Seller Reputation and Trust in Pre-Trade Communication*

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Abstract. *It is shown that if there is adverse selection on seller's ability in experience goods market, credible communication can be sustained by reputation motives in spite of the inherent conflict of interests between sellers and buyers. Reputation motives are explained as a consequence of the interplay between the market's perception on a seller's ability to deliver quality and the level of trust it places on the information he provides. In our model, honest communication is less costly for high ability sellers and as a result they signal themselves by communicating in a more trustworthy manner. This model is applied to examine the extent to which consumer rating systems may discipline sellers in honestly informing buyers about the quality of their product. The price dynamics and the information acquired by the market on sellers differ substantially with pre-trade communication and without. We then show that the mechanism is qualitatively robust to the possibility that sellers may restart as new traders by obtaining new identities.* (*JEL* Codes: C73, D82, D83, L14)

Keywords: cheap talk, consumer rating system, reputation, trust.

1 Introduction

In online markets buyers cannot physically inspect the items for sale but only rely on the descriptions provided by the sellers on the quality of both the item itself and delivery service. Since payment will have been already made when the buyer learns the quality (upon delivery), effective functioning of online markets hinges critically on there being a mechanism that would warrant a sufficient level of trust amongst traders in an ocean of strangers.

In this paper we highlight a new reputation mechanism that capitalizes on an endogenous link between the market's confidence in pre-trade communication

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(trust) and its belief on the seller's ability to deliver high quality goods (reputation). It points to a virtuous cycle by which the interaction between the two perceptions helps both short-term communication (on the sale item's quality) and long-term learning process (on the seller's true ability).

This reputation mechanism has a particular relevance for the role of feedback systems widely adopted by online markets, such as eBay, that allow buyers and sellers to leave publicly available comments about their counterparts. However, the principal insight has general applicability for experience good markets.

Traditionally, word-of-mouth reputation systems have played an important role in experience goods market when legal systems of contract enforcement were absent or do not work effectively. What is attempted via feedback systems amounts to engineering this old wisdom in a new environment with certain special issues.[1] In a recent survey paper, Bajari and Hortacsu (2004) convey that existing empirical studies appear to point to existence of a correlation between the "feedback score" of an online seller and the average quality delivered by that seller,[2] yet the primary causes for the observed correlation remain inconclusive.

Our approach stems from the observation that there is a clear, albeit subtle, distinction between information contained in the feedback scores of a seller and that reflected in the prices he fetches: The former is the level of "trust" the market places on the soft information provided by the seller prior to transaction concerning the quality, suitability, etc., of the particular item for trade, while the latter reflects the market's belief on the item's quality expected from him. Thus, the role of feedback system is "to promote honest trade rather than to distinguish sellers who sell high quality products from those that sell low quality products" as pointed out by Dellarocas (2005).

Therefore, the observed correlation between the market's trust on a seller's honesty (feedback score) and its belief on his ability to deliver quality (price) is an indirect link that is yet to be accounted for. As far as we are aware, this paper is one of the first theoretical treatments that map out the endogenous link between these two separate dimensions of "reputation", which we believe exists in marketplaces more generally, i.e., beyond online markets.

Specifically, we analyze a model in which a seller randomly draws an item of either good or bad quality in each period and announces this quality as cheap

[1] See Dellarocas (2003 and 2005) and references therein for a discussion of reputation issues on Internet, and Bar-Isaac and Tadelis (2006) for a survey of the literature on seller reputation.

[2] According to Bajari and Hortacsu (2004), a presence of 5-12% price premium for an "established" seller (with hundreds or thousands of mostly positively feedback scores) relative to a seller with no track record appears to be the most robust findings reported, e.g., by Melnik and Alm (2002), Livingston (2002), Kalyanam and McIntyre (2001), and Resnick et al. (2003). This positive effect of seller reputation on price is confirmed when more detailed data are used, by Houser and Wooders (2006) who control heterogeneous item descriptions and by Canals-Cerda (2008) who analyze panel (as opposed to cross-sectional) data.

talk. Each seller is of one of two private types, high or low ability: a high-type seller draws a good quality item more frequently. Each item is traded at a price that is equal to the expected quality based on the market's belief on the seller's true ability, termed his "reputation," and his announcement. The buyer learns the true quality and publicly reveals the truthfulness of the seller's announcement (feedback system), which updates the seller's reputation level accordingly. Here, we postulate fully reliable feedback comments because we are interested in the extent to which feedback mechanisms help elicit trustworthy pre-trade communication from sellers.

Seller's announcements being cheap talk, there always exists a "babbling" equilibrium in which all announcements by sellers are ignored. This is the benchmark equilibrium in which feedback comments induce only pure learning through simple observation of past quality, but have no strategic effect on eliciting honest pre-trade communication. In this equilibrium, price fluctuates following a martingale that reflect the evolution of the belief on the seller's ability but not the true quality of the item for sale.

As the polar benchmark, we focus on equilibria in which high-type sellers always announce truthfully, and we establish that there is a unique equilibrium of this kind, which we refer to as the "communication equilibrium." In this equilibrium, each and every truthful announcement increases the seller's reputation, raising the price he receives in the next period if he claims his item to be of a good quality. Low-type sellers falsely claim bad quality items to be good with a positive probability for short-term gain, because the cost of continuing with truthful announcements is larger for them since they anticipate more "bad news" that they will have to disclose honestly. The probability of lying by a low-type seller is a continuous but non-monotonic function of the prevailing reputation level, reaching 1 above a certain threshold reputation level.

We also show that low-type sellers lie less frequently as they become more patient for all levels of reputation. This has two polar effects on information transmission. Firstly, it enhances the informational content of the announcements on the item's quality, so that the prices reflect the true quality of the good more accurately. Secondly, it reduces the informational content on the seller's type and as a result, the market learns the seller's type more slowly.

Note that pre-trade communication would be useless without the feedback system in our model (hence, no information gets transmitted) due to the inherent conflict of interests between sellers and buyers. Thus, we delineate two ways in which a feedback system can improve information transmission. One is pure learning on the seller's true ability from observation of realized qualities. The other, which is absent in the babbling equilibrium, is differing degrees of truthful announcements between the two types, which is sustainable in equilibrium because the value of retaining reputation differs between types. It is via this second effect that feedback systems foster trust in pre-trade communication and thus establish the aforementioned link between trustworthiness and reputation on ability. High-ability sellers are thus more reliable in disclosing nonverifiable information to buyers honestly. Consequently, buyers learn the seller's ability more quickly in the communication equilibrium and more information is incorporated in the price of the item.

Observe that the second effect evaporates if the two types behaved in the same way in pre-trade communication, because then announcements would have no impact on the market's update of the seller's reputation and consequently, either type would pursue short-term gain by lying. This proclaims a key characteristic of our model that pre-trade announcements cannot carry information on the item's quality unless it also carries information on the seller's ability. In particular, if the seller's ability is publicly known in our model, repeated interactions would not suffice for any credible communication to take place, however patient the seller may be. It may also be worth noting that adverse selection on an arbitrarily small difference in ability is enough to actuate credible communication via reputation mechanism if the seller is sufficiently patient.

A well-known problem with feedback systems is that a trader may start afresh by obtaining a new identity after "milking" his/her reputation (in the same or another marketplace).[3] To examine the extent to which such a possibility may undermine online reputation mechanism, we extend the model to allow for entry and exit of sellers. We characterize stationary equilibrium when sellers may opt out at any time for a fixed exit value, as well as when they can freely restart with a clean record. We find that the reputation mechanism is robust to these possibilities in the sense that stationary equilibrium exists that exhibits all the aforementioned features of the communication equilibrium. However, we also confirm the insight that reputation effects are weakened: Allowing restarts increases cheating incentives by limiting the damage from abusing reputation and as a result, low-type sellers lie more frequently across all reputation levels than when fresh restarts are infeasible. In addition, restarters slow down reputation building process by "contaminating" the pool of new-comers.

Bar-Isaac (2003) studies a model similar to ours but without the scope of pre-trade communication because even the seller does not observe the quality. In a context where the seller incurs a cost to trade in each period, he shows that the seller's decision to continue to trade is a signal that facilitates learning process and thus, enhances the seller's reputation. In contrast to our model, though, lacking means to cash in their reputation by duping buyers, both types of sellers benefit in the same way from high reputation and thus, do not separate

[3]Potential ways to make restarting with a clean record more difficult and costly have been discussed by some authors, such as Friedman and Resnick (2001), who term the issue as "cheap pseudonyms," and Dellarocas (2005). However, it is not yet known how damaging cheap pseudonyms may be to the functioning of feedback systems.

when reputation is above a certain threshold.

Mailath and Samuelson (2001) analyze a moral hazard version of our model without pre-trade communication and show, *inter alia,* that for reputation effects to arise the seller's type needs to be subject to continual random changes. Our results extend to the case of moral hazard (see Section 6) and thus, suggest that pre-trade communication may restore reputation effects in their model even if the seller's type is fixed through time.

At a general theoretical level, the current paper builds on the reputation literature initiated by Kreps and Wilson (1982) and Milgrom and Roberts (1982), and further developed by Diamond (1989), Fudenberg and Levine (1989), Mailath and Samuelson (2001), Ely and Valimaki (2003), and Cripps, Mailath and Samuelson (2004), among others.[4] In this literature reputation arises either when an agent strategically mimics a certain behavior so as to be pooled with agents of another type who are committed to that behavior (pooling reputation), or when an agent strategically diverts from a certain behavior so as not to be mistaken for another type who is committed to that behavior (separating reputation) as in Mailath and Samuelson (2001).

In our model, reputation arises from the presence of adverse selection on seller's ability in delivering quality, without the existence of a type committed to a certain behavior. In particular, both types of sellers are strategic and have conflicting interests with the buyer; however, the more able type always behaves trustworthily in equilibrium, while the less able type finds doing so worthwhile when his reputation is low but too costly when it is above a certain threshold. As such, the motives for both the pooling and separating reputation coexist in our model, the former (latter) motive for sellers of low (high) ability type.

More specifically, our paper contributes to the literature on cheap-talk reputation which, due to the nature of the issue, often concerns experts, advisors and certifiers. Sobel (1985) shows that an "enemy" (an informed agent who has a completely opposing preference to the decision maker) may build a reputation by mimicking the honest reporting of a "friend" (who has a perfectly aligned preference with the decision maker). Generalizing this model to noisy information, Benabou and Laroque (1992) study the reputation of financial experts. In a model where an enemy is biased in one direction, Morris (2001) shows that even a friend may have a reputational incentive to lie. Ottaviani and Sorensen (2001, 2006) study reputational cheap talk in a different model where experts are motivated by exogenous payoff that increases in their perceived ability, akin to the career concerns literature.[5] Mathis, McAndrews and Rochet (2009) examine the extent to which reputation concerns discipline rating agencies.[6]

[4]Mailath and Samuelson (2006) provide an extensive review of the literature.

[5]Although the mechanism differs for experts, they also find that often adverse selection on experts' quality enhances meaningful communication.

[6]These papers use the adverse selection approach to reputation. Papers (on cheap-talk

Our model is akin to Sobel (1985) and Morris (2001) in that the model does not assume an inherently honest type. In contrast to these papers, however, a "friendly" seller cannot exist in our model because sellers have the same monotonic preferences over prices, i.e., both types of sellers are "enemies." As a consequence, knowing the seller's type would exacerbate the communication problem for all types in our context, whilst it would solve the problem for the friendly type in theirs. In this respect, our contribution differs substantially from theirs.

Finally, the current paper also makes a methodological innovation in establishing existence and uniqueness of the equilibrium when Blackwell's condition for a contraction mapping does not apply, as detailed in Section 4.2.

The next section describes the main model and defines equilibrium. Section 3 presents some preliminary results. Section 4 analyzes the reputation mechanism and characterizes the unique communication equilibrium in which the high type sellers always trade truthfully. Section 5 analyzes an extended model in which sellers may opt out or restart with a new identity. Section 6 discusses some further extensions. Appendix contains some technical details.

2 Model

We consider a marketplace (or website) on which sellers interact with a large set of buyers in infinite periods $t = 1, 2, \cdots$. Initially, we focus on a single cohort of sellers all entering the marketplace at date $t = 1$ and stay there forever. We analyze this baseline model until Section 5 where we introduce the options for sellers to opt out for a fixed exit value or to freely restart with a clean record.

Sellers differ in abilities and a representative seller is either of a high type ($\theta = h$) or a low type ($\theta = \ell$) where $0 < \ell < h < 1$. The seller's type θ is private information. The seller's perceived ability in each period t is captured by his *reputation* $\mu_t \in [0, 1]$, the common belief that the prospective buyers attach to the seller being of a high type at the beginning of that period.

In each period t, a seller with reputation μ_t draws one item for sale of a random quality q_t which is good (g) with probability θ and bad (b) with probability $1-\theta$ where $\theta \in \{h, \ell\}$ is the seller's type. Observing the quality of the item, the seller publicly makes a cheap talk announcement $m_t \in \{G, B\}$ about its quality, where $m_t = G$ (B) is interpreted as announcing the quality to be g (b).[7] We say

reputation) also exist that use the so-called "bootstrap" approach based on the folk theorem argument, e.g., Park (2005) on cheap talk reputation of differentiated experts and McLennan and Park (2007) on auditor reputation.

[7]Alternatively, we may model that each seller posts a price p at which buyers either buy or not, and the purchaser of the item reports whether satisfied ($q \geq p$) or not ($q < p$). Our equilibrium continues to be an equilibrium in this alternative model. We don't consider the possibility that the seller announces his type θ, although we conjecture that this would not change our results.

that the agent *lies* if he announces B when $q_t = g$ or G when $q_t = b$, and *tells the truth* if he announces G when $q_t = g$ or B when $q_t = b$.[8]

Buyers value an item at its quality. So, we normalize as $g = 1$ and $b = 0$. The prospective buyers are myopic and try to maximize the expected quality minus the price paid. We assume a competitive demand side so that each item is traded at a price that is equal to the expected quality.[9] In particular sellers are not competing either because their goods are non-rival or because there are more buyers than sellers. (For instance, sellers run an auction between multiple buyers with the same valuation for the good and common beliefs.) At the end of the trading period, the purchaser observes the true quality q_t and honestly reports it publicly.[10] The seller's reputation is revised from μ_t to μ_{t+1} based on m_t and q_t, and the period $t + 1$ starts. The seller's objective is to maximize the discounted sum of its revenue stream with discount factor $\delta \in (0, 1)$. At any date t, the full history of messages and items' quality of the seller is publicly known. The structure of this game, denoted by Γ, is common knowledge.

Our equilibrium concept is Markov perfect equilibrium with state variable μ_t, i.e., we focus on Perfect Bayesian equilibria such that the equilibrium strategies in each period depend only on the seller's reputation level of that period. Thus, the seller's equilibrium strategy specifies the probability that a seller "lies" as a function of his type θ, the reputation level μ and the quality q of the item drawn.

Given a seller's strategy, a "price profile" $p^*_m(\mu)$ is defined as the posterior probability that the item is of a good quality ($q = g$) when the seller with a reputation level μ announced $m \in \{G, B\}$. Being the expected quality, $p^*_m(\mu)$ is also the price at which the item will be traded.

A "transition rule" is a function $\pi^*_{mq}(\mu)$ that specifies the posterior probability that $\theta = h$ in the next period, when in the current period the reputation is μ and the seller sells an item of quality q and announces m.

In equilibrium, the seller's strategy is optimal given the price profile and the transition rule, where the price profile and the transition rule are obtained by Bayes rule from the seller's strategy whenever possible.

Before turning to the characterization of the equilibria with adverse selection and cheap-talk communication, we discuss a few properties of our model.

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[8] Of course the labelling of the messages is somewhat arbitrary, but it will be unambiguous when we introduce the communication equilibrium.

[9] This is in line with Mailath and Samuelson (2001) and Bar-Isaac (2003).

[10] We assumed that the quality, although observable by the buyer, cannot be verified ex-post, so that no warranty contract is feasible.

3 Preliminary Considerations

The term "reputation" in the economic literature encompasses several notions, two of which are present in our model. First, reputation may refer to the beliefs concerning the average quality provided by the seller to the market. In our model this corresponds to the beliefs μ_t on the seller's type θ. Second, the notion of reputation may refer to the level of confidence that consumers have on the truthfulness of the announcement of the seller concerning the quality of the good. This notion thus refers more to trust than to beliefs on the type. As shown below, however, the two concepts are closely related.

We use the term "learning" to refer to the fact that the mere observation of the history of quality q_t helps consumers improve their knowledge on the seller's type, in a non-strategic manner.

3.1 The learning equilibrium

Suppose that there is no communication, say because the seller doesn't observe the quality of the good. Then, in every period a seller's item is traded at a price equal to the expected quality

$$p_t = E(q|\mu_t) = \mu_t h + (1 - \mu_t)\ell,$$

and buyers' belief on a seller's type evolves according to the simple Bayes rule:

$$\begin{aligned}
\mu_{t+1} &= \frac{\mu_t h}{\mu_t h + (1-\mu_t)\ell} > \mu_t \quad \textit{if} \;\; q_t = g \\
\mu_{t+1} &= \frac{\mu_t (1-h)}{\mu_t (1-h) + (1-\mu_t)(1-\ell)} < \mu_t \quad \textit{if} \;\; q_t = b.
\end{aligned}$$

Beliefs and prices follow a martingale, so that the price increases or declines depending on whether the quality delivered last period was good or bad.

Notice that this equilibrium remains as an equilibrium, the so-called "babbling equilibrium," in the game Γ described in Section 2. For instance such an equilibrium obtains when the seller always announces G and thus, the message m_t, containing no information content, is ignored. The beliefs and the price evolve as in the learning equilibrium above and since announcement doesn't affect the continuation game, it is trivially optimal for the seller to announce G.

3.2 A single type

Now suppose that there is a single type, say type ℓ. (As we shall see below, this is different from saying that the buyers' beliefs assign probability 1 to $\theta = \ell$.) Due to risk-neutrality and certainty to trade, our model has the feature that there is a zero value for the seller of transmitting information to the buyer in this case. The reason is that the ex-ante payoff is equal to the expected price which always coincides with the expected quality. An implication is that repeated interaction cannot help to foster communication in this set-up, as explained below.

Consider any equilibrium of our game when a seller's type is publicly known to be ℓ. Because the price is the expected value conditional on the information available at date t, the ex-ante expected price must be equal to ℓ. Thus, any equilibrium generates an expected payoff of $\frac{\ell}{1-\delta}$.

In addition, there cannot be any information transmitted through communication. To see this, suppose to the contrary that the message is informative in some period. This would mean that the probability of announcing G when $q = b$ is different from the probability of doing so when $q = g$. Then, the prices would differ for the two messages. But, since the expected payoff from the next period on must be equal to $\frac{\ell}{1-\delta}$ independently of the message to be sent as verified above, the seller would announce with certainty the message that would fetch the highest price, irrespective of q, contradicting the assertion that he would announce differently contingent on q.

Thus, when the type of the seller is common knowledge, the unique equilibrium outcome is the no communication equilibrium outcome.

3.3 On communication in equilibrium

We say that an equilibrium involves communication if there is a positive probability that at some date the message conveys some information. In our model, there are two types of information that can be transmitted: information about the quality of the current item, q, and information about the seller's type θ.

In our set-up, messages convey information about q in some period with prevailing reputation level μ_t, if the two messages, G and B, generate different posterior beliefs on q by Bayes rule. This is the case if $p^*_G(\mu_t) \neq p^*_B(\mu_t)$ and both messages are sent with positive probability in equilibrium. Similarly, messages convey information about θ if the Bayesian updating of the seller's reputation will depend on the message to be sent by the seller as well as on the quality of the item to be reported.

We asserted above that communication about the quality of the item is not possible if there is a single type. This observation extends to the following property when there are multiple types, i.e., in a setting of adverse selection:

Property 1: Messages cannot convey information on the quality of the good unless they convey information on the type of the seller.

To see this, suppose that no information on θ is transmitted by messages in equilibrium. Then, a seller's reputation will be updated as a function only of the history of delivered quality as asserted above and consequently, in any period t the expected future payoff of the seller is independent of the current message to be sent. If $p^*_G(\mu_t) \neq p^*_B(\mu_t)$, therefore, both types of seller must send the same message (the one that fetches a higher price) with probability 1 regardless of q, establishing that the messages be uninformative on q.

Therefore, adverse selection and signalling about the type are necessary ingredients for messages to be a credible signal of quality in our environment.

4 Communication Equilibrium

We now turn to analysis of equilibrium with reputation. Given seller's strategy and the associated price profile and transition rule, we define the value function $V^*_\theta(\mu) : [0,1] \to \mathbb{R}$, as the expected discounted sum of revenue stream of a seller of type θ and reputation μ. As we wish to study the extent to which reputation motives help to induce truthful revelation of the quality of the product, we focus on equilibria with the following properties:

Condition R:

[**R-a**] An h-type seller always tells the truth regardless of q so long as $\mu > 0$;
[**R-b**] The value function $V^*_\theta(\mu)$ is non-decreasing in μ for $\theta = h, \ell$.

The property [R-a] lends the idea that beliefs about the types will generate trust in messages. The intuition behind this property is that building/maintaining reputation through truthful announcement of the quality is less costly for an h-seller because he knows he will have more good draws than an ℓ-seller, hence he should announces the truth with a larger probability.

The property [R-b] states that higher reputation raises a seller's prospects and thus expected profit. Notice that the expected quality of the product, $\mu h + (1-\mu)\ell$, increases with μ. This would be the payoff of a seller in a one-shot game or if $\delta = 0$. The property states that this monotonicity property extends to our dynamic setting, which seems natural.

We refer to an equilibrium satisfying condition 1 as a "communication equilibrium," although we will use the term equilibrium without qualification when there is no ambiguity. As some results are rather technical, we derive them formally in appendix and present them below in a more heuristic manner.

4.1 Announcement strategy and value function of ℓ-type

Notice that since the unconditional expected price at any date is at least ℓ (since the expected quality in the market is equal to some weighted average of h and ℓ), the continuation value is at least $\ell/(1-\delta)$ for a seller with any level of reputation μ. Note that this lower bound, $\ell/(1-\delta)$, would be the equilibrium payoff of an

ℓ-seller if he were to always tell the truth, because then the equilibrium prices would be $p_G^* \equiv 1$ and $p_B^* \equiv 0$. Since a seller would always announce G in this case because it maximizes current payoff without reducing the continuation value, we deduce that it is not possible that an ℓ-seller is always truthful in equilibrium.

But, due to property [R-b], there should be no incentive to misreport good quality as bad, since this would reduce the current price without enhancing next period's reputation. Indeed, an ℓ-seller is truthful whenever $q = g$ as we show in appendix, whence under Condition R, the seller announces the quality truthfully unless his type is low and the quality is bad. Thus, we characterize equilibrium strategy by the probability, denoted $y^*(\mu)$, that an ℓ-seller lies when $q = b$.

We start with an ℓ-seller with extreme reputation. Once a seller's reputation falls to $\mu = 0$, he cannot increase his reputation above 0, because Bayes rule dictates that $\pi_{mq}^*(0) = 0$ for any message that is sent with a positive probability. Therefore, a seller with reputation 0 announces the message that gives the highest price regardless of q, which implies that the seller gets the same equilibrium price, ℓ, regardless of q. This is the case when an ℓ-seller's announcement strategy is independent of q when $\mu = 0$. Since labeling of the messages is inconsequential due to the costless nature of cheap talk messages, we make the convention that an ℓ-seller announces G regardless of q when $\mu = 0$, i.e., $y^*(0) = 1$. This implies that $p_G^*(0) = \ell \geq p_B^*(0)$. An immediate consequence is that the equilibrium value at $\mu = 0$ is $V_\ell^*(0) = \ell/(1-\delta)$.

Consider an ℓ-seller with the other extreme reputation $\mu = 1$. If he were to tell the truth upon drawing $q = b$ with positive probability in equilibrium, his payoff would be $V_\ell^*(1) = \ell + \delta V_\ell^*(1)$, i.e., $V_\ell^*(1) = \ell/(1-\delta) = V_\ell^*(0)$. This would mean that it would be optimal to lie when $q = b$ contrary to the postulated equilibrium strategy. Thus, we deduce that $y^*(1) = 1$. This conclusion is irrespective of the reputation level that lying would take to, $\pi_{Gb}^*(1)$, which is undefined by Bayes rule. Without loss of generality, we take the convention that $\pi_{Gb}^*(1) = 0$ because this ensures continuity of equilibrium variables (V_θ^* and π_{Gb}^*) at $\mu = 1$ without influencing the equilibrium characterization as we show in Appendix (Lemma **??**). Then, $V_\ell^*(1) = 1 + \delta\left(\ell V_\ell^*(0) + (1-\ell)\, V_\ell^*(1)\right)$ so that

$$V_\ell^*(1) = V_\ell^*(0) + \Delta \quad \text{where} \quad \Delta := \frac{1-\ell}{1-\delta\ell} < 1. \tag{1}$$

For $\mu \in [0, 1]$, we define

$$p_G(\mu, y) := \frac{\mu h + (1-\mu)\ell}{\mu h + (1-\mu)(\ell + (1-\ell)y)}, \tag{2}$$

the expected quality of the product claimed as good ($m = G$) by a seller with reputation μ if an ℓ-seller would falsely claim so with a probability $y \in [0, 1]$. Then, the equilibrium prices are given by

$$p_B^*(\mu) = 0 \quad \text{and} \quad p_G^*(\mu) = p_G(\mu, y^*(\mu)).$$

For $y > 0$ and $\mu < 1$, $p_G(\mu, y)$ is strictly increasing in μ and strictly decreasing in y. Thus, the more an ℓ-seller lies, the lower is the short-run benefits from lying.

Consider now the transition rules $\pi^*_{mq}(\mu)$. Let us define

$$\pi_{Bb}(\mu, y) := \frac{\mu(1-h)}{\mu(1-h) + (1-\mu)(1-\ell)(1-y)}. \tag{3}$$

Then, for all $\mu > 0$, Bayesian updating of reputation prescribes

$$\pi^*_{Gg}(\mu) = \frac{\mu h}{\mu h + (1-\mu)\ell} \quad \text{and} \quad \pi^*_{Bb}(\mu) = \pi_{Bb}(\mu, y^*(\mu)). \tag{4}$$

Again, since π_{Bb} is strictly increasing in $\mu < 1$ and $y < 1$, the more an ℓ-seller lies, the higher is the reputational gain from telling the truth.

Given that a good quality is never claimed to be bad, we can set $\pi^*_{Bg}(\mu) = 0$ without loss of generality. Moreover, if $\mu < 1$ then $\pi^*_{Gb}(\mu) = 0$ so long as $y^*(\mu) > 0$, which we show is the case in Appendix. Thus, we set $\pi^*_{Gb}(\mu) = \pi^*_{Bg}(\mu) = 0$ for all $\mu \in [0, 1]$.

The reader should have noticed that a false claim of good quality triggers a downward jump from $\mu = 1$ to $\mu = 0$. Such a fall in reputation, although it needs not be so drastic, is essential for a communication equilibrium. The reason is that for a seller of "maximal reputation" $\mu = 1$ not to lie upon drawing $q = b$, the foregone value associated with lost reputation, $V^*_\theta(1) - V^*_\theta(\pi^*_{Gb}(1))$ must be sufficient. One interpretation is that buyers having classified the seller as high type with certainty based on past records may reconsider their interpretation of the records upon arrival of new evidence inconsistent with this classification.

Thus the communication equilibrium is characterized by the function $y^*(\mu)$ which must be optimal relative to the associated value function and transition rule. The value function V^*_ℓ of ℓ-seller is characterized by

$$\begin{aligned} V^*_\ell(\mu) \;=\; & [\ell + (1-\ell)y^*(\mu)]p_G(\mu, y^*(\mu)) \\ & +\delta(\ell V^*_\ell(\pi^*_{Gg}(\mu)) + (1-\ell)[y^*(\mu)V^*_\ell(0) + (1-y^*(\mu))V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)))]\big) \end{aligned} \tag{5}$$

where

$$y^*(\mu) \in \arg\max_{0 \le y \le 1} y p_G(\mu, y^*(\mu)) + \delta y V^*_\ell(0) + (1-y)\,\delta V^*_\ell(\pi_{Bb}(\mu, y^*(\mu))). \tag{6}$$

Following the recursive approach we characterize V^*_ℓ as a fixed point of a mapping, T, defined on the set $\mathcal{F}$ of all non-decreasing functions $V : [0, 1] \to \mathbb{R}$ such that $V(0) = \ell/(1-\delta)$ and $V(1) = \ell/(1-\delta) + \Delta$.

Observe that $p_G(\mu, y)$ is strictly decreasing in y (except when $\mu = 1$) and $V(\pi_{Bb}(\mu, y))$ increases in y. For any continuous function $V \in \mathcal{F}$, therefore, we define the unique "pseudo-best-response" as a function $y_V : [0,1] \to [0,1]$ by

$$y_V(\mu) = \begin{cases} 0 & \text{if} \quad p_G(\mu,0) < \delta(V(\pi_{Bb}(\mu,0)) - V(0)) \\ 1 & \text{if} \quad p_G(\mu,1) > \delta(V(\pi_{Bb}(\mu,1)) - V(0)) = \delta\Delta \\ y & \text{s.t.} \quad p_G(\mu,y) = \delta(V(\pi_{Bb}(\mu,y)) - V(0)), \text{ otherwise.} \end{cases} \tag{7}$$

We extend the pseudo-best-response to discontinuous functions in Appendix.

The value on the RHS of the inequalities in (7) is the gain from enhanced reputation that accrues to a seller truthfully announcing a bad quality, which is bounded by $\delta\Delta$. Hence, $y_V(\mu)$ is optimal for ℓ-type seller given V, $p_G(\mu, y_V(\mu))$ and $\pi_{Bb}(\mu,_V(\mu))$. Since $p_G(\mu,0) = 1 > \delta\Delta$, it must be the case that $y_V(\mu) > 0$ for all μ. From $p_G(1,1) = 1 > \delta\Delta$ it further follows that $y_V(\mu) = 1$ for all $\mu > \bar{\mu}$ where

$$\bar{\mu} := \inf\{\mu \in [0,1] \mid p_G(\mu,1) > \delta\Delta\} < 1. \tag{8}$$

Note that the threshold $\bar{\mu}$ may be zero, in which case an ℓ-seller lies whenever $q = b$.

Using the fact that $y_V(\mu)>0$ for all μ, we define a mapping $T : \mathcal{F} \to \mathcal{F}$ by

$$T(V)(\mu) := p_G(\mu, y_V(\mu)) + \delta\left(\ell V(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\right). \tag{9}$$

As we elaborate in Appendix, the equilibrium value function V^*_ℓ is a fixed point of the operator T and the equilibrium strategy is $y^* = y_{V^*_\ell}$. We obtain:

Proposition 1 *There exists a unique fixed point V^*_ℓ of T. The value function V^*_ℓ is continuous and strictly increasing on $[0,1]$; and the strategy $y^* = y_{V^*_\ell}$ is continuous on $[0,1]$.*

Existence follows from continuity of the operator T. Astute readers may notice that a fixed point theorem is not directly applicable to T because $\mathcal{F}$ is not compact. Since we can show that any fixed point of T is a continuous (hence, right-continuous) function and T preserves right-continuity, we solve this problem by restricting T on the set of all right-continuous functions in $\mathcal{F}$ before applying the Fan-Glicksberg Fixed Point Theorem.

Uniqueness is obtained separately from existence by using the properties of the fixed point. It stems from the observation that the value function is uniquely determined for $\mu > \bar{\mu}$. Using the fact that any truthful announcement enhances reputation along equilibrium path (Lemma 6), it is then possible to show that there is a unique way to "unravel" the value function by backward induction from the (stochastic) date when reputation jumps above $\bar{\mu}$.

Thus, our result is distinguished from that of Benabou-Laroque (1992): In a model of financial experts who can manipulate the market by distorting information, they obtain existence and uniqueness by applying Blackwell's Theorem.[11]

[11] Morris (2001) and Bar-Isaac (2003) also use Blackwell's Theorem.

This is not applicable in our model because our sellers may benefit by distorting information in only one direction, unlike their model where information holders can benefit by distorting the market in either direction, which generates symmetry in the model. More precisely, T is not non-decreasing in V and hence, Blackwell's condition for a contraction is not applicable. A second key difference is that Benabou and Laroque assume continuity whereas we do not restrict a priory to continuous functions. Mathis, McAndrew and Rochet (2009) exploit an idea akin to ours in obtaining a constructive proof of existence in a model of rating agencies. Their proof relies on the fact that only positive claims generate trade and thus can be verified, which simplifies the analysis greatly. In our model both positive and negative claims can be verified.

4.2 Optimality for h-type and equilibrium

To fully verify equilibrium conditions, it remains to show that it is indeed optimal for the h-seller to announce truthfully, given the strategy y^*, price profile p^*_m, and the transition rule π^*_{mq} identified in the previous section. Let V^*_h be the value function of an h-seller for such an equilibrium.

Note that $\pi^*_{Bb}(0)$ is undefined by Bayes rule. We proceed our analysis by presuming that $\pi^*_{Bb}(0) = 0$, in which case it straightforwardly follows that

$$V^*_h(0) = \frac{\ell}{1-\delta} \quad \text{and} \quad V^*_h(1) = \frac{h}{1-\delta}. \tag{10}$$

It is possible that $\pi^*_{Bb}(0) > 0$, hence $V^*_h(0) > V^*_\ell(0)$ in equilibrium, which would be feasible only if h-seller is presumed to announce $q = b$ truthfully when $\mu = 0$.[12] Since what h-seller would do when $\mu = 0$ is postulation of off-equilibrium behavior anyway, such an equilibrium would not enlarge the set of equilibrium outcomes. In this sense, our presumption that $\pi^*_{Bb}(0) = 0$ is inconsequential.

An h-seller, if followed the same strategy as an ℓ-seller, would obtain a higher expected payoff because he would get a better sequence of draws on average, i.e., $V^*_h(\mu) > V^*_\ell(\mu)$ for $\mu > 0$. So, it is clear that an h-seller announces truthfully whenever $q = g$. In addition, this means that the value of building reputation is higher for h-seller than for ℓ-seller. But, the value of burning the reputation by falsely claiming good quality is the same for both and is equal to $p^*_G(\mu)+\delta V^*_h(0)$. Hence, whenever an ℓ-seller is indifferent between lying or not, which is the case when $\mu \leq \bar{\mu}$ and $q = b$, an h-seller prefers announcing truthfully.

When $\mu > \bar{\mu}$, upon drawing $q = b$, an h-seller gets $\delta V^*_h(1)$ by reporting truthfully and $p^*_G(\mu) + \delta V^*_h(0)$ by reporting untruthfully because $\pi^*_{Bb}(\mu) = 1$ and $\pi^*_{Gb}(\mu) = 0$. Thus, it is optimal for an h-seller to report truthfully if and

[12] Such an equilibrium exists if h is large and the seller is patient enough (see Appendix).

only if $\delta(V_h^*(1) - V_h^*(0)) \geq 1 = \max p_G^*(\mu)$, or equivalently,

$$h - \ell \geq \frac{1-\delta}{\delta} \quad \Longleftrightarrow \quad \delta \geq \delta_h := \frac{1}{h - \ell + 1}. \tag{11}$$

The discussions up to now are summarized in the first main result below.

Theorem 1 *There exists an equilibrium satisfying Condition R if and only if* $\delta \geq \delta_h$*. The equilibrium outcome is unique.*

Proof. See Appendix. ■

4.3 Discussion of the equilibrium

The communication equilibrium exhibits several interesting features which we discuss below (and prove in Appendix). We also highlight the contrast to the learning (babbling) equilibrium in their price and learning dynamics.

First, the value of building reputation is bounded for ℓ-sellers however patient they may be (whereas it approaches infinity for h-sellers as δ tends to 1):

Property 2 The value Δ of good reputation for an ℓ-seller is smaller than the good/bad quality differential $g - b = 1$.

To understand this property, recall that an ℓ-seller strictly prefers to lies when $\mu = 1$ in the communication equilibrium. This is because otherwise, the value at $\mu = 1$ would be equal to that at $\mu = 0$, removing any incentive to be truthful. This means that the short-term gain from lying when $\mu = 1$, which is $g - b$, exceeds the value of lost reputation, $\delta\Delta$. The willingness to pay, Δ, for high reputation of an ℓ-seller known as such is decomposed as follows: moving from $\mu = 0$ to $\mu = 1$ raises the expected price of the first period from $(1-\ell)b + \ell g$ to g, and it yields $\delta\Delta$ in future value when the current quality turns out to be g, which occurs with probability ℓ. Thus, $\Delta = g - (1-\ell)b - \ell g + \ell\delta\Delta < g - b$.

We established that an ℓ-seller lies with certainty upon drawing a bad quality item if $\mu \geq \bar{\mu}$. The idea behind this is that as μ gets large it pays off to cash in sufficiently high reputation. Note from (8) that $\bar{\mu} > 0$ if and only if $\delta\Delta > \ell = p_G(0, 1)$, or equivalently,

$$\delta \; > \; \delta_\ell := \frac{\ell}{1 - \ell + \ell^2}. \tag{12}$$

For $0 < \mu < \bar{\mu}$, we deduced that $y^*(\mu) < 1$ since $y^*(\mu) = 1$ would lead to a contradictory conclusion that short term gain of cheating $p_G(\mu, 1)$ falls short of the foregone value of reputational gain, $\delta\Delta$. However, since $\delta\Delta$ is bounded away from the short-term gain when ℓ-seller is fully honest, there is a uniform upper bound on how honest ℓ-sellers may be regardless of δ as is verified in Appendix:

$$y^*(\mu) > \hat{y} := \frac{h-\ell}{1-\ell} \text{ for all } \mu. \tag{13}$$

Thus, the market never trusts sellers at a level approaching full confidence however patient they may be. In conjunction with (12), we obtain

Property 3 If $\delta \leq \delta_\ell$, then $y^*(\mu) = 1$ for all $\mu \in [0,1]$. If $\delta > \delta_\ell$, then $y^*(\mu)$ is a continuous function assuming values $y^*(\mu) = 1$ at $\mu = 0$ and $\mu \geq \bar{\mu}$ and $y^*(\mu) \in (\hat{y}, 1)$ at $\mu \in (0, \bar{\mu})$.

The diagram below illustrates typical $y^*(\mu)$ for $\delta > \delta_\ell$.

[**Figure 1 about here**]

The result (13) places an upper bound on the market price for items claimed to be good: $p_G^*(\mu) < p_G(\mu, \hat{y}) = \mu + (1-\mu)\frac{\ell}{h}$. However, the aforementioned insight that price and reputation are correlated is verified as proved in Appendix:

Property 4 $p_G^*(\mu)$ is strictly increasing in μ on $\mu < 1$.

This property is at the heart of our equilibrium analysis. Higher reputation results in higher prices which is the driver of the incentives to build reputation through honest communication with buyers. Moreover, as intuition suggests, more patient sellers are more trustworthy at all levels of reputation, stated as the next property (proved in Appendix). Consequently, more patient sellers fetch higher price by announcing a good quality.

Property 5 $\bar{\mu}$ increases in δ; $p_G^*(\mu)$ (resp. $y^*(\mu)$) strictly increases (resp. decreases) in δ for all $\mu \in (0, \bar{\mu})$.

Notice that our model involves a trade-off between the short-term transmission of information on q through honest announcement by sellers and the speed of learning on θ through the observation of past records: When $y^*(\mu)$ decreases, the price $p_G^*(\mu)$ increases but the updated reputation level for honest announcement of a bad quality, $\pi_{Bb}^*(\mu)$, is lower. Indeed, truthful announcements of bad quality carry less information on the seller's type since false announcements are less frequent. Increasing the discount factor fosters credible communication at a cost of slower learning.

In any case, the lower bound on $y^*(\mu)$ in (13) implies a lower bound on the informational content on the seller's type conveyed by equilibrium messages. In particular, trustful behavior by sellers is always "good news" for their reputation.

Property 6 $\pi^*_{Bb}(\mu) > \mu$ all $\mu > 0$.

If $\delta_h \leq \delta \leq \delta_\ell$, therefore, an equilibrium have a simple characterization: ℓ-sellers always lie upon drawing a bad item (and are honest otherwise).[13] Hence, the trading price p_t increases over time until a bad draw occurs, at which point the type is revealed. Then, p_t drops to ℓ for good if $\theta = \ell$; whilst $p_t = q_t$ in all subsequent periods if $\theta = h$, i.e., p_t reflects the true quality.

If $\delta > \max\{\delta_h, \delta_\ell\}$, on the other hand, an ℓ-seller with a reputation below $\bar{\mu}$ randomizes between announcing truthfully and untruthfully upon drawing $q = b$. As long as he tells the truth, he builds reputation and thus benefits from higher future prices for items he will announce to be good.

Thus, in either case the seller's reputation increases over time until one of two events occurs: *i*) the seller falsely announces $m = G$ when $q = b$, in which case his type is revealed to be ℓ and the price drops to ℓ for good; or *ii*) the seller truthfully announces $m = B$ when his reputation is $\mu > \bar{\mu}$, in which case his type is revealed to be h and the price reflects the true quality in the future. In particular, the seller's type gets revealed whenever a bad quality item is drawn once his reputation exceeded $\bar{\mu}$. Since the reputation level goes above μ within a finite number of periods unless it collapses to zero due to false announcement,

Property 7 The seller's type is revealed within a finite time with probability 1.

As such, a salient characteristic of the equilibrium with credible communication is that information on seller's type is revealed much faster than in the case without communication, where convergence occurs only asymptotically. Compared to the case of pure learning in Section 3.1, a key difference concerns updating of reputation following a bad draw ($q = b$). In the learning equilibrium it is given by $\mu_{t+1}(b) = \pi_{Bb}(\mu_t, 0) < \mu_t$. In the communication equilibrium, reputation improves as long as the seller announces truthfully. While a bad quality is always interpreted as "bad news" in the absence of communication, it is perceived as "good news" in our model if truthfully announced and facilitates learning. Since this extra learning effect stems from h-seller's desire to separate through more trustworthy behavior, communication helps to mitigate the asymmetric information problem along two interrelated dimensions:

i) it helps credible communication of the true quality, and

ii) it helps consumers learn the true type of the seller.

The price dynamics also differ substantially. In a communication equilibrium, the price for items announced to be good increases over time roughly in line with the reputation until the seller's type is revealed, while the price stays constant at zero for items announced bad. In contrast, the price in the learning equilibrium

[13]In this case the equilibrium may vary in $V^*_\ell(1)$ and $\pi_{Gb}(1,1)$ subject to $V^*_\ell(1) = 1 + \delta(\ell V^*_\ell(1) + (1-\ell)V^*_\ell(\pi_{Gb}(1,1)))$, but they all generate the same equilibrium outcome.

follows a martingale where the price of date t is independent of the realization of the quality at date t.

5 Outside Option and New-life

Up to now we have assumed that sellers stay in one marketplace forever and that memory is infinite. One of the key issues surrounding the reputation mechanisms based on feedback systems is that sellers may find ways to escape from the bad consequences of damaged reputation. For instance, sellers may change to another marketplace. As emphasized for instance by Friedman and Resnick (2001), even within a given marketplace it may be difficult to keep track of the identity of a seller, in which case a seller may have at any date an option to erase his history by changing his identity and start again as a new-comer.[14]

One may then be concerned that such a possibility may destroy the fundamental reputation mechanism of feedback systems elaborated in Section 3. We show in this section that this is not the case, although the effectiveness of the mechanism in fostering honest communication by the seller is reduced.

Addressing this issue requires delineating additional equilibrium dynamic interactions because the incentive to change identity depends on the market's belief concerning new-comers, and this belief depends on equilibrium strategies.

Thus, we extend the analysis to a full dynamic setting obtained by augmenting our baseline model with a stationary entry and exit of sellers in every period: There is a constant measure 1 of sellers on the platform in each period. Each seller dies with probability $\chi \in (0, 1)$ at the end of each period. These deaths are replaced by measure χ of newborn sellers at the beginning of the next period. Each new born seller is of h-type with probability $\mu_i \in (0, 1)$. We maintain the assumptions that there is a single platform to which each (living) seller brings an item (of either good or bad quality) for sale in each of infinite periods and that the past record of each seller is publicly known. Additionally, at the beginning of each period, each seller has an option of erasing his past history and restart as if a newborn seller. A "new-comer" refers to both a newborn and a restarter.

In stationary equilibria of this model, a unique "initial" value is associated for each new-comer and a seller would restart as a new-comer if the value associated with his current reputation level falls short of the initial value. From the perspective of each seller, therefore, restarting is equivalent to taking an outside option of this value and leaving the platform. In addition, a model in which sellers has an outside option of a fixed value (but cannot restart) is of interest

[14] The ability to do so depends on the technology used by the platforms. This is known to be an issue with eBay for instance (see Delarocas 2006), but would be less of an issue when the platform controls the bank coordinates or the social status of companies, for then it would involve creating a new firm which is costly.

on its own. Therefore, we start with such a model which proves useful.

5.1 Equilibrium with an outside option

Suppose that sellers, instead of being able to restart, may take an outside option of a fixed value $v_o > 0$ and leave the market at the beginning of any period. We assume that this decision is irreversible so that a seller who exists never reenters (this would be the case for instance if there are many platforms that do not allow reentry). Moreover, we assume that the exit value is the same for both types of sellers, although our results extend easily to type-dependent exit values provided that the difference is not too large (see our 2009 working paper).

Note that the analysis is equivalent to that in Section 3 if $v_o \leq \ell/(1-\delta)$ because then sellers never exit the platform. Indeed the equilibrium derived there applies to each seller, starting from the date of arrival when his initial reputation starts at μ_i and period t is interpreted as the age or seniority of the seller (the number of trading periods since joining the platform). Since all sellers exit immediately if $v_o \geq 1/(1-\delta)$ as will be verified below, we assume $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$.

Given that seniority is observed, we can analyze this case by examining the equilibrium for a representative seller who is born at date 1. An equilibrium then describes an entry decision at birth, and announcement strategy and exit decision of the seller conditional on entry. As before we focus on stationary equilibria with state variable μ_t. We denote the value function of type θ by $V_\theta^\dagger$ to distinguish from the previous section. Note that there are some levels of reputation that will never be observed on the platform. Still, we need to derive the value function for all reputation levels so as to understand entry/exit decisions. So, $V_\theta^\dagger(\mu)$ denotes the value for a seller of type θ when his reputation is μ and he is committed to stay at least one period in the platform.

Again we are concerned with the existence and properties of equilibrium that satisfies Condition R. The difference from the previous case is that the seller will exit the market as soon as his reputation falls to low enough a level that the continuation value is below v_o. But, once this is accounted for in the value function, the analysis is similar to the case with no outside option. As detailed in Appendix, the proof of existence and equilibrium characterization extend to this case by replacing $V_\ell^*(0)$ with v_o.

Proposition 2 *For* $v_o \in \left(\frac{\ell}{1-\delta}, \frac{1}{1-\delta}\right)$*, the continuation game starting from entry of a seller admits an equilibrium satisfying Condition R if* $\delta\left(\frac{h}{1-\delta} - v_o\right) > 1$.

Proof. In Appendix. ■

The same argument as before verifies straightforwardly that an ℓ-seller truthfully announces when $q = g$. Given $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, let $y^\dagger(\cdot)$ denote the equilibrium probability of lying by an ℓ-seller upon drawing $q = b$. While the outside option affects the values of equilibrium variables, the nature of equilibrium remain unchanged. The values at boundaries can easily be computed to be

$$V_\ell^\dagger(0) \; = \; \ell + \delta v_o \in \left(\frac{\ell}{1-\delta}, v_o\right) \quad \text{and} \quad V_\ell^\dagger(1) = v_o + \Delta_{v_o} \; > \; V_\ell^*(0) \qquad (14)$$

$$\text{where} \quad \Delta_{v_o} \; := \; \frac{1-(1-\delta)v_o}{(1-\delta\ell)} \; < \; \Delta\,.$$

Thus, the extreme values are higher for ℓ-seller ($V_\ell^\dagger(\mu) > V_\ell^*(\mu)$ for $\mu = 0, 1$) but the value of a good reputation (the difference of the two extreme values) is lower with an outside option than without. Note also that $V_\ell^\dagger(1) < v_o$ if $v_o \geq 1/(1-\delta)$ and thus, all sellers would exit immediately as asserted above.

It still is the case that, unless $y^\dagger \equiv 1$, there exists some critical level of reputation, $\bar{\mu}^\dagger < \bar{\mu}$, below which the ℓ-seller truthfully announces a bad quality with positive probability. Moreover,

Proposition 3 *For* $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,

$$y^\dagger(\mu) > y^*(\mu) \; \forall \mu \in (0, \bar{\mu}) \quad \textit{and} \quad y^\dagger(\mu) = y^*(\mu) = 1 \; \forall \mu \in [\bar{\mu}, 1]. \qquad (15)$$

Proof. In the Appendix. ■

Thus, availability of an option to exit and obtain an outside value larger than the value attached to bad reputation, results in a uniform increase in the probability that a bad item is falsely claimed as good. As a consequence, the price for items announced as good is lower at all levels of reputation. At the same time, note that learning takes place faster than in the equilibrium without such an option for two reasons. First, it is more likely that an ℓ-seller reveals his type by falsely announcing a good quality. Second, reputation gets updated to higher levels following truthful announcements, which also implies that a reputation level is reached sooner at which the seller's type is revealed for sure if a bad item is drawn (because an ℓ-seller would definitely lie).

As already pointed out, our model realizes a balance between the short-run reliability of communication and the speed of revelation of the type of the seller. The outside option shifts this balance toward faster separation of the type.

There is a critical level $\mu_o^\dagger$ defined by $V_\ell^\dagger(\mu_o^\dagger) = v_o$ below which an ℓ-seller would choose to exit the market. As before, every truthful announcement increases the reputation until there is a lie by an ℓ-seller. We also verify that truthful announcement of a bad quality always results in a reputation level above $\mu_o^\dagger$. Thus, once in the platform a seller will only opt out after a lie on the quality, which characterizes the exit decision.

So far we have set aside the decision of a seller on the day he was born as to whether to enter the platform or to stay out and take the value v_0. It is trivial that he should enter irrespective of his type if born with reputation $\mu_i > \mu_o^\dagger$.

But, if $\mu_i < \mu_o^\dagger$ then both types entering with certainty is not viable since it would be suboptimal for ℓ-type. In this case the two types enter with different probabilities as stated below, and entry decision acts as another signal of type.

Proposition 4 *For $v_o \in \left(\frac{\ell}{1-\delta}, \frac{1}{1-\delta}\right)$, the game admits an equilibrium satisfying Condition R, in which an h-type enters with probability 1 at birth and the initial reputation upon entry is $\mu_1 = \min\{\mu_i, \mu_o^\dagger\}$.*

Proof. This is immediate if $\mu_i \geq \mu_o^\dagger$. For $\mu_i < \mu_o^\dagger$, postulate that an h-seller enters with certainty and an ℓ-seller enters with a probability $\frac{\mu_i(1-\mu_o^\dagger)}{\mu_o^\dagger(1-\mu_i)} \in (0,1)$, so that the initial reputation upon entry is $\mu_o^\dagger$ by Bayes rule. Then, since $V_h^\dagger(\mu_o^\dagger) > V_\ell^\dagger(\mu_o^\dagger)$, the postulated entry strategy is optimal for both types. ■

Thus, when the initial proportion of low type sellers is high, the initial stage induces some screening through self-selection. An immediate implication is that a platform could improve the screening by charging a positive price to join.

5.2 New-life

We now extend the analysis to the case in which there is no outside option, but sellers can freely restart in any period by erasing their history and obtaining a new identity. We assume that buyers cannot distinguish a newborn seller from an old seller restarting afresh, and focus on stationary equilibria satisfying Condition R, that is, the proportion of ℓ-sellers restarting afresh is constant every period, while h-sellers never lie and consequently, never change their identities.

In a stationary equilibrium, a constant mass, denoted by χ_1, of new sellers (newborns and restarters) appear on the platform in each period and they start with a reputation level set at an endogenous default level μ_1 which reflects the mix of genuine newborn sellers (of which a proportion μ_i are of type h) and equilibrium mass of sellers who restart. Let v_1 denote the value of an ℓ-seller starting at the "default reputation level" μ_1. Notice that the value function depends on v_1, while the fraction μ_1 depends on both the value function and v_1.

To determine μ_1 and v_1 endogenously, we start by treating v_1 as a parameter representing an outside option value to be obtained when an ℓ-seller exits the market, so that we can apply the result from the previous section to determine the equilibrium strategy and value function of ℓ-seller that are consistent with v_1, which we denote by $y_{v_1}^\dagger$ and $V_{v_1}^\dagger$, respectively, to emphasize their dependence on v_1.

From Section 5.1 we know that in the equilibrium identified in Proposition 2 (with v_1 playing the role of v_o), an ℓ-seller will change identity only when his reputation has fallen to $\mu_t = 0$ after a false claim. Based on this, we derive in

Appendix the proportion $\Lambda(v_1)$ of ℓ-sellers starting at a given period who will change identity and restart in some future period as

$$\Lambda(v_1) := (1-\ell)(1-\chi)\sum_{k=1}^{\infty}\sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k) y_{v_1}^{\dagger}(\pi(\mathbf{h}^k)). \tag{16}$$

Here, $\Pr(\mathbf{h}^k)$ is ex ante probability that an ℓ-seller remains in the market without having cheated after a k-period quality history $\mathbf{h}^k \in H^k := \{g, b\}^k$, and $\pi(\mathbf{h}^k)$ is posterior reputation for a seller who has survived the history $\mathbf{h}^k$ without cheating, updated according to $y_{v_1}^{\dagger}$ from initial reputation μ_1.

Since the mass of new ℓ-sellers at each date is $\chi_1(1-\mu_1)$, the mass of sellers who change their identities in a stationary equilibrium is $\chi_1(1-\mu_1)\Lambda(v_1)$. Stationarity, therefore, dictates that the mass of new sellers is

$$\chi_1 = \chi + \chi_1(1-\mu_1)\Lambda(v_1). \tag{17}$$

Since only ℓ-sellers restart, Bayes rule dictates that in a stationary state

$$\mu_1 = \frac{\chi\mu_i}{\chi_1}. \tag{18}$$

Solving (17) and (18) simultaneously, we define a mapping $\mu_1^{\dagger} : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to (0,1)$ as

$$\mu_1^{\dagger}(v_1) = \frac{\mu_i - \mu_i\Lambda(v_1)}{1-\mu_i\Lambda(v_1)} < \mu_i \tag{19}$$

where the inequality follows from $0 < \Lambda(v_1) < 1$. This mapping determines the unique initial reputation level of new sellers, that is consistent with a given value v_1 of a new seller of type ℓ. Thus, the equation $v_1 = V_{v_1}^{\dagger}(\mu_1^{\dagger}(v_1))$ must hold in a stationary equilibrium. By solving this equation for v_1, we show that

Theorem 2 *If sellers can freely change identity, there exists a stationary equilibrium satisfying Condition R if h and δ are sufficiently large (but less than 1). In this equilibrium, the default reputation level is lower than μ_i and ℓ-type sellers lie more than when identity cannot be changed.*

Proof. See appendix. ■

The main conclusion is thus that allowing sellers to change identity at no cost doesn't invalidate our result that equilibria exist in which sellers of high ability always communicate truthfully. Of course it impedes somewhat the power of incentives: Since Proposition 3 applies in any stationary equilibrium, sellers' announcements are less reliable than when fresh restart with a new identity is not possible. However, this does not mean that untruthful announcements are more frequent in the market when restarts are possible than when they are not: ℓ-sellers who have lied once, rather than keep lying forever when $q = b$, would start afresh and announce according to $y^{\dagger}_{v_1}(\mu_1)$. In fact, when δ is close to 1 there will be more truthful announcements in the market when sellers are allowed to restart with a new identity.

Nevertheless, h-sellers tend to suffer more due to untrustworthy behavior of ℓ-sellers when restarts are possible for two reasons. First, as in the case of an outside option, the decrease in the reliability of ℓ-sellers results in lower prices for h-sellers. Notice that the price profile is the same as if the seller had an exogenous outside option with value $v_o = v_1$. Second, newborn h-sellers suffer from depressed reputation at birth due to the restarters boosting the fraction of ℓ-type in the pool of new sellers ($\mu_1 < \mu_i$).

In particular, contrary to the case of an outside option, the decline in confidence and in prices doesn't necessarily get translated to a faster revelation of the seller's type. Since sellers start with a lower initial reputation level, it may take longer for an h-seller to be identified as such by the market. Overall, the possibility to restart one's activity on a platform with a new identity reduces the credibility of communication without necessarily enhancing the separation dynamics of seller types.

6 Concluding Remarks

In this paper we investigate the extent to which the quality of product can be credibly communicated to prospective buyers in experience good markets. We show that if there is adverse selection on seller's ability (in supplying good quality items), credible communication can be sustained by reputational motives in spite of the inherent conflict of interests between sellers and buyers. In addition, if sellers can restart with a new identity, a stationary equilibrium exists but the reliability of sellers' announcements deteriorates uniformly across all reputation levels.

To focus on the reputational incentives in pre-trade communication, we carried out our analysis in a model of pure adverse selection on seller's ability. However, the analysis can be extended to situations that involve moral hazard. To see this, modify the baseline model in such a way that in each period a seller draws an item of good quality with a probability h if he exerted high effort at a cost of $c_\theta > 0$ that depends on the seller's type $\theta \in \{h, \ell\}$, but he draws a good item with a probability ℓ if he exerted low effort at zero cost. Note that our communication equilibrium continues to be an equilibrium in this modified model if c_h is small enough for an h-seller to find it worthwhile to exert high effort, but c_ℓ is large so that an ℓ-seller finds otherwise.[15] If pre-trade communication

[15]This is the case if $\delta(V^*_h(\pi_{Bb}(\mu, y^*(\mu)) - \frac{c_h}{1-\delta} - V^*_h(0)) \geq p_G(\mu, y^*(\mu))$ for all μ, and the

is not possible, this model is equivalent to the baseline model of Mailath and Samuelson (2001) without replacement of types, for which they show that high effort cannot be induced unless discontinuous strategies are allowed (Proposition 2, p424). Our result suggests that pre-trade communication may motivate the more efficient type to exert high effort by facilitating the learning process in the market.

We anticipate that our analysis can be extended in other directions as well. For instance, in the context of internet markets, to examine the effect of competition between trading websites appears as an interesting task from the market design perspective, particularly because rival websites would influence each other by providing exit values as our results suggest. Analysis of such competition may also carry implications on the market segmentation between trading websites and their pricing strategies. We intend to address these issues in the future.

APPENDIX

Proof of Proposition 1. For completeness, we state and prove (as Lemmas) all the assertions made in the informal analysis preceding the statement of Proposition 1, apart from the off-equilibrium specifications, (20) below, that we argued we may impose without loss of generality. Let $z^*(\mu)$ denote the equilibrium probability that an ℓ-seller of reputation μ lies upon drawing $q = g$.

Lemma 1 *In any communication equilibrium,* $z^*(\mu) = 0$ *for all* $\mu \in (0, 1]$.

Proof. First, we show that $p^*_G(\mu) \geq p^*_B(\mu)$ for all $\mu > 0$. If $y^*(\mu) = 0$, then $p^*_G(\mu) = 1 > p^*_B(\mu)$ is immediate by Condition [R-a] and Bayes rule. If $y^*(\mu) > 0$, then $p^*_G(\mu) + \delta V^*_\ell(\pi^*_{Gb}(\mu)) \geq p^*_B(\mu) + \delta V^*_\ell(\pi^*_{Bb}(\mu))$ and $\pi^*_{Gb}(\mu) = 0 < \pi^*_{Bb}(\mu)$ by [R-a] and Bayes rule. These two inequalities, together with Condition [R-b], imply that $p^*_G(\mu) \geq p^*_B(\mu)$ as desired.

Now, to prove the claim of the Lemma by contradiction, suppose $z^*(\mu) > 0$ for some $\mu > 0$, which would imply $\pi^*_{Bg}(\mu) = 0 < \mu < \pi^*_{Gg}(\mu)$ by [R-a] and Bayes rule and thus, $p^*_G(\mu) + \delta V^*_\ell(\mu') \leq p^*_B(\mu) + \delta V^*_\ell(0)$ where $\mu' = \pi^*_{Gg}(\mu)$. Hence, from [R-b] and $p^*_G(\mu) \geq p^*_B(\mu)$ shown above, we further deduce that $p^*_G(\mu) = p^*_B(\mu)$ and $V^*_\ell(\mu') = V^*_\ell(0)$. Since $\mu < \mu'$, we would also have $V^*_\ell(\mu) = V^*_\ell(0)$ by [R-b]. In addition, note that $y^*(\mu) > 0$ because $y^*(\mu) = 0$ would mean that $p^*_G(\mu) = 1 = p^*_B(\mu)$, an impossibility. Hence, $V^*_\ell(\mu) = p^*_G(\mu) + \delta V^*_\ell(0)$. Then, since $V^*_\ell(\mu') \geq p^*_m(\mu') + \delta V^*_\ell(0)$

inequality is reversed if c_ℓ replaces c_h, where $V^*_h(0) = \frac{\ell}{1-\delta} = V^*_\ell(0)$ and y^*, V^*_ℓ and V^*_h are as derived in Section 3. Such values of c_h and c_ℓ exist because *i)* $\delta(V^*_h(\pi_{Bb}(\mu, y^*(\mu)) - V^*_h(0)) > p_G(\mu, y^*(\mu))$ for $\mu \geq \bar{\mu}$ if $\delta > \delta_h$ due to (11), *ii)* $\delta(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)) - V^*_\ell(0)) = p_G(\mu, y^*(\mu))$ by definition of V^*_ℓ, and *iii)* $V^*_h(\pi_{Bb}(\mu, y^*(\mu)) > V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)) + \zeta$ for some $\zeta > 0$ due to Lemma 40, continuity of V^*_θ for $\mu > 0$, and $\lim_{\mu\to 0} \pi_{Bb}(\mu, y^*(\mu)) > 0$, where the last inequality is implied by $\lim_{\mu\to 0} \delta\big(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)) - V^*_\ell(0)\big) = \ell$.

holds for either $m \in \{G, B\}$, the equality $V_\ell^*(\mu') = V_\ell^*(\mu)$ obtained above would imply that $p_m^*(\mu') \le p_G^*(\mu)$ must hold for both $m = G, B$. But, this is impossible because some weighted average of $p_G^*(\mu')$ and $p_B^*(\mu')$ is the expected quality of an item drawn by a seller of reputation μ', hence must be strictly greater than $p_G^*(\mu) = p_B^*(\mu) = \mu h + (1-\mu)\ell$. Therefore, we have to conclude that $z^*(\mu) = 0$ for all $\mu > 0$. ∎

Given Lemma 1, as established already, without loss of generality we set

$$y^*(0) = 1,\ p_G^*(0) = \ell, \text{ and } p_B^*(\mu) = \pi_{Gb}^*(\mu) = \pi_{Bg}^*(\mu) = 0\ \forall \mu \in [0,1] \tag{20}$$

except for the value of $\pi_{Gb}^*(1)$. Determining $\pi_{Gb}^*(1)$ is a little delicate because it determines the value of $V_\ell^*(1)$ and thereby, the optimality of $y^*(\mu) = 1$ for $\mu < 1$ via determining the deviation value when an ℓ-type seller with reputation μ announced truthfully upon drawing $q = b$, which would induce belief $\pi_{Bb}^*(\mu) = 1$. The value of $\pi_{Gb}^*(1)$ also plays a salient role for a seller of "maximal reputation" $\mu = 1$, who, upon drawing $q = b$, has a choice between maintaining its reputation with a low current price ($p_B^*(1) = 0$), and a high current price of $p_G^*(1) = 1$ followed by a drop of future profits from $V_\theta^*(1)$ to $V_\theta^*(\pi_{Gb}^*(1))$ due to lost reputation. The next lemma characterizes what happens for an ℓ-type seller at $\mu = 1$ in equilibrium.

Lemma 2 *In any communication equilibrium,*
(*i*) $y^*(1) = 1$ *and* $y^*(\mu)$ *is continuous at* $\mu = 1$;
(*ii*) $\lim_{\mu \to 1} V_\ell^*(\mu) = \overline{V}_\ell := \dfrac{1 - \delta(1 - \ell + \ell^2)}{(1-\delta)(1-\delta\ell)}$;
(*iii*) *if* $y^*(\mu) < 1$ *for some* μ, *then* $V_\ell^*(1) = \overline{V}_\ell$ *and* $V_\ell^*(\pi_{Gb}^*(1)) = V_\ell^*(0)$.
(*iv*) *if* $y^*(\mu) \equiv 1$, *then there exists an equilibrium with the same strategy of seller* (*i.e.,* $y^*(\mu) \equiv 1$) *and* $\pi_{Gb}^*(1) = 0$ *and* $V_\ell^*(1) = \overline{V}_\ell$.

Proof. (i) If $y^*(1) < 1$, we would have

$$V_\ell^*(1) = \ell\big(1 + \delta V_\ell^*(1)\big) + (1-\ell)\big(y^*(1)(1 + \delta V_\ell^*(\pi_{Gb}^*(1)) + (1 - y^*(1))\delta V_\ell^*(1)\big)$$

along with $1 + \delta V_\ell^*(\pi_{Gb}^*(1)) \le \delta V_\ell^*(\pi_{Bb}^*(1)) = \delta V_\ell^*(1)$, whence

$$\begin{aligned} V_\ell^*(1) &\le \ell\big(1 + \delta V_\ell^*(1)\big) + (1-\ell)\delta V_\ell^*(1) = \ell + \delta V_\ell^*(1) \\ \implies &\ V_\ell^*(1) \le \ell/(1-\delta) = V_\ell^*(0) \implies V_\ell^*(1) = V_\ell^*(0), \end{aligned}$$

i.e., V_ℓ^* would be constant. Since this would contradict the inequality asserted above, $1 + \delta V_\ell^*(\pi_{Gb}^*(1)) \le \delta V_\ell^*(\pi_{Bb}^*(1))$, we conclude that $y^*(1) = 1$.

Next, suppose that $\lim_{\mu\to 1} y^*(\mu) \ne 1$. Then, the following holds for some $\eta > 0$: for any $\epsilon > 0$ there is $\mu_\epsilon < 1$ such that $1 - \epsilon < \mu_\epsilon$ and $y^*(\mu_\epsilon) < 1 - \eta$ and thus,

$$V_\ell^*(\mu_\epsilon) = \ell p_G(\mu_\epsilon, y^*(\mu_\epsilon)) + \delta\big(\ell V_\ell^*(\pi_{Gg}^*(\mu_\epsilon)) + (1-\ell)V_\ell^*(\pi_{Bb}(\mu_\epsilon, y^*(\mu_\epsilon)))\big). \tag{21}$$

Since $p_G(\mu_\epsilon, y^*(\mu_\epsilon)) \to 1$, $\mu_\epsilon \to 1$, $\pi_{Gg}^*(\mu_\epsilon) \to 1$, and $\pi_{Bb}(\mu_\epsilon, y^*(\mu_\epsilon)) \to 1$ as $\epsilon \to 0$, (21) would imply $\lim_{\mu\to 1} V_\ell^*(\mu) = \ell + \delta \lim_{\mu\to 1} V_\ell^*(\mu)$, i.e., $\lim_{\mu\to 1} V_\ell^*(\mu) = \frac{\ell}{1-\delta} =$

$V_\ell^*(0)$. Then, $V_\ell^*(\mu) = V_\ell^*(0)$ for all μ which would imply that an ℓ-seller with a bad quality product would always lie as $p_G(\mu_\epsilon, y^*(\mu_\epsilon)) + \delta V_\ell^*(0) > \delta V_\ell^*(0)$, contradicting $y^*(\mu_\epsilon) < 1$. Hence, we conclude that $\lim_{\mu\to 1} y^*(\mu) = 1$.

(ii) Since $y^*(\mu) > 0$ by (i) for all sufficiently large $\mu < 1$, we have $V_\ell^*(\mu) = p_G(\mu, y^*(\mu)) + \delta\big(\ell V_\ell^*(\pi_{Gg}^*(\mu)) + (1-\ell)V_\ell^*(0)\big)$ as $\mu \to 1$, and thus,

$$\lim_{\mu\to 1} V_\ell^*(\mu) = \frac{p_G(1,1) + \delta(1-\ell)V_\ell^*(0)}{1-\delta\ell} = \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)}.$$

(iii) With a view to reach a contradiction, suppose that $y^*(\mu) < 1$ for some μ, yet $\overline{V_\ell} < V_\ell^*(1)$. Since $V_\ell^*(0) = \ell/(1-\delta)$ due to (20), $\delta(\overline{V_\ell} - V_\ell^*(0)) = \delta\Delta < 1$ from (1).

If $\delta(V_\ell^*(1) - V_\ell^*(0)) > \ell$, then there must be some $\mu \in (0,1)$ such that $\delta(V_\ell^*(1) - V_\ell^*(0)) > p_G(\mu, 1) > \delta(\overline{V_\ell} - V_\ell^*(0))$ so that $y^*(\mu)$ cannot be equal to 1 because $\delta(V_\ell^*(1) - V_\ell^*(0)) > p_G(\mu, 1)$, nor can it be less than 1 because $\delta(V_\ell^*(\pi_{Bb}(\mu, y)) - V_\ell^*(0)) < p_G(\mu, y)$ for all $y < 1$, an impossibility.

If $\delta(V_\ell^*(1) - V_\ell^*(0)) \leq \ell$, on the other hand, $y^*(\mu) = 1$ must hold for all $\mu \in (0,1]$ because $p_G(\mu, y) > \ell \geq \delta(V_\ell^*(1) - V_\ell^*(0)) \geq \delta(V^*(\pi_{Bb}(\mu, y)) - V^*(\pi_{Gb}(\mu, y)))$ for any y. Since $y^*(0) = 1$ from (20), we have encountered a contradiction to the supposition that $y^*(\mu) < 1$ for some μ.

Therefore, we conclude that $\overline{V_\ell} = V_\ell^*(1)$ if $y^*(\mu) < 1$ for some μ. This, together with $V_\ell^*(1) = 1 + \delta\left(\ell V_\ell^*(1) + (1-\ell)V_\ell^*(\pi_{Gb}^*(1)\right)$, implies that $V_\ell^*(\pi_{Gb}^*(1)) = V_\ell^*(0)$.

(iv) Suppose we reset $\pi_{Gb}^*(1) = 0$ in an equilibrium with $y^*(\mu) \equiv 1$. The value function $V_h^*(.)$ is unchanged, while the incentive compatibility condition for type h at 1 is still verified: $1 + \delta V_h^*(1) \geq \delta V_h^*(0)$ holds if it holds for $\pi_{Gb}^*(1) > 0$. For the ℓ-seller, the value $V_\ell^*(1)$ is lower while $V_\ell^*(\mu)$ is unchanged for $\mu < 1$. Hence, the incentive compatibility is preserved for $\mu < 1$ and consequently, for $\mu = 1$ as well since $p_G^*(\mu)$ is continuous and $\pi_{Gb}^*(\mu) = 0$ for $\mu < 1$. Thus, it still constitutes an equilibrium when $\pi_{Gb}^*(1)$ is reset at 0. It is trivial to verify that $\overline{V_\ell} = V_\ell^*(1)$ holds in this equilibrium. ∎

Lemma 2 implies that without loss of generality we can set $\pi_{Gb}^*(1) = 0$ in the sense that it is inconsequential for the equilibrium outcome. Thus, as discussed prior to (1),

$$V_\ell^*(0) = \frac{\ell}{1-\delta} \quad \text{and} \quad V_\ell^*(1) = V_\ell^*(0) + \Delta. \tag{22}$$

Next, to characterize V_ℓ^* as a fixed point of the operator defined on $\mathcal{F}$, we extend the definition of the "pseudo-best-response" function y_V to all function $V \in \mathcal{F}$. Note that since $\bar{\mu}$ is independent of V, (7) determines $y_V(0) = y_V(\mu) = 1$ for all $\mu \geq \bar{\mu}$. For $0 < \mu < \bar{\mu}$, we extend the definition of $y_V(\mu)$ to be the unique $y \in (0,1)$ that satisfies

$$\delta \lim_{y'\uparrow y} \big(V(\pi_{Bb}(\mu, y')) - V(0)\big) \leq p_G(\mu, y) \leq \delta \lim_{y'\downarrow y} \big(V(\pi_{Bb}(\mu, y')) - V(0)\big). \tag{23}$$

This uniquely determines the pseudo-best-response function y_V as

$$y_V(\mu) = \begin{cases} 1 & \text{if} \quad \mu > \bar{\mu} \\ \text{the unique } y \text{ that satisfies (23)} & \text{if} \quad 0 < \mu \leq \bar{\mu} \\ 1 & \text{if} \quad \mu = 0. \end{cases} \tag{24}$$

Lemma 3 *For any $V \in \mathcal{F}$, $y_V(\mu)$ is continuous and strictly positive on $[0,1]$ and $p_G(\mu, y_V(\mu))$ is nondecreasing in μ.*

Proof. For each $\mu \in (0, \bar{\mu}]$, by construction, $y_V(\mu)$ is the value of y at which the graph of $p_G(\mu, y)$ intersects with the "connected" graph of $\delta(V(\pi_{Bb}(\mu, y)) - V(0))$, i.e., the latter graph is connected vertically at every discontinuity points by the shortest distance. Since both of the graphs are uniformly continuous as functions of μ, the intersection point changes continuously in μ, i.e., $y_V(\mu)$ is continuous on $\mu \in (0, \bar{\mu}]$. Since $p_G(\mu, 0) = 1 > \delta\Delta$ from (2), the intersection takes place at some $y > 0$, establishing that $y_V(\mu) > 0$ for $\mu \in (0, \bar{\mu}]$ as well as when $\mu = 0$ and $\mu > \bar{\mu}$ as per (24).

Furthermore, note that $y_V(\mu) \to 1$ as $\mu \to 0$ because, for every $y < 1$, $\pi_{Bb}(\mu, y) \to 0$ as $\mu \to 0$ and thus, $\delta(V(\pi_{Bb}(\mu, y)) - V(0)) < \ell \leq p_G(\mu, y)$ for all μ sufficiently small. Since $y_V(\bar{\mu}) = 1$ by construction (using $y_V(0) = 1$ if $\bar{\mu} = 0$), it follows that $y_V(\mu)$ is continuous on $[0, 1]$.

For $\mu \geq \bar{\mu}$, we have $p_G(\mu, y_V(\mu)) = p_G(\mu, 1)$ which increases in μ by (2). For $\mu \in (0, \bar{\mu})$, the two aforementioned graphs move upward as μ increases because both $p_G(\mu, y)$ and $\pi_{Bb}(\mu, y)$ increases in μ by (2) and (3), respectively. Hence, the height of the intersection point also increases, i.e., $p_G(\mu, y_V(\mu))$ weakly increases in μ. ∎

Thus, the operator T is well-defined on $\mathcal{F}$ by (9). The next result establishes that the value function V_ℓ^* of an equilibrium can be computed as a fixed point of T.

Lemma 4 *$T(\mathcal{F}) \subset \mathcal{F}$ and for any communication equilibrium, V_ℓ^* is a fixed point of T and $y^*(\mu) = y_{V_\ell^*}(\mu)$.*

Proof. Since $\pi_{Gg}^*(\mu)$ increases in μ and $p_G(\mu, y_V(\mu))$ is non-decreasing in μ by Lemma 3, it follows that $T(V)$, defined in (9), is non-decreasing in μ.

Next, since $y_V(0) = 1$ and $\pi_{Gg}^*(0) = 0$, we have $T(V)(0) = p_G(0, 1) + \delta(\ell V(0) + (1 - \ell)V(0))$ which yields $T(V)(0) = \ell/(1 - \delta)$.

Also, since $y_V(1) = \pi_{Gg}^*(1) = 1$ and $p_G(1, 1) = 1$, we have $T(V)(1) = 1 + \delta(\ell V(1) + (1 - \ell)V(0))$ which yields $T(V)(1) = \ell/(1 - \delta) + \Delta$.

To prove that V_ℓ^* is a fixed point, first notice from the conditions (6) and (7) that $y^*(\mu) = y_{V_\ell^*}(\mu)$ for all μ, i.e., $y^* = y_{V_\ell^*}$. Since $y^*(\mu) = y_{V_\ell^*}(\mu) > 0$ by Lemma 3, we deduce from (5), (6) and (9) that $T(V_\ell^*)(\mu) = V_\ell^*(\mu)$ for all μ. ∎

Although $T : \mathcal{F} \to \mathcal{F}$ is well-defined, fixed point theorems may not be applied directly because $\mathcal{F}$ is not a compact set. To resolve this problem, we need two lemmas.

Lemma 5 *If $T(V) = V$ then V is continuous and strictly increasing.*

Proof. Since y_V is continuous at 0 by Lemma 3, (9) implies that V is continuous at 0 because

$$\lim_{\mu \to 0} V(\mu) = \frac{p_G(0, 1) + \delta(1 - \ell)V(0)}{1 - \delta\ell} = V(0).$$

Since $y_V(\mu) = 1$ for all $\mu \geq \bar{\mu}$ and $\pi_{Gg}^*(\mu)$ is continuous with $\pi_{Gg}^*(1) = 1$, we verify from (9) that V is continuous at 1 as well:

$$\lim_{\mu \to 1} V(\mu) = \frac{1 - \delta(1 - \ell + \ell^2)}{(1 - \delta)(1 - \delta\ell)} = V(1).$$

Next, since $y_V(\mu)$ is continuous in μ by Lemma 3, (9) implies that

$$|V^+(\mu) - V^-(\mu)| \le \delta \max_{0<\mu<1} |V^+(\mu) - V^-(\mu)|$$

holds for all $\mu \in (0,1)$ where $V^+(\mu) = \lim_{\mu' \downarrow \mu} V(\mu')$ and $V^-(\mu) = \lim_{\mu' \uparrow \mu} V(\mu')$. This implies that $|V^+(\mu) - V^-(\mu)| \equiv 0$ since $\delta < 1$. Thus, V is continuous.

Finally, it is clear from (9) that V is strictly increasing on $(\bar{\mu}, 1)$ because $p_G(\mu, 1)$ is strictly increasing and $\pi^*_{Gg}(\mu)$ is nondecreasing. If V were not strictly increasing on $(0,1)$, one could find the highest value $\mu' \le \bar{\mu}$ such that $V(\mu)$ is constant on some interval $(\mu' - \varepsilon, \mu')$. For sufficiently small $\varepsilon > 0$, we would have $\pi^*_{Gg}(\mu) > \mu'$ if $\mu \in (\mu' - \varepsilon, \mu')$ and thus, $V(\pi^*_{Gg}(\mu)) < V(\pi^*_{Gg}(\mu'))$. Since $p_G(\mu, y_V(\mu))$ is nondecreasing by Lemma 3, we would then have

$$\begin{aligned} V(\mu') &= p_G(\mu, y_V(\mu')) + \delta(\ell V(\pi^*_{Gg}(\mu')) + (1-\ell)V(0)) \\ &> p_G(\mu, y_V(\mu)) + \delta(\ell V(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)) = V(\mu), \end{aligned}$$

contradicting the supposition that $V(\mu)$ is constant on $(\mu' - \varepsilon, \mu')$. This proves that V is strictly increasing. ∎

Lemma 6 *If $T(V) = V$, then $\pi_{Bb}(\mu, y_V(\mu)) > \mu$ all $\mu > 0$.*

Proof. To reach a contradiction, suppose that there exists μ such that $\pi_{Bb}(\mu, y_V(\mu)) \le \mu$. Since y_V is continuous (Lemma 3) and $y_V(\mu) = 1$ for $\mu \ge \bar{\mu}$, there exists

$$\tilde{\mu} = \max\{\mu < 1 \mid \pi_{Bb}(\mu, y_V(\mu)) \le \mu\} < \bar{\mu}. \tag{25}$$

Note that $\pi_{Bb}(\tilde{\mu}, y_V(\tilde{\mu})) = \tilde{\mu}$. Since it is easily verified from (3) that

$$\pi_{Bb}(\mu, y) \ge \mu \iff y \ge \hat{y} = (h - \ell)/(1 - \ell), \tag{26}$$

it must be the case that $y_V(\tilde{\mu}) = \hat{y}$ and thus

$$p_G(\tilde{\mu}, \hat{y}) = \delta(V(\tilde{\mu}) - V(0)). \tag{27}$$

Expanding $V(\mu) = p_G(\mu, y^*(\mu)) + \delta\big(\ell V(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\big)$ by applying an analogous equation to $V(\pi^*_{Gg}(\mu))$ repeatedly, we get

$$\begin{aligned} V(\mu) &= \left[\sum_{t=0}^{\infty} \ell^t \delta^t p_G\left(\pi^t_{Gg}(\mu), y_V(\pi^t_{Gg}(\mu))\right)\right] + \delta V(0)(1-\ell)\sum_{t=0}^{\infty} \ell^t \delta^t & (28) \\ &= \sum_{t=0}^{\infty} \delta^t \ell^t (p_G(\pi^t_{Gg}(\mu), y_V(\pi^t_{Gg}(\mu))) - \ell) + V(0). & (29) \end{aligned}$$

where $\pi^1_{Gg}(\mu) = \pi^*_{Gg}(\mu)$ and $\pi^t_{Gg}(\mu) = \pi^*_{Gg}\big(\pi^{t-1}_{Gg}\big)$ recursively for $t \ge 2$ so that

$$\pi^t_{Gg}(\mu) = \frac{\mu h^t}{\mu h^t + (1-\mu)\ell^t}. \tag{30}$$

Note from (30) that $\pi^t_{Gg}(\tilde{\mu}) > \tilde{\mu}$ for $t \geq 1$ and thus, $\pi_{Bb}(\pi^t_{Gg}(\tilde{\mu}), y_V(\pi^t_{Gg}(\tilde{\mu}))) > \pi^t_{Gg}(\tilde{\mu})$ by (25). Consequently, $y_V(\pi^t_{Gg}(\tilde{\mu})) > \hat{y}$ by (26). Therefore, since $p_G(\mu, y) \leq 1$ and $p_G(\mu, y)$ decreases in y, (29) implies that

$$V(\tilde{\mu}) - V(0) < \sum_{t=0}^{\infty} \big(p_G(\pi^t_{Gg}(\tilde{\mu}), \hat{y}) - \ell\big)\delta^t \ell^t. \tag{31}$$

Since

$$p_G(\mu, \hat{y}) = \frac{\mu h + (1-\mu)\ell}{h} \tag{32}$$

from (2), we further deduce from (31) that

$$\begin{aligned} V(\tilde{\mu}) - V(0) &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \Big(\frac{\pi^t_{Gg}(\tilde{\mu})(h-\ell) + \ell(1-h)}{h}\Big)\delta^t \ell^t \\ &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \Big(\frac{(h-\ell) + \ell(1-h)}{h}\Big)\delta^t \ell^t \\ &= p_G(\tilde{\mu}, \hat{y}) - \ell + (1-\ell)\frac{\delta\ell}{1-\delta\ell} \\ &= p_G(\tilde{\mu}, \hat{y}) - \frac{(1-\delta)\ell}{1-\delta\ell} < p_G(\tilde{\mu}, \hat{y}) \end{aligned}$$

where the second inequality follows from $\pi^t_{Gg}(\tilde{\mu}) < 1$. Thus, we have reached a contradictory conclusion that (27) cannot hold at $\tilde{\mu}$. ∎

By combining Lemmas 3–5, at this point we have proved all claims of Proposition 1 apart from uniqueness of the fixed point. We are now ready to prove the existence and uniqueness of fixed point of T. Lemma 6 is key to this analysis because it implies that along any equilibrium path, the reputation level, and thus the price p^*_G, increases until the seller, in case he is of an ℓ-type, reveals his type by falsely claiming a high quality, at which point the reputation level drops to 0 and the price to ℓ.

<u>*Proof of existence.*</u> Define $\mathcal{F}^r$ as the subset of all right-continuous functions in $\mathcal{F}$, endowed with the topology of the weak convergence. The set $\mathcal{F}^r$ is convex and compact (Theorem 5.1, Billingsley, 1999). By Fan-Glicksberg Fixed Point Theorem,[16] therefore, T has a fixed point in $\mathcal{F}^r$ if

$$T(\mathcal{F}^r) \subset \mathcal{F}^r \text{ and } T \text{ is continuous on } \mathcal{F}^r. \tag{33}$$

We now show (33). First note that $T(V)$ is right-continuous if V is right-continuous because $y_V(\mu)$ is continuous in μ, which proves $T(\mathcal{F}^r) \subset \mathcal{F}^r$.

Next, consider a sequence V_n, $n = 1, 2, \cdots$, in $\mathcal{F}^r$ that weakly converges to $V \in \mathcal{F}^r$. To prove continuity of T, we show below that $T(V_n)$ weakly converges to $T(V)$,

[16] This theorem (Fan, 1952; Glicksberg, 1952) states that an upper hemi-continuous convex valued correspondence from a nonempty compact convex subset of a convex Hausdorff topological vector space has a fixed point.

i.e., $T(V_n)(\mu)$ converges to $T(V)(\mu)$ at all continuity points of $T(V)$ (Theorem 2.1, Billingsley, 1999).

Let Ω be the set of all points where $V(\pi^*_{Gg}(\mu))$ is continuous. Since $\pi^*_{Gg}(\mu)$ is increasing, $[0,1]\backslash\Omega$ is countable. Since V is continuous at $\pi^*_{Gg}(\mu)$ if $\mu \in \Omega$ by continuity of π^*_{Gg}, it follows that $V_n(\pi^*_{Gg}(\mu))$ converges to $V(\pi^*_{Gg}(\mu))$ on Ω.

Next, Let $y_V(\mu)$ be as defined in (24) for V and $y_{V_n}(\mu)$ for V_n. Let Λ be the set of points where $V(\pi_{Bb}(\mu, y_V(\mu)))$ is continuous. Since $\pi_{Bb}(\mu, y_V(\mu))$ is non-decreasing on $(0,1]$ as verified in the proof of Lemma 3, $[0,1]\backslash\Lambda$ is countable. We now show that $y_{V_n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Consider $\mu \in \Lambda$. That $y_{V_n}(\mu) \to y_V(\mu)$ is trivial from (24) if $\mu = 0$ or $\mu > \bar{\mu}$. Hence, suppose $0 < \mu \le \bar{\mu}$ so that, with V_n^- denoting the left limit,

$$\delta\big(V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big) \le p_G(\mu, y_{V_n}(\mu)) \le \delta\big(V(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big). \quad (34)$$

By taking a subsequence if necessary, we may assume that $y_{V_n}(\mu)$ converges to a limit y'. To reach a contradiction, suppose $y' \ne y_V(\mu)$. First, consider the case that $y' < y_V(\mu)$. Then, since $p_G(\mu, y)$ decreases with μ there exists $\varepsilon > 0$ such that

$$p_G(\mu, y_{V_n}(\mu)) > p_G(\mu, y_V(\mu)) + \varepsilon = \delta\big(V(\pi_{Bb}(\mu, y_V(\mu))) - V(0)\big) + \varepsilon$$

for sufficiently large n, where the equality follows because $\mu \in \Lambda$. From this we further deduce that

$$\begin{aligned} p_G(\mu, y_{V_n}(\mu)) &> \delta(V_n(\pi_{Bb}(\mu, y_V(\mu))) - V(0)) + \varepsilon/2 \\ &> \delta(V_n(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)) + \varepsilon/2 \end{aligned}$$

for sufficiently large n, where the first inequality follows because $V_n(\pi_{Bb}(\mu, y_V(\mu))) \to V(\pi_{Bb}(\mu, y_V(\mu)))$ for $\mu \in \Lambda$ and the second because $\pi_{Bb}(\mu, y)$ increases in y and $y_{V_n}(\mu) \to y' < y_V(\mu)$. However, this contradicts (34).

For the case $y' > y_V(\mu)$, we can apply the same reasoning using $V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) \ge V_n(\pi_{Bb}(\mu, y_V(\mu)))$ for n large to reach an analogous contradiction:

$$p_G(\mu, y_{V_n}(\mu)) < \delta\big(V_n^-(\pi_{Bb}(\mu, y_{V_n}(\mu))) - V(0)\big) - \varepsilon/2.$$

Hence, we conclude that $y_{V_n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Together with the earlier result that $V_n(\pi^*_{Gg}(\mu)) \to V(\pi^*_{Gg}(\mu))$ for all $\mu \in \Omega$, this establishes for all $\mu \in \Omega \cap \Lambda$ that

$$\begin{aligned} T(V_n)(\mu) &= p_G(\mu, y_{V_n}(\mu)) + \delta\big(\ell V_n(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\big) \\ &\to p_G(\mu, y_V(\mu)) + \delta\big(\ell V(\pi^*_{Gg}(\mu)) + (1-\ell)V(0)\big) = T(V)(\mu) \end{aligned}$$

as $n \to \infty$. Finally, to verify this convergence at every continuity point of $T(V)(\mu)$, observe first that this convergence is trivial from (24) at $\mu = 0, 1$. For any other $\mu \notin \Omega \cap \Lambda$ at which $T(V)$ is continuous, one can find $\mu_1 \in \Omega \cap \Lambda \cap (0, \mu)$ arbitrarily close to μ and $\mu_2 \in \Omega \cap \Lambda \cap (\mu, 1)$ arbitrarily close to μ because $\Omega \cap \Lambda$ is dense in $[0,1]$.

Since $T(V_n)(\mu_1) \leq T(V_n)(\mu) \leq T(V_n)(\mu_2)$ and $T(V)(\mu_1) \leq T(V)(\mu) \leq T(V)(\mu_2)$, taking the limits we get

$$T(V)(\mu_1) \leq \liminf T(V_n)(\mu) \leq \limsup T(V_n)(\mu) \leq T(V)(\mu_2), \quad \text{and}$$

$$\sup_{\substack{\mu_1 \in \Omega \cap \Lambda \\ \mu_1 < \mu}} T(V)(\mu_1) = T(V)(\mu) = \inf_{\substack{\mu_2 \in \Omega \cap \Lambda \\ \mu_2 > \mu}} T(V)(\mu_2),$$

which imply, as desired, that $T(V_n)(\mu)$ converges to $T(V)(\mu)$ at every continuity point of $T(V)(\mu)$. This proves (33), thus completes the proof of existence.

<u>*Proof of uniqueness.*</u> To reach a contradiction, suppose there are two fixed points V^1 and V^2. Notice that V^1 and V^2 are continuous by Lemma 5 and

$$V^i(\mu) = p_G(\mu, 1) + \delta\left(\ell V(1) + (1-\ell)V(0)\right) \quad \forall \mu \geq \bar{\mu}, \quad i = 1, 2, \tag{35}$$

in particular, $V^1(\mu) = V^2(\mu)$ for all $\mu \geq \bar{\mu}$. Thus, the following is well-defined:

$$\hat{\mu} := \min\{\mu \,|\, V^1(\mu') = V^2(\mu') \;\forall \mu' \geq \mu\} \in (0, \bar{\mu}]. \tag{36}$$

A "segment" for $i = 1, 2$, is a nonempty interval $I_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) > V^j(\mu)$ for all $\mu \in (x, z)$ and $V^i(\mu) = V^j(\mu)$ for $\mu = x, z$, where $j \neq i$. A "region" for $i = 1, 2$, is a nonempty interval $R_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) \geq V^j(\mu)$ for all $\mu \in I_i$ and there are $x', z' \in R_i$ such that $[x, x']$ and $[z', z]$ are segments for i. Let

$$p^i_G(\mu) := p_G(\mu, y_{V^i}(\mu)) \quad \text{and} \quad \pi^i_{Bb}(\mu) := \pi_{Bb}(\mu, y_{V^i}(\mu)) \quad \text{for} \quad i = 1, 2. \tag{37}$$

Recall that in the proof of Lemma 3, we have shown that both $p^i_G(\mu)$ and $\pi^i_{Bb}(\mu)$ weakly increase in μ. Since V^i strictly increases in μ by Lemma 5, the same reasoning establishes that

[A] $p^i_G(\mu)$ and $\pi^i_{Bb}(\mu)$ strictly increase in μ.

Next, we establish the following:

[B] If $V^1(\pi^i_{Bb}(\mu)) = V^2(\pi^i_{Bb}(\mu))$ for some $\mu > 0$ and some $i = 1, 2$, then $y^1(\mu) = y^2(\mu)$ and consequently, $p^1_G(\mu) = p^2_G(\mu)$ and $\pi^1_{Bb}(\mu) = \pi^2_{Bb}(\mu)$. If, in addition, $V^1(\pi^*_{Gg}(\mu)) = V^2(\pi^*_{Gg}(\mu))$ holds, then $V^1(\mu) = V^2(\mu)$.

Note that this observation is trivial for $\mu \geq \bar{\mu}$. Since

$$p^i_G(\mu) = \delta\big(V^i(\pi^i_{Bb}(\mu)) - V^i(0)\big) \quad \forall \mu \in (0, \bar{\mu}], \tag{38}$$

$V^1(\pi^i_{Bb}(\mu)) = V^2(\pi^i_{Bb}(\mu))$ implies $p^1_G(\mu) = p^2_G(\mu)$, which in turn implies $y^*_{V^1}(\mu) = y^*_{V^2}(\mu)$, from which the remaining claims of [B] follow.

Finally, since $\pi^i_{Bb}(\hat{\mu}) > \hat{\mu}$ by Lemma 6 and $\pi^*_{Gg}(\hat{\mu}) > \hat{\mu}$ by (4), due to continuity, there is $\mu' < \hat{\mu}$ such that $V^1(\mu') \neq V^2(\mu')$, $\pi^i_{Bb}(\mu') > \hat{\mu}$ and $\pi^*_{Gg}(\mu') > \hat{\mu}$. Then, $V^1(\pi^i_{Bb}(\mu')) = V^2(\pi^i_{Bb}(\mu'))$ by (36) and thus, $V^1(\mu') = V^2(\mu')$ by [B], a contradiction to the earlier assertion that $V^1(\mu') \neq V^2(\mu')$. This completes the proof of uniqueness, hence of Proposition 1. ■

Proof of Theorem 1. By construction optimality of ℓ-seller strategy is satisfies for $y^* = y_{V_\ell^*}$ where V_ℓ^* is the unique fixed point of T. Let V_h^* denote the value function obtained from the h-seller's optimal strategy given the associated price schedule p_m^* and transition rules π_{mq}^*. Since $V_h^*(0) = \ell/(1-\delta)$ and $\pi_{Bg}^*(\mu) = 0$, upon drawing $q = g$ it is clearly optimal for an h-seller is to announce $m = G$ truthfully. It remains to show optimality of truthful announcement upon drawing $q = b$ as well for $\mu > 0$.

For $\mu \in [\bar{\mu}, 1]$, this follows from (11) because $\pi_{Bb}(\mu, y^*(\mu)) = 1$ and $p_G(\mu, y^*(\mu)) \leq 1$. For $\mu \in (0, \bar{\mu})$, observe from (23) and (24) that

$$\delta\big(V_\ell^*(\pi_{Bb}^*(\mu)) - V_\ell^*(0)\big) = p_G^*(\mu). \tag{39}$$

Since $V_\ell^*(0) = V_h^*(0)$ while

$$V_h^*(\mu) > V_\ell^*(\mu) \quad \forall \mu > 0 \tag{40}$$

as verified below, it follows that

$$\delta\big(V_h^*(\pi_{Bb}^*(\mu)) - V_h^*(0)\big) > p_G^*(\mu). \tag{41}$$

This proves optimality of truthful announcement upon drawing $q = b$ for $\mu > 0$.

Then, the value function for the h-type is given by

$$V_h^*(\mu) = \sum_{t=0}^{\infty} \sum_{\mathbf{h}^t \in H_g^t} \delta^t \rho(\mathbf{h}^t) p_G\left(\pi(\mathbf{h}^t, \mu), y^*(\pi(\mathbf{h}^t, \mu))\right) \tag{42}$$

where $H_g^t := \{g, b\}^{t-1} \times \{g\}$ is the set of all possible realizations of q for t periods with the requirement that $q = g$ in period t; $\rho(\mathbf{h}^t)$ is the ex ante probability that $\mathbf{h}^t \in H_g^t$ realizes; $\pi(\mathbf{h}^t, \mu)$ is the posterior belief at the beginning of period t calculated by Bayes rule from the prior belief μ along $\mathbf{h}^t$. Observe that $V_h^*(\mu)$ is increasing in μ because $p_G(\mu, y^*(\mu))$, $\pi_{Gg}^*(\mu)$ and $\pi_{Bb}(\mu, y^*(\mu))$ all increase in μ as verified earlier.

We already showed that lying when $\mu = 0$ and $q = b$ is optimal for h-seller with $\pi_{Bb}^*(0) = 0$, which completes description of a communication equilibrium. Other communication equilibria may exist that differ in what h-seller does when $\mu = 0$ and $q = b$. But since consistency requires that an h-seller starts with an initial reputation level $\mu > 0$ and an h-seller always tells the truth as per Condition [R-a], the difference pertains to off-equilibrium path. Therefore, the equilibrium outcome is unique.

Finally, we prove (40). Let $V_h(\mu)$ be the value function from the following strategy of an h-seller: always report $q = g$ truthfully and upon drawing $q = b$ for the first time report $m = G$ and get $V_h^*(0)$ in the continuation subgame. Then,

$$V_h(\mu) = \Big[\sum_{t=0}^{\infty} h^t \delta^t p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu)))\Big] + \delta V_h^*(0)(1-h) \sum_{t=1}^{\infty} h^t \delta^t \tag{43}$$

where $\pi_{Gg}^t(\mu)$ is as defined in (30). Since $V_h^*(\mu) \geq V_h(\mu)$ is clear from definition of V^*, it suffices to show $V_h(\mu) - V_\ell^*(\mu) > 0$. From equation (28) in the proof of Lemma 6,

$$V_\ell^*(\mu) = \Big[\sum_{t=0}^{\infty} \ell^t \delta^t p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu)))\Big] + \delta V_\ell^*(0)(1-\ell) \sum_{t=0}^{\infty} \ell^t \delta^t. \tag{44}$$

Subtracting (44) from (43),

$$\begin{aligned} V_h(\mu) - V_\ell^*(\mu) &= \Big[\sum_{t=0}^{\infty}(h^t - \ell^t)\delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big)\Big] \\ &\quad + \delta\Big(\frac{1-h}{1-\delta h} - \frac{1-\ell}{1-\delta\ell}\Big)V_\ell^*(0). \end{aligned}$$

Since $p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) > \ell$ for $\mu > 0$, (40) follows from

$$V_h(\mu) - V_\ell^*(\mu) > \frac{\delta(h-\ell)\ell}{(1-\delta h)(1-\delta\ell)} - \frac{\delta(1-\delta)(h-\ell)}{(1-\delta h)(1-\delta\ell)}V_\ell^*(0) = 0.$$

This completes proof of Theorem 1. For later use, however, we also prove the following nested result:

[S] If $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$ and δ is large enough, there exists a communication equilibrium in which h-seller announces truthfully even when $\mu = 0$.

To prove this, let

$$V_h^o(\mu) := h\sum_{t=0}^{\infty} h^t\delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) \quad \forall\mu > 0$$

so that

$$V_h^*(\mu) = V_h^o(\mu) + (1-h)\delta\sum_{t=0}^{\infty} h^t\delta^t V_h^*\big(\pi_{Bb}(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) \quad \forall\mu > 0. \tag{45}$$

In conjunction with (28), we have

$$V_h^o(\mu) - V_\ell^*(\mu) = \Big[\sum_{t=0}^{\infty}(h^{t+1} - \ell^t)\delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big)\Big] - \delta V_\ell^*(0)\frac{1-\ell}{1-\delta\ell}$$

and thus,

$$\frac{dV_h^o(\mu)}{d\mu} - \frac{dV_\ell^*(\mu)}{d\mu} = \sum_{t=0}^{\infty}(h^{t+1} - \ell^t)\delta^t\frac{\partial p_G\big(\pi_{Gg}^t(\mu), 1\big)}{\partial\mu}\frac{d\pi_{Gg}^t(\mu)}{d\mu} \tag{46}$$

$$\begin{aligned} &= \sum_{t=0}^{\infty}\delta^{2t}\Bigg[(h^{2t+1} - \ell^{2t})\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial\mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} \\ &\qquad + \delta(h^{2t+2} - \ell^{2t+1})\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial\mu}\frac{d\pi_{Gg}^{2t+1}(\mu)}{d\mu}\Bigg] \\ &> \sum_{t=0}^{\infty}\delta^{2t}\ell^{2t}\Bigg[(h-1)\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial\mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} \\ &\qquad + \delta(h^2 - \ell)\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial\mu}\frac{d\pi_{Gg}^*(\pi_{Gg}^{2t}(\mu))}{d\mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu}\Bigg] \end{aligned} \tag{47}$$

for $\mu \geq \bar{\mu}$ because $y^*(\mu) = 1$ for $\mu \geq \bar{\mu}$. By routine calculation, we get

$$\begin{aligned}(h-1)&\frac{\partial p_G\big(\mu,1\big)}{\partial \mu} + (h^2-\ell)\frac{\partial p_G\big(\pi^*_{Gg}(\mu),1\big)}{\partial \mu}\frac{d\pi_{Gg}(\mu)}{d\mu} \\ &= -\frac{h(1-h)(1-\ell)}{(1-(1-h)\mu)^2} + \frac{h^2(h^2-\ell)(1-\ell)\ell}{(\ell(1-\mu)+h^2\mu)^2},\end{aligned} \tag{48}$$

the derivative of which is

$$-2(1-\ell)\frac{h(1-h)^2}{(1-(1-h)\mu)^3} - \frac{\ell(h^3-h\ell)^2}{(\ell(1-\mu)+h^2\mu)^3} < 0. \tag{49}$$

If $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, it is routinely verified that (48) evaluated at $\mu = 1$ is positive and thus, (48) is positive for all μ due to (49). This further implies that (47) is positive for all $\mu \geq \bar{\mu}$ and consequently, from (45),

$$\frac{dV^*_h(\mu)}{d\mu} \geq \frac{dV^*_\ell(\mu)}{d\mu} \quad \forall \mu \geq \bar{\mu} \tag{50}$$

when $\delta < 1$ is sufficiently close to 1, provided that $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$.

Next, let $\mu_m = \min\{\mu | \pi^*_{Gg}(\mu) \geq \bar{\mu}$ and $\pi_{Bb}(\mu, y^*(\mu)) \geq \bar{\mu}\}$ and consider $\mu \in [\mu_m, \bar{\mu}]$. Note that $\mu_m < \bar{\mu}$ due to Lemma 6. Since

$$\begin{aligned} V^*_h(\mu) &= hp_G(\mu, y^*(\mu)) + \delta\big(hV^*_h(\pi^*_{Gg}(\mu)) + (1-h)V^*_h(\pi_{Bb}(\mu, y^*(\mu)))\big) \text{ and} \\ V^*_\ell(\mu) &= p_G(\mu, y^*(\mu)) + \delta\big(\ell V^*_\ell(\pi^*_{Gg}(\mu)) + (1-\ell)V^*_\ell(0)\big), \end{aligned}$$

we deduce that $\frac{dV^*_h(\mu)}{d\mu} - \frac{dV^*_\ell(\mu)}{d\mu}$, which exists almost everywhere because both $V^*_h(\mu)$ and $V^*_\ell(\mu)$ are continuous and increasing, is equal to the derivative of

$$(1-h)\big(\delta V^*_h(\pi_{Bb}(\mu, y^*(\mu))) - p_G(\mu, y^*(\mu))\big) + \delta\big(hV^*_h(\pi^*_{Gg}(\mu)) - \ell V^*_\ell(\pi^*_{Gg}(\mu))\big),$$

which is positive due to (50) because $p_G(\mu, y^*(\mu)) = \delta\big(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu))) - V^*_\ell(0)\big)$ for $\mu \leq \bar{\mu}$. Repeated application of analogous argument establishes that $\frac{dV^*_h(\mu)}{d\mu} > \frac{dV^*_\ell(\mu)}{d\mu}$ for all $\mu > 0$ when $\delta < 1$ is sufficiently close to 1 if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$.

Setting $\pi_{Bb}(0,1) = \lim_{\mu\to 0} \pi_{Bb}(\mu, y^*(\mu))$ and $V^*_h(0) = \lim_{\mu\to 0} V^*_h(\mu)$, this implies that h-seller prefers to tell the truth upon drawing $q = b$ whenever ℓ-seller is indifferent, i.e., when $\mu \in (0, \bar{\mu}]$. Then, $x^*(0, b) = 0$ is optimal by continuity of V^*_h, $p_G(\mu, y^*(\mu))$, $\pi^*_{Gg}(\mu)$, and $\pi_{Bb}(\mu, y^*(\mu))$. Finally, optimality of $x^*(\mu, g) = 0$ follows immediately from (**??**) as before. ■

Proofs of Properties 2–7. Properties 2 and 7 are already proved in the main text. For Property 3, it only remains to prove (13). This is done in the proof of Lemma 6 which also proves Property 6. Property 4 is proved by applying the argument in the proof of Lemma 3 of verifying monotonicity of $p_G(\mu, y_V(\mu))$ to V^*_ℓ which is continuous and strictly increasing by Lemma 5.

We now prove Property 5. It is clear from (1), (2) and (8) that $\bar{\mu}$ strictly increases in δ. Consider $0 < \delta < \delta' < 1$ and let $y^*(\cdot|\delta)$ and $y^*(\cdot|\delta')$ denote $y^*(\cdot)$ for different δ and similarly for other equilibrium variables. To reach a contradiction, suppose that $y^*(\mu|\delta) \leq y^*(\mu|\delta')$ for some $\mu \in (0, \bar{\mu})$ where $\bar{\mu}$ is associated with δ. Then, $\mu' = \max\{\mu < \bar{\mu} \,|\, y^*(\mu|\delta) \leq y^*(\mu|\delta')\}$ is well-defined. Note that $y^*(\mu'|\delta) = y^*(\mu'|\delta') < 0$ and thus, $p_G^*(\mu'|\delta) = p_G^*(\mu'|\delta')$ and $\delta(V_\ell^*(\mu'|\delta) - V_\ell^*(0)) = \delta'(V_\ell^*(\mu'|\delta') - V_\ell^*(0))$. However, since $y^*(\mu|\delta) \geq y^*(\mu|\delta')$ for all $\mu \geq \mu'$, from (29) we derive a contradiction:

$$\begin{aligned}\delta(V_\ell^*(\mu'|\delta) - V_\ell^*(0)) \;&=\; \sum_{t=0}^{\infty} \delta^{t+1}\ell^t \big(p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu)|\delta)) - \ell\big) \\ &< \sum_{t=0}^{\infty} (\delta')^{t+1}\ell^t \big(p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu)|\delta')) - \ell\big) \;=\; \delta'(V_\ell^*(\mu'|\delta') - V_\ell^*(0)).\end{aligned}$$

This completes the proof. ■

Proof of Proposition 2. Recall that a communication equilibrium is characterized by the probability $y^\dagger(\mu)$ that an ℓ-seller announces $m = G$ when $q = b$ and value functions $V_\theta^\dagger$ for $\theta \in \{h, \ell\}$. Note that $V_\ell^\dagger(0) \geq p_G(0, y^\dagger(0)) + \delta v_o > \frac{\ell}{1-\delta}$. If $y^\dagger(0) < 1$ then $V_\ell^\dagger(0) = \ell p_G(0, y^\dagger(0)) + \delta(\ell V_\ell^\dagger(\pi_{Gg}^*(0)) + (1-\ell)(V^\dagger(\pi_{Bb}(0, y^\dagger(0)))) \leq \ell + \delta V_\ell^\dagger(0)$, which would contradict $V_\ell^\dagger(0) > \frac{\ell}{1-\delta}$. Hence, we conclude that $y^\dagger(0) = 1$ and, therefore, $V_\ell^\dagger(0) = p_G(0,1) + \delta \max\{v_o, V_\ell^\dagger(0)\} = \ell + \delta v_o$. In addition, an argument analogous to the proof of Lemma 2 establishes, so long as $y^\dagger(\mu) < 1$ for some μ, that

$$\lim_{\mu \to 1} y^\dagger(\mu) = y^\dagger(1) = 1, \quad V_\ell^\dagger(1) = \frac{1 + \delta(1-\ell)v_o}{(1-\delta\ell)}, \text{ and } V_\ell^\dagger(\pi_{Gb}^\dagger(1)) \leq v_o. \tag{51}$$

Thus, we have verified (14).

Define $\mathcal{F}_{v_o}$ to be the set of all non-decreasing functions V on $[0,1]$ such that $V(0) = V_\ell^\dagger(0)$ and $V(1) = V_\ell^\dagger(1)$. Define $y_V^\dagger(\mu)$ in the same manner as in (23) and (24) with $V(0)$ replaced by v_o and $\bar{\mu}$ replaced by $\bar{\mu}^\dagger := \inf\{\mu \mid p_G(\mu, 1) > \delta\Delta_{v_o}\} < \bar{\mu}$ where the last inequality follows from $\delta\Delta_{v_o} < \Delta$. As long as $\delta\Delta_{v_o} > \ell$ so that $y_V^\dagger(\mu) < 1$ for some μ, which we assume below, we have $y_V^\dagger(\mu) \in (0,1)$ for $\mu \in (0, \bar{\mu}^\dagger)$ with $\lim_{\mu \to 0} y_V^\dagger(\mu) = 1$ because $\delta\big(\max\{V(\pi_{Bb}(\mu, y)), v_o\} - v_o\big)$ approaches $\delta\Delta_{v_o} > \ell$ as $y \to 1$ while it approaches 0 as $\mu \to 0$ for all $y < 1$. Note that this implies

$$V(\pi_{Bb}(\mu, y_V^\dagger(\mu))) - v_o > p_G(\mu, y_V^\dagger(\mu)) \quad \forall \mu \in (0, \bar{\mu}^\dagger). \tag{52}$$

Furthermore, $y_V^\dagger(\mu)$ is clearly continuous and assumes 1 for $\mu \geq \bar{\mu}^\dagger$. Define $T_{v_o} : \mathcal{F}_{v_o} \to \mathcal{F}_{v_o}$ as

$$T_{v_o}(V)(\mu) := p_G(\mu, y_V^\dagger(\mu)) + \delta\big(\ell \max\{v_o, V(\pi_{Gg}^*(\mu))\} + (1-\ell)v_o\big). \tag{53}$$

It is straightforward to verify that $T_{v_o}(V) \in \mathcal{F}_{v_o}$.

Then, Lemmas 5 and 6 and Theorem 1 extend to T_{v_o}, establishing that, for any $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, there is a unique fixed point of T_{v_o} and it is continuous and strictly increasing. We omit the proofs because they are analogous with straightforward changes due to the seller opting to restart whenever his reputation level is so low that the continuation value falls short of v_o.[17]

Since the outside option value is v_o for an h-seller as well, optimality of truth-telling for h-type can be verified by an argument analogous to that leading to Theorem 1, with δ_h replaced by the threshold δ_{v_o} that solves

$$\delta_{v_o}\Big(\frac{h}{1-\delta_{v_o}} - v_o\Big) = 1.$$

Thus, a unique equilibrium outcome exists if $\delta > \delta_{v_o}$ when sellers can exit for an outside option v_o.[18] ■

Proof of Proposition 3. It is immediate from the definition of $\bar{\mu}^\dagger$ that $y^\dagger(\mu) = 1$ for all $\mu \geq \bar{\mu}^\dagger$. Hence, we consider $\mu < \bar{\mu}^\dagger$ ($< \bar{\mu}$) below.

Recall that $y^\dagger$ is continuous by construction (which is analogous to (24)) and $y^\dagger(\mu) \in (0,1)$ for $\mu < \bar{\mu}^\dagger$. To reach a contradiction, suppose $y^\dagger(\mu') = y^*(\mu')$ for some $\mu' < \bar{\mu}^\dagger$ and $y^\dagger(\mu) > y^*(\mu)$ for all $\mu \in (\mu', \bar{\mu})$. Then,

$$\begin{aligned}\delta\big(V_\ell^*(\pi_{Bb}(\mu', y^*(\mu'))) - V_\ell^*(0)\big) &= p_G(\mu', y^*(\mu')) \\ &= p_G(\mu', y^\dagger(\mu')) = \delta\big(V_\ell^\dagger(\pi_{Bb}(\mu', y^\dagger(\mu'))) - v_o\big)\end{aligned}$$

and thus,

$$V_\ell^*(\tilde{\mu}) - V_\ell^*(0) = V_\ell^\dagger(\tilde{\mu}) - v_o \quad \text{where} \quad \tilde{\mu} := \pi_{Bb}(\mu', y^*(\mu')) > \mu' \tag{54}$$

and the inequality is from Lemma 6. Furthermore, since

$$V_\ell^*(\tilde{\mu}) = p_G(\tilde{\mu}, y^*(\tilde{\mu})) + \delta\big(\ell V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)V_\ell^*(0)\big) \quad \text{and} \tag{55}$$

$$V_\ell^\dagger(\tilde{\mu}) = p_G(\tilde{\mu}, y^\dagger(\tilde{\mu})) + \delta\big(\ell V_\ell^\dagger(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)v_o\big) \tag{56}$$

while $p_G(\tilde{\mu}, y^*(\tilde{\mu})) \geq p_G(\tilde{\mu}, y^\dagger(\tilde{\mu}))$, (54)-(56) would imply

$$\delta\ell\big[\big(V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) - V_\ell^*(0)\big) - \big(V_\ell^\dagger(\pi_{Gg}^*(\tilde{\mu})) - v_o\big)\big] \leq (\delta-1)\big(v_o - V_\ell^*(0)\big) < 0. \tag{57}$$

Since $V_\ell^*(1) - V_\ell^*(0) = \Delta > \Delta_{v_o} = V_\ell^\dagger(1) - v_o$, there must exist $\mu'' \in (\tilde{\mu}, 1)$ such that $V_\ell^*(\mu'') - V_\ell^*(0) \leq V_\ell^\dagger(\mu'') - v_o$ and $V_\ell^*(\mu) - V_\ell^*(0) > V_\ell^\dagger(\mu') - v_o$ for all $\mu > \mu''$. However, since $p_G(\mu'', y^*(\mu'')) \geq p_G(\mu'', y^\dagger(\mu''))$ and $\pi_{Gg}^*(\mu'') > \mu''$, (55) and (56) evaluated at $\mu = \mu''$ imply that $V_\ell^*(\mu'') - \delta V_\ell^*(0) > V_\ell^\dagger(\mu'') - \delta v_o$ and consequently, $V_\ell^*(\mu'') - V_\ell^*(0) > V_\ell^\dagger(\mu'') - v_o$, contradicting the definition of μ''. ■

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[17] In the proof of lemma 6, $V(\mu) = \sum_{t=0}^{\infty} \delta^t \ell^t (p_G(\pi_{Gg}^t(\mu), y_V(\pi_{Gg}^t(\mu))) - \ell) + \frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell}$, which implies that $V(\tilde{\mu}) - v_o < \sum_{t=0}^{\infty}(p_G(\pi_{Gg}^t(\tilde{\mu}), \hat{y}) - \ell)\delta^t\ell^t$ because $\frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell} < v_o$.

[18] Note that an h-seller with any reputation $\mu > 0$ does not exit after trading a bad quality item because the value of updated reputation exceeds v_o as (52) indicates. However, both types of seller may exit after trading a good quality item in the initial period if the value of the updated reputation, $V_\theta^\dagger(\pi_{Gg}^*(\mu_i))$, falls short of v_o.

Proof of Theorem 2. Recall that μ_1 and χ_1 denote the default reputation level and stationary mass of new sellers, respectively; and v_1, $y^\dagger_{v_1}$ and $V^\dagger_{v_1}$ denote ℓ-seller's default value, strategy and value function, respectively, in equilibrium.

Let $\rho_\theta(q)$ denote the probability that a seller of type θ draws $q \in \{g, b\}$, i.e., $\rho_\theta(g) = \theta = 1 - \rho_\theta(b)$. For any k-period quality history $\mathbf{h}^k = (q_1, \cdots, q_k) \in H^k := \{g, b\}^k$, let $\rho_\theta(\mathbf{h}^k)$ be the ex ante probability that $\mathbf{h}^k$ realizes for a seller of type θ. We use $\mathbf{h}^k_j = (q_1, \cdots, q_j)$ to denote the first j-entry truncation of $\mathbf{h}^k$.

Given a default reputation $\mu_1 > 0$, let $\pi(\mathbf{h}^k_j)$ denote the posterior reputation for a seller who has survived the history $\mathbf{h}^k_j$ without cheating, updated according to $y^\dagger_{v_1}$. Setting $\pi(\mathbf{h}^k_0) = \mu_1$, we can define $\pi(\mathbf{h}^k_j)$ recursively by:

$$\pi(\mathbf{h}^k_j) = \frac{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j)}{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j) + (1 - \pi(\mathbf{h}^k_{j-1}))\rho_\ell(q_j)(1 - y^\dagger_{v_1}(\pi(\mathbf{h}^k_{j-1}), h_j))}, \tag{58}$$

where $y^\dagger_{v_o}(\mu, g) = 0$ and $y^\dagger_{v_o}(\mu) = y^\dagger_{v_o}(\mu, b)$ for all μ. Then, the ex ante probability that an ℓ-seller remains in the market without having cheated after k-period history $\mathbf{h}^k$ is

$$\Pr(\mathbf{h}^k) = \prod_{j=1}^{k} [\rho_\ell(q_j)(1 - y^\dagger_{v_1}(\pi(\mathbf{h}^k_{j-1}), q_j))(1 - \chi)]. \tag{59}$$

Consequently, in a stationary state, the measure of nominally k-period old ℓ-sellers who restart in period $k + 1$ for $k \geq 1$, is

$$\chi_1(1 - \mu_1)\Big(\sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k)(1 - \ell) y^\dagger_{v_1}(\pi(\mathbf{h}^k), b)(1 - \chi) \Big).$$

This implies that the total measure of old ℓ-sellers who restart in an arbitrary period is $\chi_1(1 - \mu_1)\Lambda(v_1)$ where $\Lambda(v_1)$ is as defined in (16).

Now, as verified in the discussion preceding Theorem 2, the value of v_1 in a stationary equilibrium is a fixed point that satisfies $v_1 = V^\dagger_{v_1}(\mu^\dagger_1(v_1))$ where $\mu^\dagger_1(v_1)$ is defined in (19). To show that such a fixed point exists, we need the next lemma which we prove later for uninterrupted flow of argument.

Lemma 7 *Let $\psi : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to \mathcal{C}_{[0,1]}$ be a mapping such that $\psi(v_1) = V^\dagger_{v_1}$ where $\mathcal{C}_{[0,1]}$ is the set of all continuous functions on $[0, 1]$. Then, ψ is continuous in v_1 under the sup norm at any $v_1 > \frac{\ell}{1-\delta}$.*

Note that, as $v_1 \to \frac{\ell}{1-\delta}$, $\mu^\dagger_1(v_1)$ converges to a limit strictly greater than 0. Since the right derivative of $p_G(\mu, y^\dagger_{v_1}(\mu))$ with respect to μ is uniformly bounded away from 0 at $\mu = 0$, so is the right derivative of $V^\dagger_{v_1}(\mu)$ and consequently, $V^\dagger_{v_1}(\mu^\dagger_1(v_1)) > v_1$ for v_1 sufficiently close to $\frac{\ell}{1-\delta}$. On the other hand, as $v_1 \to \frac{1}{1-\delta}$, since $\mu_i < 1$ we have $V^\dagger_{v_1}(\mu^\dagger_1(v_1)) \leq V^\dagger_{v_1}(\mu_i) < V^\dagger_{v_1}(1) \leq v_1$ for v_1 sufficiently close to $\frac{1}{1-\delta}$. Then, since $\mu^\dagger_1(v_1)$ is continuous in v_1 from (19) and ψ is continuous by Lemma 7, we must have $V^\dagger_{v_1}(\mu^\dagger_1(v_1)) = v_1$ for at least one $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$.

Let μ_1 and v_1 denote a pair of stationary default reputation level and value, i.e., $v_1 = V^\dagger_{v_1}(\mu_1)$ and $\mu_1 = \mu^\dagger_1(v_1)$. Note that to establish a stationary equilibrium, we still need to show that it is optimal for h-sellers to always report truthfully as long as $\mu \geq \mu_1$. Since the continuation value of h-seller after cheating is the equilibrium value of the default level μ_1, $V^\dagger_h(\mu_1)$, rather than $V^\dagger_h(0)$, the optimality condition of h-seller is more difficult to verify than when restarting is impossible. In fact, it has not been proved that for all stationary pair of μ_1 and v_1, truthful announcement for all $\mu \geq \mu_1$ is optimal for h-sellers when ℓ-sellers announce according to $y^\dagger_{v_1}(\mu)$ for $\mu \geq \mu_1$.

However, the proof of [S] included in the proof of Theorem 1 relies on $V^*_\ell(0)$ being a constant, rather than $V^*_\ell(0) = \frac{\ell}{1-\delta}$ and consequently, applies analogously to $V^\dagger_h(\mu)$ defined as per (42) with y^* replaced by $y^\dagger_{v_1}$ for $\mu > \mu_1$. As a result, if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, it constitutes an equilibrium for ℓ-sellers to announce according to $y^\dagger_{v_1}(\mu)$ and h-sellers honestly for $\mu \geq \mu_1$ for any stationary pair μ_1 and v_1, provided that $\delta < 1$ is sufficiently large so that, in particular, $\delta(V^\dagger_h(1) - V^\dagger_h(\mu_1)) \geq 1$.[19] It may be worth mentioning that this is a sufficient condition, so stationary equilibria in which h-sellers behave honestly may exist in a wider class of environments.

Finally we prove Lemma 7. Since continuity under the sup norm requires uniform convergence, the possibility of a fixed point having unbounded derivative poses a potential problem. The bulk of the proof evolves around how to circumvent this problem. We start with two preliminary lemmas asserting that $p^*_G(\mu)$ is of bounded variation on $[\varepsilon, 1]$ for any $\varepsilon > 0$ (Lemma A1) and consequently, so is the fixed point $V^\dagger_{v_1}$ (Lemma A2).

Lemma A1 *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\forall v_1 \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, $\forall V \in \mathcal{F}_{v_1} \cap \mathcal{C}_{[0,1]}$, $\forall \mu$ and $\mu' \in (\varepsilon, \bar{\mu}^\dagger)$,*

$$\frac{p_G(\mu', y^\dagger_V(\mu')) - p_G(\mu, y^\dagger_V(\mu))}{\mu' - \mu} \leq M_\varepsilon. \tag{60}$$

Proof. Note from (2) that we can find $k > 0$ such that $\frac{\partial p_G}{\partial \mu} > 0$ is bounded above uniformly by k, and $\frac{\partial p_G}{\partial y} < 0$ is bounded below uniformly by $-k$. Suppose $\mu < \mu'$ without loss of generality.

If $y^\dagger_V(\mu') \geq y^\dagger_V(\mu)$, then $\frac{p_G(\mu', y^\dagger_V(\mu')) - p_G(\mu, y^\dagger_V(\mu))}{\mu' - \mu} < k$ because p_G decreases in y, proving (60).

Now suppose that $y^\dagger_V(\mu') < y^\dagger_V(\mu)$. Note that one can find k_ε, $\tilde{k}_\varepsilon > 0$ such that

$$\begin{aligned} \frac{\partial \pi_{Bb}(\mu, y)}{\partial \mu} &= \frac{(1-h)(1-\ell)(1-y)}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} < k_\varepsilon \\ \frac{\partial \pi_{Bb}(\mu, y)}{\partial y} &= \frac{(1-h)(1-\ell)(1-\mu)\mu}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} > \tilde{k}_\varepsilon \end{aligned}$$

[19] The proof is omitted because it is the same as the proof of Lemma **??** with obvious changes, such as v_1 and $\bar{\mu}^\dagger$ in place of $V^*_\ell(0)$ and $\bar{\mu}$, respectively.

for all $\mu > \varepsilon$ and $y \in [0,1]$. Thus, recalling that $\pi_{Bb}\big(\mu, y_V^\dagger(\mu)\big)$ is nondecreasing, we deduce that

$$0 \leq \pi_{Bb}(\mu', y_V^\dagger(\mu')) - \pi_{Bb}(\mu, y_V^\dagger(\mu)) < k_\varepsilon\,(\mu' - \mu) + \tilde{k}_\varepsilon\big(y_V^\dagger(\mu') - y_V^\dagger(\mu)\big),$$

using the facts that $y_V^\dagger(\mu') < y_V^\dagger(\mu)$ and $\mu < \mu'$, and consequently,

$$y_V^\dagger(\mu') - y_V^\dagger(\mu) > -\frac{k_\varepsilon}{\tilde{k}_\varepsilon}\,(\mu' - \mu)\,.$$

Therefore, we have

$$\begin{aligned} p_G(\mu', y_V^\dagger(\mu')) - p_G(\mu, y_V^\dagger(\mu)) &< k(\mu' - \mu) - k\big(y_V^\dagger(\mu') - y_V^\dagger(\mu)\big) \\ &< k\Big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\Big)(\mu' - \mu). \end{aligned}$$

We complete the proof by setting $M_\varepsilon = k\big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\big)$. □

Lemma A2 *For any $\varepsilon > 0$ and $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,*

$$D^+V_{v_1}^\dagger(\mu) := \lim_{\mu' \downarrow \mu} \sup \frac{V_{v_1}^\dagger(\mu') - V_{v_1}^\dagger(\mu)}{\mu' - \mu} \;\leq\; \frac{M_\varepsilon}{1 - \delta h} \quad \text{if} \quad \mu > \varepsilon, \tag{61}$$

where $V_{v_1}^\dagger$ is the fixed point of T_{v_1}.

Proof. For given v_1 there exists $\underline{\mu} > 0$ defined by $V_{v_1}^\dagger(\pi_{Gg}^*(\underline{\mu})) = v_1$, so that

$$V_{v_1}^\dagger(\mu) = \begin{cases} p_G(\mu, y_{v_1}^\dagger(\mu)) + \delta v_1 & \text{if} \quad \mu \leq \underline{\mu} \\ p_G(\mu, y_{v_1}^\dagger(\mu)) + \delta\left(\ell V_{v_1}^\dagger(\pi_{Gg}^*(\mu)) + (1-\ell)v_1\right) & \text{if} \quad \mu \geq \underline{\mu}. \end{cases} \tag{62}$$

To reach a contradiction, suppose that for any $K > 0$ one can find $\mu_1 > \varepsilon$ such that $D^+V_{v_1}^\dagger(\mu_1) > K$. Then, since $\pi_{Gg}^*(\mu)$ is differentiable and $\frac{\ell}{h} \leq \frac{\partial \pi_{Gg}^*(\mu)}{\partial \mu} \leq \frac{h}{\ell}$, (60) and (62) would imply that $\mu_1 > \underline{\mu}$ when K is sufficiently large and that one can construct a sequence $\mu_n \to 1$ where $\mu_n = \pi_{Gg}^*(\mu_{n-1})$. Since there is $\tau < \infty$ such that $\pi_{Gg}^\tau(\mu) > \bar{\mu}^\dagger$ for any $\mu > \varepsilon$, by choosing K arbitrarily large, one can ensure that $D^+V_{v_1}^\dagger(\pi_{Gg}^\tau(\mu))$ is arbitrarily large. But, this is impossible because $D^+V_{v_1}^\dagger(\mu)$ is bounded for $\mu > \bar{\mu}^\dagger$ as can be verified from

$$V_{v_1}^\dagger(\mu) = \Big[\sum_{t=0}^{\infty} \ell^t \delta^t p_G(\pi_{Gg}^t(\mu), 1)\Big] + \delta v_1(1-\ell)\sum_{t=0}^{\infty} \ell^t \delta^t, \tag{63}$$

a formula adapted from (44) for $V_{v_1}^\dagger(\mu)$ for $\mu > \bar{\mu}^\dagger$. Hence, we conclude that $D^+V_{v_1}^\dagger(\mu)$ is uniformly bounded for $\mu > \varepsilon$ and thus, (60) and (62) imply

$$\begin{aligned} D^+V_{v_1}^\dagger(\mu) &\leq M_\varepsilon + \ell\delta\Big(\sup_{\mu > \varepsilon} D^+V_{v_1}^\dagger(\mu)\Big)\Big(\max_{\mu} \frac{\partial \pi_{Gg}^*(\mu)}{\partial \mu}\Big) \\ &\leq M_\varepsilon + h\delta\Big(\sup_{\mu > \varepsilon} D^+V_{v_1}^\dagger(\mu)\Big) \end{aligned}$$

for $\mu > \varepsilon$. Thus, $D^+V^\dagger_{v_1}(\mu) \le \frac{M_\varepsilon}{1-\delta h}$ if $\mu > \varepsilon$. $\square$

Next, choose $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$. Notice that for a sufficiently small $\eta > 0$, in particular smaller than $v_1 - \frac{\ell}{1-\delta}$, the operator T_{v_1} can be extended to $\mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ where $\mathcal{F}^\eta_{v_1} := \cup_{v_1-\eta \le v \le v_1+\eta} \mathcal{F}_v$. As an intermediate step, we need

Lemma A3 *For $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, the operator*

$$T_{v_1} : \mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]} \to \mathcal{C}_{[0,1]} \text{ is continuous in sup norm.} \tag{64}$$

Proof. Consider $V, V' \in \mathcal{F}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ such that $\max_{\mu \in [0,1]} |V'(\mu) - V(\mu)| < \epsilon$. Since $y^\dagger_V(\mu)$ and $y^\dagger_{V'}(\mu)$ are, by construction, the solutions to

$$p_G(\mu, y) = \delta\big(\max\{v_1, V(\pi_{Bb}(\mu, y))\} - v_1\big) \tag{65}$$

and the same equation with V' instead of V, respectively, $|p_G(\mu, y^\dagger_{V'}(\mu)) - p_G(\mu, y^\dagger_V(\mu))| < \epsilon$. From (53), therefore, we deduce that

$$\max_{\mu \in [0,1]} |T_{v_1}(V')(\mu) - T_{v_1}(V)(\mu)| < \epsilon + \delta\epsilon,$$

which establishes (64). $\square$

Given $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$ and η small as specified above, consider small $|\kappa| < \eta/2$ and any $V \in \mathcal{F}^\eta_{v_1} \cap \mathcal{F}^\eta_{v_1+\kappa} \cap \mathcal{C}_{[0,1]}$. By (65), the value of $p_G(\mu, y^\dagger_V(\mu))$ differs when calculated for T_{v_1} and when calculated for $T_{v_1+\kappa}$, and the difference is at most $\delta\kappa$. Thus, from (53),

$$T_{v_1}(V)(\mu) - 2|\delta\kappa| \le T_{v_1+\kappa}(V)(\mu) \le T_{v_1}(V)(\mu) + 2|\delta\kappa| \quad \forall \mu \in [0,1]. \tag{66}$$

In particular, observe that

$$T_{v_1}(V^\dagger_{v_1+\kappa})(\mu) - 2|\delta\kappa| \le T_{v_1+\kappa}(V^\dagger_{v_1+\kappa})(\mu) = V^\dagger_{v_1+\kappa}(\mu) \le T_{v_1}(V^\dagger_{v_1+\kappa})(\mu) + 2|\delta\kappa|.$$

Finally, to prove continuity of ψ at v_1, we decompose the argument into two parts: First, we prove uniform convergence of functions $\psi(v_1 + \kappa) = V^\dagger_{v_1+\kappa}$ to $\psi(v_1) = V^\dagger_{v_1}$ as $\kappa \to 0$ on intervals $[\varepsilon, 1]$, then do the same separately on $[0, 2\varepsilon]$. The continuity will be established by combining the two parts.

We know from Lemma A2 that on the interval $[\varepsilon, 1]$, the function $V^\dagger_{v_1+\kappa}$ is K_ε-Liptchitz where $K_\varepsilon = \frac{M_\varepsilon}{1-\delta h}$. Then from Ascoli-Arzelà Theorem (see Royden (1988)), the subset consisting of all K_ε-Lipschitz function of $\mathcal{F}^\eta_{v_1}$ is compact under the sup norm. Hence, there exists a sequence of fixed points $V^\dagger_{v_1+\kappa}$ such that, when restricted to the domain $[\varepsilon, 1]$, it converges as $\kappa \to 0$ to a limit, denoted by $W^{[\varepsilon,1]}_{v_1}$, where $W^{[\varepsilon,1]}_{v_1}$ is continuous on $[\varepsilon, 1]$ and

(c0) $V^\dagger_{v_1+\kappa} \overset{unif}{\longrightarrow} W^{[\varepsilon,1]}_{v_1}$ under the sup norm on $[\varepsilon, 1]$ for any $\varepsilon > 0$.

Let $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote $V^{\dagger}_{v_1+\kappa}$ restricted on $[\varepsilon, 1]$ and let $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote the continuous linear extension of $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ on $[0, \varepsilon]$. Then, by (64) and (66),

$$T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq \lim_{\kappa\to 0} T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu). \tag{67}$$

Note that $T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu)$ for each μ is fully determined by $\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ restricted on $[\mu, 1]$ according to (53), and the same is true for $T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})$. Since $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa} = V^{\dagger}_{v_1+\kappa}$ on $[\varepsilon, 1]$ by definition, therefore, (c0) and (67) imply that

$$T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \leq \widetilde{W}^{[\varepsilon,1]}_{v_1}(\mu) \leq T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \quad \text{for all} \quad \mu \in [\varepsilon, 1],$$

where $\widetilde{W}^{[\varepsilon,1]}_{v_1}$ is the continuous linear extension of $W^{[\varepsilon,1]}_{v_1}$ on $[0, \varepsilon]$. Since $\varepsilon > 0$ is arbitrary and $V^{\dagger}_{v_1}$ is the only function V that satisfies $T_{v_1}(V)(\mu) = V(\mu)$ on $[\varepsilon, 1]$ for all $\varepsilon \in (0, 1)$ by uniqueness of the fixed point of T_{v_1}, it further follows that $\widetilde{W}^{[\varepsilon,1]}_{v_1} = V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$, i.e., $W^{[\varepsilon,1]}_{v_1}$ coincides with $V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$. From (c0), therefore,

(c1) $V^{\dagger}_{v_1+\kappa} \overset{unif}{\longrightarrow} V^{\dagger}_{v_1}$ under the sup norm on $[\varepsilon, 1]$ for any $\varepsilon > 0$.

Note, however, that this is not sufficient for uniform convergence on $[0, 1]$. Hence, choose $\breve{\mu} > 0$ such that $V^{\dagger}_{v_1}(\pi^*_{Gg}(\breve{\mu})) < v_1$. Then, because $V^{\dagger}_{v_1+\kappa}$ converges to $V^{\dagger}_{v_1}$ under the sup norm on $[\frac{\breve{\mu}}{2}, 1]$ by (c1), we have $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(\breve{\mu})) < v_1 + \kappa$ for sufficiently small κ. But this implies that $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(\mu)) < v_1 + \kappa$ for all $\mu \leq \breve{\mu}$ for sufficiently small κ, and consequently, $V^{\dagger}_{v_1+\kappa}(\mu) = p_G(\mu, 1) + \delta(v_1 + \kappa)$ on $[0, \breve{\mu}]$, which converges uniformly to $V^{\dagger}_{v_1}(\mu) = p_G(\mu, 1) + \delta v_1$. Thus,

(c2) $V^{\dagger}_{v_1+\kappa} \overset{unif}{\longrightarrow} V^{\dagger}_{v_1}$ under the sup norm on $[0, \breve{\mu}]$.

Combining (c1) and (c2), we obtain uniform convergence under the sup norm on the entire domain $[0, 1]$, which proves continuity of ψ at v_1 and thus, Lemma 7. This completes the proof of Theorem 2. ■

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Figure 1 : $y*(\mu)$