the bubble has not burst until then).

# 3 An Example for a Bubble

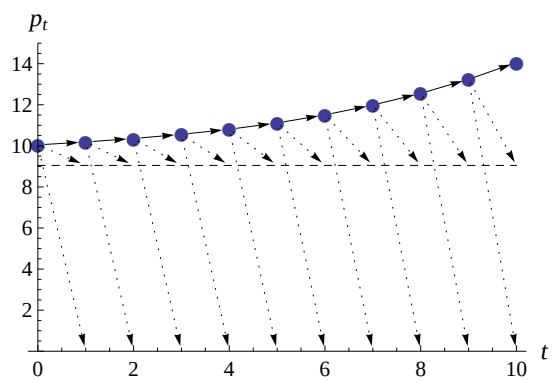
Assume that the price $p_t$ is above $\bar{p}$ at some date $t$. The only conceivable reason to buy is that managers expect the price to rise even further, at least with some probability. Otherwise, as shown above, it would be a dominant strategy for managers to store rather than to invest in the asset. However, a price increase to some $p_{t+1} > p_t$ could also require more liquidity than investor's aggregate endowment. In this case, the price would hit a ceiling, no more price increase will be expected, and the bubble would have to collapse back to $\tilde{p}_{t+1} = \bar{p}$. Alternatively, if the underlying firm goes bust, the price will drop to $\tilde{p}_{t+1} = 0$. As a consequence, the simplest process that can exhibit a bubble is trinomial. Let us hence look at a process with

$$\tilde{p}_{t+1} = \begin{cases} 0, & \text{with probability } 1-q \\ \bar{p}, & \text{with probability } q - Q_t \\ p_{t+1}, & \text{with probability } Q_t \end{cases} \tag{5}$$

[7] The proofs for this remark and all propositions are in the appendix.

with $Q_t \leq q$. Note the notational difference between $\tilde{p}_{t+1}$ and $p_{t+1}$. $\tilde{p}_{t+1}$ is the stochastic price at date $t+1$ that can assume three different values. $p_{t+1}$ is the largest of these values, $p_{t+1} > \bar{p} > 0$. A possible price process is depicted in figure 3 (with parameters as above). The process starts at some price $p_0 > \bar{p}$, and the bubble potentially grows further and further. However, it can hit the ceiling $N$ at any time and burst. $N$ cannot be pictured in the figure, since it is unknown. The ceiling will be hit with probability 1, but the date at which the bubble bursts is (and must be) unknown.

Figure 3: A Trinomial Price Process with a Bubble




For a price increase from $p_t$ to $p_{t+1}$, the probability of a continuation (*non*-collapse) of the bubble is

$$Q_t = q\,\frac{1-F(p_{t+1})}{1-F(p_t)} = q\,\frac{N_0^\gamma/p_{t+1}^\gamma}{N_0^\gamma/p_t^\gamma} = q\,p_t^\gamma/p_{t+1}^\gamma. \tag{6}$$

Hence, $q$ is the probability that a firm continues to operate, and $Q_t$ is the probability that the firm's asset price continues to rise. The probability that the bubble just bursts although the firm is still solvent is thus $1 - Q_t - (1-q) = q - Q_t$.

If the share price falls because the firm is insolvent, then the price will drop to zero and no dividends will be paid. The payment to the manager is then

$$\alpha\,\max\left\{\frac{0}{p_t} + \frac{0}{p_t} - \beta; 0\right\} = 0. \tag{7}$$

If the share price falls because a bubble bursts, the price will drop to $\bar{p}$, and dividends will still be paid. The payment to the manager is then

$$\alpha\,\max\left\{\frac{d}{p_t} + \frac{\bar{p}}{p_t} - \beta; 0\right\} = \alpha\,\max\left\{\frac{d + \frac{d\,q}{(1-\beta)\,(1-q)+r}}{p_t} - \beta; 0\right\}. \tag{8}$$

This implies that, if the price is only slightly above the steady-state price $\bar{p}$ (hence the bubble is small), the manager will earn a bonus even when the bubble bursts. The according condition is

$$p_t < \hat{p} := \Big(1 + \frac{d\,q}{(1-\beta)\,(1-q)+r}\Big)/\beta. \tag{9}$$

Otherwise, the manager gets nothing if the bubble bursts. Let us start with discussing the second case. If she invests in the risky asset, she gets a bonus with probability $Q_t$. Then a modified version of (3) must hold,

$$\begin{aligned}
\alpha\,(1+r-\beta)+S &= Q_t\,\alpha\,\big((p_{t+1}+d)/p_t-\beta\big)+S,\\
&= q\left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\alpha\,\big((p_{t+1}+d)/p_t-\beta\big)+S,\\
\frac{1+r-\beta}{q} &= \left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\left(\frac{p_{t+1}}{p_t}+\frac{d}{p_t}-\beta\right).
\end{aligned} \tag{10a}$$

If, on the other hand, $p_t$ is below $\hat{p}$ such that (9) is satisfied, another version of (3) must hold,

$$\begin{aligned}
\alpha\,(1+r-\beta)+S &= Q_t\,\alpha\,\big((p_{t+1}+d)/p_t-\beta\big)+(q-Q_t)\,\alpha\,\big((\bar{p}+d)/p_t-\beta\big)+S,\\
\frac{1+r-\beta}{q} &= \left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\frac{p_{t+1}}{p_t}+\left(1-\left(\frac{p_t}{p_{t+1}}\right)^{\gamma}\right)\left(\frac{\bar{p}}{p_t}+\frac{d}{p_t}-\beta\right).
\end{aligned} \tag{10b}$$

Equations (10a) and (10b) respectively implicitly determine a price process in a rational expectations equilibrium. To be precise, let $f(p_{t+1}, p_t)$ be defined as the right-hand side minus the left-hand side of equations (10a) and (10b), depending on whether (9) holds.

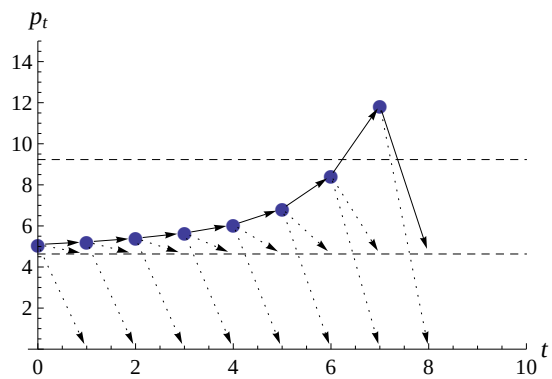
**Definition 1** *A rational-expectations equilibrium is a path of prices $\{p_t\}_{t\geq 0}$ and transition probabilities $\{(q, Q_t)\}$such that for $E_t[f(p_{t+1}/p_t)|(q, Q_t)] = 0$ for all $t \geq 0$.*

For any given $p_0 > \bar{p}$, (10a) (or 10b) implicitly define $p_1$, and (6) defines the according $Q_0$, so all variables for $\tilde{p}_1$ in (5) are defined. Then starting from $p_1$ in a next step, (10a) (or 10b) and (6) define $p_2$ and $Q_1$, so $\tilde{p}_2$ is defined. Following this procedure defines the complete process recursively. One such process is shown in the above figure 3.

However, equations (10a) and (10b) do not necessarily have a solution for any set of parameters. The higher the potential future price $p_{t+1}$, the likelier it is that the ceiling $N$ is hit and the bubble will burst. The likelier a bursting of the bubble, however, the higher a potential price increase must be in order to compensate managers for the risk they face. A multiplier effect evolves. This feedback does not necessarily reach an equilibrium price $p_{t+1}$ for all $t$. As a consequence, a bubble can burst with certainty at some date $t$, and $Q_t = 0$. If the bubble cannot be sustained at date $t+1$, managers will anticipate this

already before, and a backward induction argument shows that the bubble will not be sustainable right from the start. An example is given in figure 4 (with $r = 20\%$, all other parameters as above). At date 7, the price has risen too high, i. e. above the dashed line, and the bubble can no longer be sustained. Consequently, the according initial price $p_0$ cannot be part of an rational expectations equilibrium process in the first place.

Figure 4: A Trinomial Price Process with a Non-sustainable Bubble




We are interested in conditions under which a bubble can or cannot be sustained. In order to be sustainable, the implicit equations (10a) and (10b) must have a solution for any date $t$. Rewrite (10a) and (10b), defining the auxiliary variable $\phi_t = p_{t+1}/p_t$ as the relative price increase,

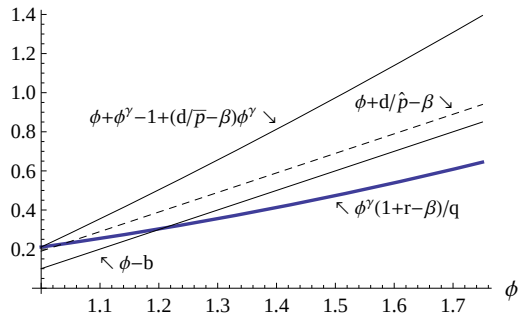
$$\phi_t^\gamma \frac{1+r-\beta}{q} = \phi_t + \frac{d}{p_t} - \beta \quad \text{for} \quad p_t > \hat{p}, \tag{11a}$$

$$\phi_t^\gamma \frac{1+r-\beta}{q} = \phi_t + (\phi_t^\gamma - 1)\frac{\bar{p}}{p_t} + \phi_t^\gamma \left(\frac{d}{p_t} - \beta\right) \quad \text{otherwise.} \tag{11b}$$

The value of $p_{t+1} = \phi_t\, p_t$ is implicitly defined by (11b) if $p_t < \hat{p}$, and otherwise by (11a). The right-hand side of the equation is always the same, the left-hand side is depends on the starting point $p_t$. The following figure 5 shows the right-hand side (thick), and the left-hand side for a couple of parameters. First, $p_t = \bar{p} < \hat{p}$. In this case, the right-hand side of (11b) becomes $\phi_t + (\phi_t^\gamma - 1) + \phi_t^\gamma \left(d/\bar{p}_t - \beta\right)$. From the figure, one can see that the only intersection with the thick curve is at $\phi_t = 1$, which implies that $p_{t+1} = \phi_t\, p_t = p_t$, hence there is no price increase. Starting with $p_t = \bar{p}$, we are of course in the steady state, and the price does not change over time. There is no bubble.

But if the initial price is slightly above $\bar{p}$, the curve bends downward, implying that it intersects with the curve at some $\phi_t > 1$. In the next period, the price will be higher still, and hence the intersection $\phi_{t+1}$ will be even higher. A bubble emerges, and the speed $\phi_t = p_{t+1}/p_t$ increases with time. When the price $p_t = \hat{p}$ is reached, the right-hand sides of (11a) and (11b) are equal, and we are at the dashed line in the figure. The intersection

Figure 5: Possibility of a Bubble




is again at some $\phi_t > 1$. This implies that the price will increase even more, resulting in a further parallel shift downwards of the line. For an infinite price $p_t$, the limiting line $q(\phi - \beta)$ is reached. From the figure, one can see that the intersection point moves right as $p_t$ increases. As a result, over time (with increasing $p_t$), the bubble becomes less and less stable, the probability of a burst increases.

**Remark 2** *In a bubble process, the relative price increase $\phi_t = p_{t+1}/p_t$ grows over time, $Q_t$ falls over time, and the bubble becomes less stable.*

As a consequence, in order to show that a bubble can be sustained in a market, it suffices to consider large prices $p_t$. Hence, we may concentrate on the case $p_t > \hat{p}$. In the limit $p_t \to \infty$, equation (11a) simplifies to

$$\phi^\gamma (1 + r - \beta) = q (\phi - \beta). \tag{12}$$

The equation does not depend on time, so we have dropped the index $t$. If (12) has a solution for $\phi$, the according market can sustain a bubble. For arbitrarily high prices $p_t$, there is always a price $p_{t+1}$ that is high enough to make fonds managers buy at date $t$. If (12) does not have a solution for $\phi$, then there is exists a price $p_t$ that is so high that a further increase is impossible. Nobody will buy, and the bubble will burst. Hence, using backward induction, the bubble cannot get started at date $t = 0$. The only possible initial price is then $p_0 = \bar{p}$.

**Numerical Examples.** Unfortunately, this innocent looking equation (12) has no closed-form solution for $\phi$. Because $\gamma > 1$, the left-hand side of (12) exceeds the right-hand side for large $\phi$. The above figure 5 shows the left and right side of (12) for the numerical example $\gamma = 2$, $\beta = 0.9$, $q = 95\%$, $d = 1$, and $r = 10\%$. There is a solution at $\phi = 1.21$

(and, for completeness, another at $\phi = 3.54$).[8] Let us briefly explain this number. For these parameters, (3) yields $\bar{p} = q/\big((1-\beta)(1-q)+r\big) = 9.05$. Hence, the minimum asset price would be much above the fundamental value of $q/(1-q+r) = 6.33$. However, a price of $9.05$ would be stable. Each period, with probability $1-q = 5\%$, the firm would stop to pay dividends, in which case the price would drop to zero.

Now if, as a zero probability event, the price of the asset moves above $\bar{p} = 9.05$, this new price is the starting point of a bubble. Figure 3 shows a bubble that starts at $\bar{p}+0.8 = 9.85$. At the starting point of the bubble, the probability of a burst is $1-Q = 1-q\,(p_t/p_{t+1})^{\gamma} \approx 5.7\%$, only slightly above $1-q = 5\%$. In later periods, $p_{t+1}/p_t$ converges towards $1.21$, as calculated above. The probability of a burst then converges towards $1-Q = 1-0.95\,(1/1.24)^2 \approx 34.7\%$. The bubble can burst for two reasons. First, as a fundamental reason, the underlying firm can go bankrupt. Second, as a financial reason, the resources in the market can be exhausted. Figure 3 shows these two possible developments of the market. The black curve starts with the steady-state price of $9.05$. The price never increases. With probability $1-q = 5\%$, the price drops to zero, but otherwise it remains stable. The gray curve starts slightly above the steady state price at $p = 9.25$. This price can only be rational if further price increases are expected.

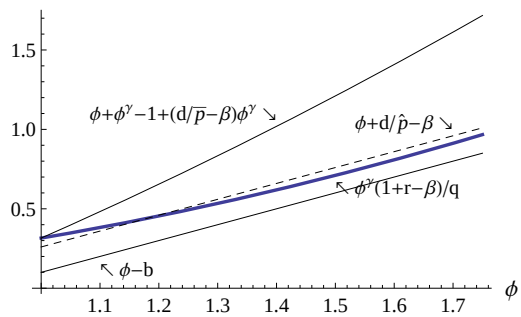
In another numerical example, let us see what happens if a bubble is not sustainable. Setting $r = 20\%$ (and letting all other parameters unchanged), we get the following figure 6. Here, because of the higher interest rate, $\bar{p}$ drops to 4.63 (the dashed and the curved line are higher). There is no solution for equation (12), so a bubble cannot be sustainable. One can calculate the maximum price $p_t$ for which (11a) has a solution, namely at $p_{\max} = 9.23$ (upper dashed line). If $p_t > 9.23$ at some date, then $p_{t+1}$ does not exist. In a bubble, prices need to rise, hence the price will reach $p_{\max}$ at some time and the bubble is not sustainable.

Figure 4 uses this parameter constellation. The price in the bubble rises. At date $t = 7$, it rises above $p_{\max} = 9.23$, so the bubble will bust no later than $t = 8$. Backward induction yields that the bubble cannot get started in the first place. The *only possible* price path is the steady state, with a price of $\bar{p} = 4.63$.

**Existence of Trinomial Bubble Processes.** The above numerical examples in figures 3 and 4 seem to suggest that lower interest rate levels support bubbles, whereas higher interest rates can punctuate a bubble. Reassuringly, this is perfectly in line with traditional intuitions of bubbles.

[8] There are at most two solutions to (12) with $\phi \geq 1$ (values of $\phi < 1$ would stand for bubbles with falling prices and, formally, negative probabilities of a burst). We do not consider the high solution in the following since the corresponding equilibrium is unstable. Note that a situation in which the straight line is above the curved one in figure 5, $\phi^{\gamma}(1+r-\beta) < q(\phi-\beta)$, implies a low probability of a burst relative to the expected gains. Hence, the price is driven up ($\phi$ falls) and we move to the left. The same argument holds for the opposite case, driving $\phi$ up. Thus, only the lower equilibrium is stable.

Figure 6: Non-Existence of Bubbles




Let us now analyze more generally under which conditions bubble processes can exist. Looking at figure 5, one can see that the solution may cease to exist if the gray line does no longer intersect with the black curve, like in figure 6. A general condition is given in the following proposition.

**Proposition 1** *The market can develop a trinomial bubble process if and only if*

$$\gamma^{\gamma}\left(\frac{\beta}{\gamma-1}\right)^{\gamma-1} \leq \frac{q}{1+r-\beta} \quad \textit{and} \tag{13}$$

$$\beta > \frac{\gamma-1}{\gamma}, \tag{14}$$

*that is, for large $q$, small $r$, small $\gamma$ or large $\beta$.*

The parameter $\gamma$ captures the uncertainty in the market. The smaller $\gamma$, the larger are mean and variance of the distribution, the more uncertain is the potential market size. For $\gamma \leq 1$, the mean is infinite, and for $\gamma \leq 2$, the variance is infinite. The parameter $N_0$ does not appear in the analysis, which shows that for the existence of a bubble only the shape of the upper tail matters. The smaller $\gamma$, the more likely a bubble can be sustained.

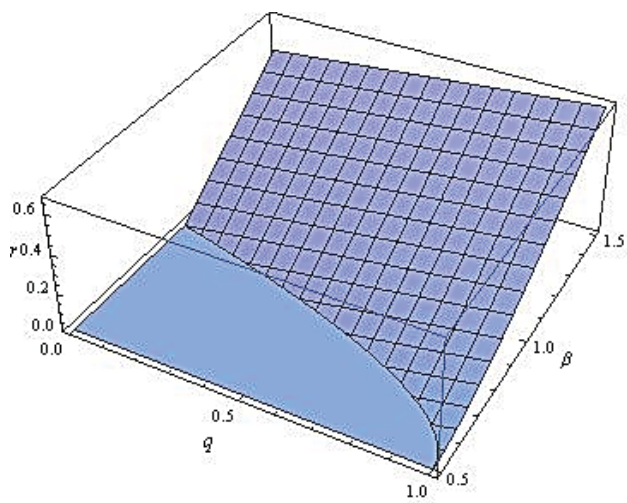
In the extreme case of $\gamma \to 1$, the expected market size becomes infinite, and $\gamma^{\gamma}\,(\beta/(\gamma-1))^{\gamma-1} \to 1$. Hence, a bubble can emerge if $q > 1+r-\beta$. On the other hand, if $\gamma \to \infty$, the market size is almost certainly $N_0$, and a bubble can never be sustained, independent of the sizes of other parameters. This is the traditional backward induction argument of Tirole (1982).

The larger the interest rate $r$, the less likely is the possible existence of a bubble. This is in line with the intuition that central banks can punctuate bubbles by increasing interest rates, and that bubbles are more likely to emerge it interest rates are low.

Bubbles can exist especially if $q$ is high, that is, if the underlying asset is rather safe. This seems to be in line with the recent housing bubble in the U. S. and other countries. Real estate itself has a bankruptcy probability of almost zero, thus $q \approx 1$.[9] Hence, as argued above, the difference between the fundamental value $\underline{p}$ and the steady-state price is higher for more risky assets, but the likelihood that a bubble emerges is larger for rather safe assets.

Finally, the parameter $\beta$ describes how steep the incentive schemes of managers are. The larger $\beta$, the later the bonus payments to the manager kick in, and the higher is the power of the contract, and the more prominent is the effect of the limited liability of the manager. Hence, we have the result that the emergence of bubbles becomes more likely when fonds manager compensation is higher powered. The following figure 7 summarizes all these observations for the case $\gamma = 2$. Condition (14) then implies that $\beta > (\gamma - 1)/\gamma = 1/2$, hence $\beta > 0.5$ in the figure. For parameters below the surface, bubbles are feasible.

Figure 7: Feasibility of a Bubble




# 4 Bubbles in General

We have argued that a very special kind of a bubble process, the trinomial bubble, exists if and only if (13) holds. We now make this result more general by showing that, if (13) fails to hold, the only rational expectations equilibrium process is the non-bubble process with price $\bar{p}$. With other words, bubble processes in general exist if and only if (13) holds.

[9] If real estate is seen as a risky investment, then mainly because real estate prices can be driven away from fundamentals, not because real estate is inherently risky.

The argument is simple. A trinomial bubble does not exist if the curves in figure 6 do not intersect, i.e. there is not solution for $\phi$ and hence prices in the bubble eventually increase too fast to be sustainable. But any other bubble, not necessarily trinomial, would have to allow for prices to increase at least as fast as in a trinomial bubble. So if, for a given set of parameters, no trinomial bubble path exists, no bubble can exist at all.

**Proposition 2** *In a rational expectations equilibrium, a price process can exhibit a bubble if and only if*

$$\gamma^{\gamma}\left(\frac{\beta}{\gamma-1}\right)^{\gamma-1} \leq \frac{q}{1+r-\beta} \quad \textit{and} \tag{15}$$

$$\beta > \frac{\gamma-1}{\gamma}, \tag{16}$$

*hence if* (13) *and* (14) *hold.*

This is the main result of our paper. If the condition holds, there are multiple rational expectations equilibria, including bubble equilibria. If the condition does not hold, there is only one steady-state (non-bubble) equilibrium price process. There are no bubbly equilibria, neither trinomial nor of any other shape.

# 5 Policy Measures

In this section, we examine whether certain policy measures that have been suggested in the public debate can prevent the creation of bubbles in our model. Specifically, we look at an asset-price augmented Taylor rule, caps on bonuses, mandatory long-term compensation, a financial transaction (Tobin-) tax, and capital requirements.

## 5.1 An Augmented Taylor Rule

We have already seen that a central bank can punctuate a bubble by increasing interest rates. Let us now analyze the impact of a preannounced interest rate increase in the case of a bubble, following a Taylor rule that takes asset price inflation into account. Specifically, assume a version of the rule used in Bernanke and Gertler (2001),

$$r_t = \bar{r} + \psi_{\pi}\left(\pi_t - \bar{\pi}\right) + \psi\left(p_t/p_{t-1} - 1\right), \tag{17}$$

where $\pi_t$ is gross consumer price index (CPI) inflation, and $p_t/p_{t-1}$ asset price inflation of the only asset in the economy as defined above. For now we neglect the influences

of asset price inflation on CPI inflation by setting CPI inflation equal to its target rate $\bar{\pi}$. The target rate of asset price inflation is assumed to be one. As in the above analysis, in a bubble, $p_{t+1}/p_t$ converges towards a constant $\phi$. Inserting (17) in equilibrium into (12) yields

$$\phi^{\gamma}\left(1+\bar{r}+\psi\left(\phi-1\right)-\beta\right)=q\left(\phi-\beta\right). \tag{18}$$

as a necessary condition for a bubble to emerge. Like for (13), we can derive a condition for parameters $\bar{r}$, $\psi$, $\beta$, $\gamma$ and $q$, determining whether (18) has a solution for $\phi$. Unfortunately, the condition is algebraically complex. An equilibrium exists if and only if

$$\begin{aligned} q(\phi-\beta) \geq \phi^{\gamma}\left(1+\bar{r}+\psi\left(\phi-1\right)-\beta\right) \text{ with} \\ \phi = \frac{1}{2\,\psi\,\gamma}\Big(1-\beta-\gamma)(1-\psi)+\bar{r}+\beta\gamma-\bar{r}\gamma+\beta\gamma\psi \\ +\sqrt{\big(\bar{r}+(1-\beta)(1-\psi)\big)\big((1-\gamma)^2(1+\bar{r}-\psi)-\beta((1+\gamma^2)(1-\psi)-2\gamma(1+\psi))\big)}\Big). \end{aligned}$$

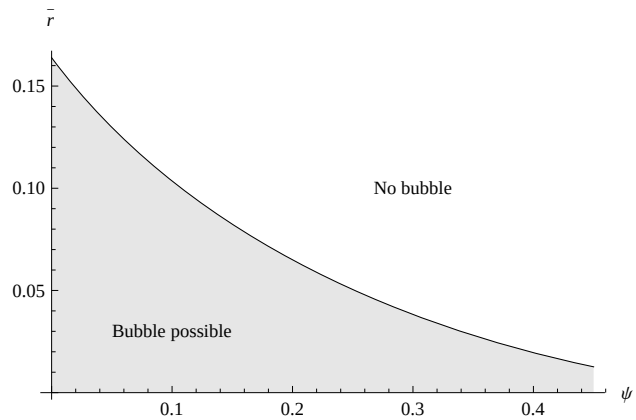
The following figure 8 shows parameters $\bar{r}$ and $\psi$ for which bubbles can exist, for $\gamma = 2$, $\beta = 0.9$ and $r = 10\%$. The figure shows that, in order to prohibit the emergence of bubbles, a regulator (central bank) can either raise the interest rate $r$, or threaten to raise interest rates in the future if a bubble should occur by committing to a Taylor rule with positive $\psi$. If the central bank opts for the Taylor rule, it never actually needs to raise interest rates: interest rates increases occur only as a consequence of asset price movements, but because of the credible announcement of this policy (with a sufficiently large $\psi$), asset prices do not rise and bubbles are prevented. This argument shows that an augmented Taylor rule can cause less distortions than direct interest policies. However, if the central bank cannot differentiate between price movements due to bubbles and changes in the underlying fundamentals (such as the probability of bankruptcy $1-q$), it faces a trade off between preventing bubbles and the risk of unnecessarily moving the interest rate in times without bubbles. A thorough examination of this trade off would require a fully specified DSGE model, which is beyond the scope of this paper.

**Remark 3** *Monetary policy that systematically reacts to asset price increases can prevent bubbles.*

## 5.2 Caps on Bonuses

The bonus payment to a manager is $B = \alpha\left((p_{t+1}+d)/p_t-\beta\right)$ if the underlying asset continues to pay off (probability $q$) and, if there is a bubble, it does not burst (probability $1-Q$). Absent a bubble, this bonus payment is a constant. In a bubble, it equals $\alpha\left(\phi_t + d/p_t-\beta\right)$. Let us first ask whether a potential cap on this bonus would bind in the early

Figure 8: Effects of the Taylor Rule




life of a bubble, hence potentially deterring a bubble from emerging in the first place, or whether it would bind in the later stadium of a bubble. In the latter case, the bubble would have to bust with probability 1 at some date $\bar{t}$, so a backward induction argument would show that the bubble could not have existed in the first place.

In the term for the bonus payment, $\phi_t$ increases over time, but $d/p_t$ decreases. In the aggregate, due to (11a), we have

$$B_t = \alpha \left(\phi_t + d/p_t - \beta\right) = \alpha\, \phi_t^{\gamma}\, (1 + r - \beta)/q.$$

Hence, bonuses increase over time in a bubble, and caps on bonus payments would become binding in later stages of a bubble. As a consequence, we can concentrate on large prices $p_t$, so that $\phi_t$ becomes a constant, and the maximum bonus is

$$B = \alpha \left(\phi - \beta\right) = \alpha\, \phi^{\gamma}\, (1 + r - \beta)/q.$$

Now assume that the regulator puts a cap $\bar{B}$ on the bonus.

There are two ways the regulation can be implemented. *First*, the compensation scheme could be adjusted such that bonuses above $\bar{B}$ are less likely to occur, for example by reducing $\alpha$ or increasing $\beta$. However, $\alpha$ does not have an effect on the existence of bubbles, and an increase in $\beta$ would forward the emergence of bubbles. Hence, this policy would backfire and make bubbles more likely.

*Second*, one could adjust the compensation to $\min\{\max\{\alpha \left((p_{t+1} + d)/p_t - \beta\right) 0\};\, \bar{B}\}$. Then, the bubble will burst with certainty at some point of its life if $\alpha \left(\phi - \beta\right) > \bar{B}$, hence if $\phi > \bar{B}/\alpha + \beta$. Consequently, for a given compensation scheme with parameters $\alpha$ and $\beta$, a cap on bonus payments $\bar{B}$ will punctuate a bubble if $\bar{B}/\alpha + \beta < \phi$, with $\phi$ defined by (12).

The implicit function theorem shows how $\phi$ depends on other exogenous parameters. For example, $d\phi/dr > 0$. To see this, define the term $T = \phi^{\gamma}(1 + r - \beta) - q(\phi - \beta)$, which is zero due to the implicit equation (12) for $\phi$. The derivative $\partial T/\partial r$ is positive, the derivative $\partial T/\partial \phi$ must be negative if we concentrate on the most moderate price path. Consequently, $d\phi/dr > 0$. This proves the following remark.

**Remark 4** *Increasing interest rates and caps on bonus payments are substitutional regulatory instruments.*

Along the same line, $\partial T/\partial q < 0$, hence $d\phi/dq < 0$. A larger $q$ can be identified with more conservative investments. For example, if the assets were securitized mortgages, then a high $q$ would stand for the prime market, and a lower $q$ would represent the subprime market. Then a cap on bonus payments would be more likely to be effective on the subprime segment. More generally, the following result would hold.

**Remark 5** *Caps on bonus payments are less effective in deterring bubbles in conservative fields of investment.*

## 5.3 Other Measures

**Long-term Compensation.** In the recent political discussion, it has often been argued that managers' incentives should be made more sustainable, such that managers concentrate more on long-term goals and avoid short-termism. The same argument might be true for the fonds managers in our model. To analyze this question, let us assume that the manager receives $\max\{0;\, \alpha(y - \beta)\}$ as before, but that she is liable with her compensation for potential future losses. Hence, she will get nothing if the yield is negative in the next period. In a steady state, the market price will then be

$$\alpha(1 + r - \beta) = q^2\, \alpha\big((p_t + d)/p_t - \beta\big),$$
$$p_t = \bar{p} := \frac{d\, q^2}{(1 - \beta)(1 - q^2) + r},$$

i.e. smaller than without long-term liability. If a bubble exists, the probability that the bubble does not burst after two periods is

$$Q = q^2\, p_t/p_{t+2} = q^2/\phi^{2\gamma}.$$

As a consequence, the one-period price increase $\phi$ is determined by

$$\alpha(1 + r - \beta) = Q\, \alpha\big(\phi - \beta\big) = q^2/\phi^{2\gamma}\, \alpha\big(\phi - \beta\big),$$
$$\phi^{2\gamma}(1 + r - \beta) = q^2\big(\phi - \beta\big).$$

The equation is similar to (12), only that $\gamma$ is substituted by $2\,\gamma$, and $q$ is substituted by $q^2$. Because bubbles exist especially for small $\gamma$ and large $q$ according to proposition 1, we find that long-term liability prevents the existence of bubbles. For an even longer liability period, the effect would be even larger.

**Remark 6** *If fonds managers are liable for future developments with their bonuses, bubbles become less likely.*

**Financial Transaction Tax.** There are different possibilities how to implement a so called Tobin tax. We concentrate on the implementations that affect the incentives of the manager, and therefore alter the probability of the emergence of bubbles. We denote the tax on the safe asset with $t$, while the potentially different tax on the risky asset is $t'$. Under such a tax regime, the no-arbitrage condition (12) changes to

$$\phi^\gamma\,[(1-t)(1+r)-\beta) = q\,[(1-t')\phi-\beta]. \tag{19}$$

The modified condition for the existence of bubbles is then

$$\gamma^\gamma\left(\frac{\beta}{\gamma-1}\right)^{\gamma-1} - \frac{q(1-t')}{(1+r)(1-t)-\beta} \leq 0. \tag{20}$$

The derivative of the above expression with respect to $t'$ is positive, i. e. increasing the tax on transactions of the risky asset can make bubbles impossible. It is important, however, how the tax is implemented. If it is levied on all financial assets, including the safe one, $t$ equals $t'$ and the derivative of the above equation with respect to the tax turns negative. In such a case the possibility of bubbles can be created by the Tobin tax.

**Remark 7** *If the financial transaction tax is levied only on the risky asset, bubbles become less likely. If, however, it is placed on the safe and the risky asset alike, bubbles become more likely.*

**Capital Requirements.** We have already argued that our fonds managers can be many kinds of financial intermediaries, for example banks. In this case, capital regulation would be the most prominent policy tool. Our model suggests that capital requirements, among other things, have the effect of preventing bubbles. The reason is straightforward. If our fonds manager is a bank, then pure equity finance would mean $\beta = 0$ and a rather low $\alpha$, whereas pure debt finance would imply a high $\beta$ and $\alpha = 1$. Hence the more equity capital a bank holds, the lower are $\beta$ and $\alpha$. According to proposition 1, the lower $\beta$ can foreclose the emergence of bubbles.

**Remark 8** *Capital requirements can also prevent bubbles.*

In this subsection, the contract parameters $\alpha$ and $\beta$ were treated as exogenous variables. However, more realistically these variables will be set optimally by the investor, who designs the contract. Therefore, we endogenize the compensation package in the following section.

## 5.4 Welfare

In order to justify any policy measure for the prevention of bubbles, it is necessary to analyze the welfare effect of bubbles. We assume that all agents are risk neutral and have identical utility functions, $u_t^i = c_{t-1}^i + \rho\, c_t^i$ for agent $i$, who is born at date $t-1$ and consumes at date $t$. The discount factor $\rho$ must satisfy $1/\rho \leq 1+r$, otherwise agents would not even have an incentive to invest into the riskfree asset. As a consequence, $c_{t-1}^i = 0$. Taking $\rho$ also as the inter-generational discount factor, we can write

$$E_0 W = E_0 \sum_{t=0}^{\infty} \sum_i \rho^t u_t^i = \sum_{t=0}^{\infty} \rho^t E_0 C_t,$$

where $C_t$ is aggregate expected consumption at date $t$. Payments between managers and investors in the same generations are mere transfers and do not directly enter the welfare function. Now, absent a bubble, the price of the asset is always $\bar{p}$. Hence, the generation that consumes at date $0$ earns $C_0 = \bar{p}$ from selling the asset. Generation 1 pays $\bar{p}$ for the asset. Because there are $N$ investors, each owning 1 dollar, the aggregate endowment of generation 1 is $N$. The investment into the riskfree asset is $N - \bar{p}$, since $\bar{p}$ is already spent on the risky asset. With probability $q$, generation 1 also gets $\bar{p}$ from selling the asset, plus the dividend $d$. Hence, the aggregate expected consumption of generation 1 is

$$E_0 C_1 = q\,(d + \bar{p}) + (N - \bar{p})\,(1+r).$$

Generation 2 buys the asset only with probability $q$; with probability $1-q$ the firm is bankrupt and there is nothing to buy. Hence

$$E_0 C_2 = q^2\,(d + \bar{p}) + (N - q\,\bar{p})\,(1+r).$$

The equations for the following generations are similar. Let us now look at the expected consumption in a bubble. For concreteness, consider the trinomial "example" bubble process of section 3. Generation 0 then gets $C_0' = p_0 > \bar{p}$ from selling the asset. Generation 1 buys the asset at price $p_0$, but expects the price to rise to $p_1$ with probability $Q_0$, to fall to $\bar{p}$ with probability $q - Q_0$, and to fall to 0 with probability $1-q$. Hence,

$$E_0 C_1' = Q_0\,(d + p_1) + (q - Q_0)\,\bar{p} + (N - p_0)\,(1+r),$$

and so on. Now consider welfare differences,

$$\begin{aligned} C_0' - C_0 &= p_0 - \bar{p}, \\ E_0(C_1' - C_1) &= Q_0\,(p_1 - \bar{p}) - (1+r)\,(p_0 - \bar{p}), \\ E_0(C_2' - C_2) &= Q_1\,Q_0\,(p_2 - \bar{p}) - Q_0\,(1+r)\,(p_1 - \bar{p}), \end{aligned}$$

and so forth. Hence the aggregate welfare difference amounts to

$$\begin{aligned} E_0 \Delta W &= (p_0 - \bar{p}) + \sum_{t=1}^{\infty} \rho^t \prod_{t'=0}^{t-2} Q_{t'} \Big( Q_{t-1}\left(p_t - \bar{p}\right) - (1+r)\left(p_{t-1} - \bar{p}\right) \Big) \\ &= \sum_{t=0}^{\infty} \rho^t \left(p_t - \bar{p}\right) \big(1 - \rho\,(1+r)\big) \prod_{t'=0}^{t-1} Q_{t'}, \end{aligned}$$

which is non-positive if $1 + r \geq 1/\rho$. Consequently, the welfare effect of a bubble is always negative, and zero only in the limiting case of $1 + r = 1/\rho$.

Alternatively, one can argue the following way. The payments of the risky asset are not affected if there is a bubble. But in a bubble, at date $t$, the young generation pays a price $p_t$ higher than $\bar{p}$ to the old generation born at date $t - 1$. This is simply a transfer of wealth between generations, with two consequences. Due to the higher price $p_t > \bar{p}$, the young generation invests less into the the safe asset, at an opportunity cost of $(1 + r)\,(p_t - \bar{p})$. The transfer is carried one period forward, hence it is discounted. But because the riskfree rate $1 + r$ exceeds the inverse discount factor $1/\rho$, the aggregate welfare effect is negative. Since bubbles always involve prices above $\bar{p}$, this argument proves the following proposition.

**Proposition 3 (Welfare)** *Assume (13) holds, and $1 + r > 1/\rho$. Then of all equilibria, the steady-state equilibrium is strictly welfare-optimal.*

A social planner would set the price of the risky asset to zero. However, this is not a feasible solution in a decentralized equilibrium.

# 6 Endogenizing the Compensation Scheme

In the above analysis, the parameters of the compensation scheme for the managers, $\alpha$, $\beta$, and $S$, are taken as exogenous. In this section we are going to explore which compensation scheme will emerge endogenously. We assume that the investor is risk averse, while the manager is risk neutral. The remaining setup is as described in the previous section, i. e. an investor delegates the investment decision to a manager, whose actions she cannot

observe. Since there are more managers than investors in the economy, the investor can make take-it-or-leave-it offers to the managers, which maximize the expected profit of the investor. In doing so, she has to consider the manager's participation constraint. Letting $y$ again denote the revenue generated by the manager, the expected profit of the investor is then

$$E_t\Pi = E_t y - E_t \max\left\{0;\, \alpha\,(y-\beta)\right\} - S.$$

We restrict the parameter $\alpha$ to be lower or equal to unity, since in the opposite case a higher $y$ can lead to a lower profit of the investor. Put differently, in the extreme a very high realization of $y$ could lead to bankruptcy of the investor under $\alpha > 1$. The manager will only accept the contract if it fulfills

$$E_t \max\left\{0;\, \alpha\,(y-\beta)\right\} + S \geq A, \tag{21}$$

where $A$ is the outside option of the manger (such as academia). Since there are more managers than investors, the investor will choose $\alpha$, $\beta$, and $S$ such that the manager will be at the limit of his participation constraint. This implies that equation (21) will hold with equality. Inserting this result in the above profit function yields $E_t\Pi = E_t y - A$. Hence, the investor maximizes her profit by reaping the complete surplus of the manager. The relation between $S$, $\alpha$, and $\beta$ can be seen by rewriting (21) as

$$S = A + Q'\,\alpha\,\beta - \alpha\,\phi' \tag{22}$$

with

$$Q' = \int_\beta^\infty f(y)\,dy \quad \text{and} \quad \phi = \int_\beta^\infty y f(y)\,dy,$$

where the probability distribution of $y$ is denoted by $f(y)$. The risk-neutral manager is indifferent between values of $S$, $\alpha$, and $\beta$, as long as this equation is fulfilled. The risk-averse investor, however, has an incentive to minimize the variance of her profits in the different states of the economy. To this end, let us rewrite the expected utility of the investor as

$$E_t U(\Pi) = \int_0^\beta U(y-S) f(y) dy + \int_\beta^\infty U\left(y\left[1-\alpha\right] + \alpha\beta - S\right) f(y) dy.$$

The investor maximizes this expression subject to (22), $S \geq 0$, and $\alpha \leq 1$. Because of her risk aversion, she tries to increase the profit in states with a low realization of $y$, relative to states with a high $y$. Therefore she chooses $\alpha = 1$, $S = 0$, and resulting from equation (22)

$$\beta = \frac{\phi' - A}{Q'}.$$

The right-hand side decreases, starting from a large number, for $\beta = 0$ to minus infinity for $\beta$ approaching infinity. Hence, a fixed point can be found. We cannot determine the

probability of a bubble, so let us take the extreme of treating it as a zero-probability event. In this case, $Q' = Q$ and $\phi' = q\,(1+r)$. Equation (22) changes to

$$\beta = \frac{(1+r)q - A}{q} < 1 + r.$$

Hence, we get the following remark.

**Remark 9** *A risk-averse investor chooses a loan contract with* $\beta < 1 + r$.

Importantly, this condition neither contradicts (13) nor (14). Consequently, with endogenous compensation contracts, proposition 2 still applies.

# 7 Conclusion

In this paper, there are two reasons why the price of an asset may deviate from its fundamental value. *First*, as also analyzed by Allen and Gale (2000), funds managers may drive up the price of risky assets due to limited liability. This effect is larger for riskier assets. *Second*, a funds manager may be willing to spend more than the fundamental value on an asset because she expects to earn even more when she sells the asset. Such an increasing bubble is more likely to emerge if the underlying asset is rather safe.

Our theory of bubbles is in line with some anecdotal evidence. During the dot-com bubble (1998–2001), phantasies about the potential of internet firms were exuberant. Possibly, the asset prices of these firms were even more exaggerated due to the limited liability of traders. Hence, the traders' limited liability let the exuberance appear as through a magnifying glass. When expectations became more realistic, assets prices collapsed because the correction of expectations was again magnified. This complete argument follows the *first* explanation, hence it is especially reasonable for risky assets, like the stock of dot-com firms.

Following the "as-long-as-the-music-is-playing-you've-got-to-get-up-and-dance" explanation for the recent U. S. housing bubble, managers bought securities because they thought they could sell them at a higher price later, driving up prices. This argument follows the *second* explanation, hence it is especially reasonable for fundamentally safe assets, like real estate. Our model can make some proposals how to avoid such bubbles. One can increase interest rates, implement a Taylor rule that reacts to asset-price developments, cap bonus payments to fonds managers (if this is done the right way), or introduce capital requirements for managers (intermediaries). Due to its relative simplicity, the model lends itself to further discussions. For example, one could consider several assets, and discuss whether a the collapse of a bubble in one market can be contagious for the other markets. One could plug bubbles into macro models and look at growth effects. Especially after the recent burst of the housing bubble, the number of possible applications seems vast.

# Appendix

**Proof of remark 1.** To see this, assume that $d$ rises and $q$ falls such that the fundamental value $\underline{p}$ remains unchanged, hence $d = \underline{p}\,(1 + r - q)/q$. The steady state price $\bar{p}$ is then

$$\bar{p} = \underline{p}\,\frac{1 + r - q}{1 + r - q - \beta\,(1 - q)},$$

which depends negatively on $q$. This implies that, for given fundamental value $\underline{p}$, the steady state price $\bar{p}$ will be higher for more risky assets. ■

**Proof of proposition 1.** We have already argued that the probability that a bubble bursts increases with $t$, or with $p_t$. But, because $p_t$ is an increasing function, a bubble is sustainable if and only if it is sustainable for $p_t \to \infty$. Hence, if (12) has a solution for $\phi$, the bubble is sustainable. Now consider the limiting case, in which the line $q\,(\phi - \beta)$ and the curve $\phi^\gamma\,(1 + r - \beta)$ will only just touch. At the touching point, the slopes must be equal,

$$(1 + r - \beta)\,\gamma\,\phi^{\gamma-1} = q,$$

which implies that the touching point is $\phi = \beta\,\gamma/(\gamma - 1)$. This $\phi$ must exceed one, otherwise prices would have to fall in the bubble; consequently $\beta > (\gamma - 1)/\gamma$, thus condition (14). Substituting the above solution into (12), we find that the limiting case is reached at

$$\Big(\frac{\beta\,\gamma}{\gamma - 1}\Big)^\gamma\,(1 + r - \beta) = q\,\Big(\frac{\beta\,\gamma}{\gamma - 1} - \beta\Big).$$

Some algebra yields (13). This condition is satisfied iff

$$(1 + r - \beta)\,\gamma^\gamma\,\Big(\frac{\beta}{\gamma - 1}\Big)^{\gamma-1} - q \le 0.$$

The derivative of this term with respect to $q$ is negative, hence the condition is more likely to be satisfied for large $q$. The derivative with respect to $r$ is positive, hence (13) holds rather for small $r$. The derivative with respect to $\gamma$ is

$$(1 + r - \beta)\,\gamma^\gamma\,\Big(\frac{\beta}{\gamma - 1}\Big)^{\gamma-1}\,\log\frac{\beta\,\gamma}{\gamma - 1}.$$

Now remember that the touching point is $\phi = \beta\,\gamma/(\gamma - 1)$. The above logarithm is therefore positive, and the complete derivative with respect to $\gamma$ is positive. A larger $\gamma$ makes bubbles less likely. Finally, the derivative with respect to $\beta$ is

$$\gamma^\gamma\,\Big(\frac{\beta}{\gamma - 1}\Big)^{\gamma-1}\,\frac{(1 + r)\,(\gamma - 1) - \beta\,\gamma}{\beta}.$$

Again, at the touching point $\phi$ must exceed 1, hence $\beta \ge (\gamma - 1)/\gamma$. For the limiting $\beta = (\gamma - 1)/\gamma$, the numerator of the above fraction becomes $(1 + r)\,(\gamma - 1) - \beta\,\gamma = -r\,(\gamma - 1) < 0$. Hence for any $\beta$ larger than the limiting $(\gamma - 1)/\gamma$, the numerator must be negative. Thus the whole derivative is negative, and a larger $\beta$ makes bubbles more likely. ■

**Proof of proposition 2.** Assume that a price process exhibits a bubble, and that $p_t > \bar{p}$ at a date $t$, and $\tilde{p}_{t+1}$ is distributed with distribution $F(\tilde{p}_{t+1})$. Then, in a rational expectations equilibrium,

$$\begin{aligned} \alpha\,(1+r-\beta)+S &= \int_0^\infty Q_t\,\alpha\,\max\left\{\frac{\tilde{p}_{t+1}+d}{p_t}-\beta;0\right\}dF(\tilde{p}_{t+1})+S, \\ \frac{1+r-\beta}{q} &= \int_0^\infty h(\tilde{p}_{t+1})\,dF(\tilde{p}_{t+1}),\ \text{where} \\ h(\tilde{p}_{t+1}) &= \max\left\{\frac{p_t^\gamma}{\tilde{p}_{t+1}^\gamma}\left(\frac{\tilde{p}_{t+1}+d}{p_t}-\beta\right);0\right\} \end{aligned} \tag{23}$$

is an auxiliary function. The $p_{t+1}$ implicitly defined by (10a) solves this equation for a distribution that has probability mass only at one point $p_{t+1}$ (and zero and $\bar{p}$). The question is, from this three-point distribution, can we shift probability mass to other prices, such that the above (23) still holds? The answer depends on the shape of $h(\tilde{p}_{t+1})$. Some straightforward analysis shows that $h(\tilde{p}_{t+1})$ is zero up to $\tilde{p}_{t+1} = \beta\,p_t - d$, then increases and decreases again. For $\tilde{p}_{t+1} \to \infty$, it again approaches zero asymptotically. The maximum of the integral is reached if all probability mass is located at

$$\tilde{p}_{t+1}^* = \gamma\,\frac{\beta\,p_t - d}{\gamma - 1} > \beta\,p_t - d.$$

Hence, a trinomial process with the possible events $p_{t+1}^*$, $\bar{p}$, and $0$ maximizes the right-hand side of (23). Shifting probability mass to other parts of $h(\tilde{p}_{t+1})$ reduces the value of the integral. Note that no bubble can emerge if the left-hand side of (23) is larger than the right side for any price path. We can therefore conclude that, if no trinomial bubble process exists, no other bubble process can exist either. On the other hand, if a trinomial bubble process exists, it is an example for a general bubble process. As a consequence, (13) is the general condition for the existence of bubble processes in rational expectations equilibrium. ■

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