The First-Order Approach to Moral Hazard Problems with Hidden Saving

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October 7, 2009

Abstract

To make moral hazard models tractable, conventional practice is to replace the incentive constraint with the associated first-order condition. The model can then be solved using Lagrangian methods. However, the first-order approach is not generally valid. In the present paper, I study a two-period moral hazard problem where, in addition to the effort decision, also the agent's consumption-saving decision is unobserved. In this setup, the standard validations of the first-order approach break down, since the agent might deviate *jointly* in both dimensions. I show that the first-order approach is valid if the following conditions hold: a) the agent has nonincreasing absolute risk aversion (NIARA) utility, b) the output technology has monotone likelihood ratios (MLR), and c) the distribution function of output is log-convex in effort (LCDF). By imposing more structure on optimal wage schemes, I also validate the first-order approach for distribution functions that are not necessarily convex.

Keywords: principal-agent problems, moral hazard, hidden savings, first-order approach, log-convexity

JEL Classification: C61, D82

*Address: Department of Economics, University of Mannheim, 68131 Mannheim, Germany. Email: skoehne@mail.uni-mannheim.de. I would like to thank Ernst-Ludwig von Thadden, Nicola Pavoni, and especially Ian Jewitt for many helpful discussions. I received further helpful comments from Martin Barbie, Kaiji Chen, Marek Kapicka, Stefan Reichelstein, Richard Suen, Harald Uhlig, and conference participants at the Workshop on Public Economics 2009 Bonn, and German Economic Association (VfS) Annual Congress 2009 Magdeburg.

1 Introduction

The study of moral hazard models is enormously simplified if one can use the first-order approach. By replacing the incentive constraint with the associated first-order condition, this approach allows the application of Lagrangian methods. The seminal works of Rogerson (1985) and Jewitt (1988) validate this procedure for the standard moral hazard problem. Very little is known, however, for more general environments. In particular, the validity of the first-order approach is not well understood for problems in which the agent can secretly save (and borrow). This class is particularly important, since observability of the consumption-saving decision appears unrealistic for many interesting applications of moral hazard models (employment relationships, insurance problems, taxation, etc).

As Kocherlakota (2004) points out, the validity of the first-order approach becomes significantly more complex in the presence of hidden saving.[1] In addition to making sure that the agent's utility is at a global maximum with respect to the effort decision, one has to show the same for the saving decision, and most importantly for *joint* deviations to different effort and saving levels. Typically, the agent would combine a reduction of effort with an increased savings level to insure against the worsened output distribution. Therefore, ruling out joint deviations is the main difficulty in proving that first-order conditions are sufficient.

The present paper derives general conditions for the validity of the first-order approach in this environment. I show that the first-order approach is valid if the agent has nonincreasing absolute risk aversion (NIARA) utility, the output technology has monotone likelihood ratios (MLR), and the distribution function of output is log-convex in effort (LCDF).[2] Note that the LCDF property requires more convexity than Rogerson's (1985) CDF condition and means that the (stochastic) returns to effort are *strongly* decreasing.

The link from these conditions to the second-order effects of joint deviations is subtle. Note that by reducing his effort, the agent increases the probability of being punished by a low wage. By increasing his saving at the same time, he alleviates the severity of the punishment, since the utility difference between high and low wages will be reduced. Decreasing returns to effort

[1] Kocherlakota (2004) provides an example in which the first-order approach to moral hazard with hidden saving fails even though the MLR and CDF conditions from Rogerson (1985) are satisfied. A similar argument shows that the conditions from Jewitt (1988) are also not sufficient for the problem with hidden saving.

[2] A function is called *log-convex* if the logarithm of that function is convex. Any log-convex functions is convex, but not vice versa.

and convex marginal utility of consumption limit the potential gains of such a strategy. The former implies that the probability of being punished increases quicker than linearly as effort is reduced; the latter implies that the reduction of the punishment diminishes quicker than linearly as saving is increased. However, since the two effects are multiplicative rather than additive, those properties are too weak. To find sufficient concavity/convexity requirements, the concept of log-convexity proves helpful.

Log-convexity is a stronger notion than convexity, but has many similar properties. For instance, log-convexity is preserved under summation or multiplication with a positive scalar. In the present paper, I exploit another useful property: The product of two univariate log-convex functions of different variables is (jointly) convex. An application of this result leads to the curvature requirements employed above—log-convexity of the agent's marginal utility of consumption, which is equivalent to NIARA, and log-convexity of the distribution function of output (LCDF).

I also derive alternative conditions for the validity of the first-order approach. As the previous reasoning suggests, one can trade convexity assumptions on the marginal utility of consumption against convexity assumptions on the distribution function. This allows a first relaxation of the LCDF condition. In addition, I relax the LCDF property by imposing more structure on optimal wage schemes, similar to the contribution by Jewitt (1988).

Previously, the first-order approach to moral hazard problems with hidden saving has only been examined under additional restrictions to the output technology or the agent's preferences. The work by Abraham and Pavoni (2009) imposes the spanning condition from Grossman and Hart (1983), whereas the paper by Koehne (2009) studies CARA utility. However, neither restriction is needed. In fact, neither restriction is particularly helpful, since the present findings contain the results by Abraham and Pavoni (2009) and Koehne (2009) as restrictive special cases. As another advancement, the present paper derives conditions for distribution functions that are not necessarily convex in effort.

The first-order approach produces a very useful characterization of optimal contracts. Questions on the monotonicity of consumption or the value of information can be answered immediately, and one finds many analogies to the model without hidden saving. One also finds some important differences between the two models, as Abraham and Pavoni (2009) describe in detail. In particular, they show that hidden saving tends to make optimal contracts more convex. This implies that the associated tax-transfer scheme is typically more regressive than in the standard setup.

The first-order approach is also important because it gives the multi-period problem a tractable recursive structure, as discussed by Werning (2001, 2002), Kocherlakota (2004), and Abraham and Pavoni (2008), among others. Analytical results for the validity of the first-order approach provide a theoretical foundation for this procedure. However, the extension from the present two-period results to the multi-period problem remains a task for future research.

The paper proceeds as follows: Section 2 describes the setup of the model. Section 3 validates the first-order approach given NIARA, MLR and LCDF. Section 4 shows how to relax the latter assumption. Section 5 collects all proofs. Section 6 concludes.

2 Setup

I study a two-period principal-agent problem. In the first period, the agent makes a hidden saving decision. In the second period, the agent exerts a hidden work effort. The contract is signed at the beginning of the first period and there is no renegotiation.

2.1 Preferences

The Principal (P) maximizes expected profits. P's discount factor is $1/R$, with $R > 0$. The Agent (A) has von-Neumann-Morgenstern preferences and maximizes the expected value of

$$u(c_1) + \beta \left(u(c_2) - v(e) \right),$$

where c_t denotes consumption and e represents effort. Consumption utility u is twice continuously differentiable and satisfies $u' > 0$, $u'' < 0$. Effort disutility v is twice continuously differentiable and satisfies $v' > 0$, $v'' \geq 0$.

2.2 Technology

In the first period, A is endowed with w_0 units of the consumption good and can save at the rate $R > 0$. Negative saving, i.e., borrowing, is allowed. The set of feasible saving choices is the real interval J, which may be bounded or unbounded.[3] A's saving decision is not observable.

In the second period, A exerts an unobservable work effort $e \in I$, where I is a real interval. This generates a publicly observable stochastic output $x \in [\underline{x}, \overline{x}]$. (All results go through for discrete output spaces as well.) The output is distributed according to the probability density $f(x, e)$, which is continuously differentiable and has full support for all $e \in I$.

2.3 Contracts

At the beginning of the first period, P proposes a **contract** $(w(\cdot), e, s)$ consisting of an output-contingent wage scheme $w(\cdot)$ and recommended choices (e, s). A's utility from rejecting the contract is $\underline{U}$. The contract is called **optimal** if it maximizes expected profits subject to the incentive compatibility constraint and the participation constraint, i.e., if it solves the following problem:

$$\max_{w(\cdot),e,s} \frac{1}{R} \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx \tag{P1}$$

s.t.

$$(e, s) \in \underset{(e',s') \in I \times J}{\operatorname{argmax}} \; u(w_0 - \frac{s'}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - \beta v(e') \tag{IC}$$

$$u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - \beta v(e) \geq \underline{U} \tag{PC}$$

2.4 First-order approach

Problem (P1) is extremely intricate. The incentive constraint (IC) consists of a two-dimensional continuum of inequalities. For all $e' \in I, s' \in J$, it requires

$$\begin{aligned} & u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - \beta v(e) \\ \geq\; & u(w_0 - \frac{s'}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s') f(x, e')\, dx - \beta v(e'). \end{aligned} \tag{1}$$

[3] The interval J may be bounded below due to a borrowing constraint and bounded above due to a nonnegativity constraint.

To obtain a problem that can be solved by standard methods, one replaces the incentive constraint by the agent's first-order necessary conditions. This gives rise to the following problem:

$$\max_{w(\cdot),e,s} \frac{1}{R} \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x, e)\, dx \tag{P2}$$

s.t.

$$\beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f_e(x, e)\, dx - \beta v'(e) = 0 \tag{FOCe}$$

$$\frac{1}{R} u'(w_0 - \frac{s}{R}) - \beta \int_{\underline{x}}^{\overline{x}} u'(w(x) + s) f(x, e)\, dx = 0 \tag{FOCs}$$

$$u(w_0 - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(w(x) + s) f(x, e)\, dx - \beta v(e) \geq \underline{U} \tag{PC}$$

Solutions to (P2) are denoted by $(w^*(\cdot), e^*, s^*)$. The associated consumption levels are denoted by $c_0^* = w_0^* - s^*/R$ and $c^*(x) = w^*(x) + s^*$.

Replacing the true problem (P1) by the first-order problem (P2) is a valid procedure only if their solutions coincide. Assuming that the solutions to (P1) are interior with respect to effort and saving, this will be the case if and only if the contracts solving (P2) are incentive compatible. A sufficient condition for incentive compatibility is that the agent's decision problem is concave at those contracts. The remainder of this paper will identify conditions under which this is the case.

3 A sufficient condition for concavity of the agent's problem

In this section, I validate the first-order approach using nonincreasing absolute risk aversion, monotonicity of the wage scheme, and an assumption on the curvature of the output distribution function. This procedure strengthens the classic approach of Mirrlees (1979) and Rogerson (1985).

Using λ, μ and ξ as the Lagrange multipliers associated with the constraints (PC), (FOCe), (FOCs), respectively, the first-order condition of the Lagrangian of problem (P2) with respect to wages is

$$0 = -\frac{1}{R} f(x, e^*) + \mu\beta u'(c^*(x)) f_e(x, e^*) - \xi\beta u''(c^*(x)) f(x, e^*) + \lambda\beta u'(c^*(x)) f(x, e^*), \quad x \in [\underline{x}, \overline{x}]. \tag{2}$$

Equivalently,

$$\frac{1}{R\beta u'(c^*(x))} = \lambda + \mu \frac{f_e(x, e^*)}{f(x, e^*)} + \xi\alpha(c^*(x)), \quad x \in [\underline{x}, \overline{x}], \tag{3}$$

where $\alpha(c) = -u''(c)/u'(c)$ is A's coefficient of absolute risk-aversion.

Expression (3) equates the principal's costs and benefits of marginally increasing the agent's utility at output x, normalized by the probability density. Compared to the standard moral hazard problem, there is now the additional term $\xi\alpha(c^*(x))$, because an increase of $u(c^*(x))$ relaxes the agent's Euler equation.[4]

I will often use the following two assumptions to give equation (3) more structure.

MLR. The likelihood ratio function, $f_e(x, e)/f(x, e)$, is continuously differentiable and nondecreasing in output x for all effort levels e.

NIARA. The agent's coefficient of absolute risk aversion, $\alpha(c) = -u''(c)/u'(c)$, is continuously differentiable and nonincreasing in consumption c.

MLR is standard and simply means that more output is indicative of higher effort. NIARA is also unproblematic, since it is satisfied by most common utility functions. NIARA implies that the multipliers λ, μ, ξ in the Kuhn-Tucker condition (3) are positive: $\lambda > 0$, $\mu > 0$, $\xi > 0$ (Abraham and Pavoni 2009). Moreover, MLR plus NIARA is sufficient for A's consumption scheme $c^*(x) = w^*(x) + s^*$ to be continuously differentiable and nondecreasing in output x; see equation (3).[5]

As noted before, the first-order approach is valid if A's objective function

$$(e, s) \mapsto u(c_0^* - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(c^*(x) + s) f(x, e)\, dx - \beta v(e) \tag{4}$$

is concave in (e, s) at the contracts that solve (P2). One can restrict attention to A's second-period consumption utility as the next result shows.

[4]Note that an increase of $\beta u(c^*(x))$ by one marginal unit costs the principal $1/(R\beta u'(c^*(x)))$ units of consumption. On the other hand, it generates a benefit of λ because the participation constraint is relaxed and a benefit (or cost) of $\mu f_e/f$ because the incentive constraint is relaxed (or tightened). In addition, there is a benefit of $\xi\alpha(c^*(x))$ because an increase of $\beta u(c^*(x))$ mitigates the agent's wish to save (Abraham and Pavoni 2009).

[5]NIARA can be relaxed. Equation (3) implies that $c^*(\cdot)$ is nondecreasing under MLR if $-(u'''u' - (u'')^2) \leq -u''(R\beta\xi)^{-1}$. This requires that the coefficient of absolute risk aversion does not increase too quickly.

Lemma 1. *A's decision problem is concave in* (e, s) *if A's second-period consumption utility*

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} u(c^*(x) + s) f(x, e)\, dx \tag{5}$$

is concave in (e, s).

By focusing on A's second-period consumption utility, I ignore the curvature generated by the effort disutility function v and by the effect of saving on first-period utility. In principle, one could obtain more general results by including these two effects. However, the terms would substantially reduce the tractability of the problem. Besides, the role of the effort disutility function is limited anyway, since effort units can always be normalized such that this function is linear.

The following lemma identifies a sufficient condition for concavity of (5).

Lemma 2. *Suppose* $c^*(\cdot)$ *is continuously differentiable and nondecreasing. Suppose the distribution function of output,* $F(x, e) = \int_{\underline{x}}^{x} f(z, e)\, dz$, *is convex in* e *and for all* $x \in [\underline{x}, \overline{x}]$, $e \in I$, $s \in J$, *we have*

$$\frac{F_{ee}(x, e) F(x, e)}{(F_e(x, e))^2} \frac{u'''(c^*(x) + s) u'(c^*(x) + s)}{(u''(c^*(x) + s))^2} \geq 1. \tag{6}$$

Then A's second-period consumption utility is concave in (e, s).

To understand condition (6), note that $F_{ee}F/(F_e)^2$ is nonnegative if and only if F is convex in e, and at least 1 if and only F is log-convex in e.[6] Hence, $F_{ee}F/(F_e)^2$ measures the convexity of the distribution function F as a function effort. This motivates the following concept.

LCDF. The distribution function of output, $F(x, e) = \int_{\underline{x}}^{x} f(z, e)\, dz$, is log-convex in effort e for all output levels x.

A necessary but not sufficient condition for LCDF is that the distribution function is convex in effort. Hence, LCDF tightens the CDF condition from Mirrlees (1979) and Rogerson (1985). To interpret LCDF, note that $F(x', e)$ equals $1 - P(x > x'|e)$. Therefore, stating that $F(x', e)$ is log-convex in effort (or highly convex, in other words) implies that the probability $P(x > x'|e)$ is highly concave in effort. For this reason, LCDF requires that the (stochastic) returns to effort

[6] A function is called *log-convex* if the logarithm of that function is convex. Any log-convex functions is convex, but not vice versa.

are strongly decreasing: The probability $P(x > x'|e)$ that output is larger than some level x' is highly concave in the agent's effort choice e for all values of x'.

Analogous to the interpretation of $F_{ee}F/(F_e)^2$, note that $u'''u'/(u'')^2$ is a measure of convexity of A's marginal utility of consumption. This measure is nonnegative if and only if u' is convex, and at least 1 if and only if u' is log-convex. Log-convexity of u' is equivalent to

$$\frac{u'''u' - (u'')^2}{(u')^2} \geq 0. \tag{7}$$

This is the case if and only if

$$\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right) \leq 0. \tag{8}$$

Hence, log-convexity of u' is equivalent to NIARA.

The main result is a now direct consequence of these observations: MLR, NIARA and LCDF validate the first-order approach.

Theorem 1. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to* (P2). *Suppose MLR, NIARA and LCDF. Then, given this contract, the agent's decision problem is concave. Hence, the contract is also a solution to* (P1).

Compared to the model without hidden saving, Theorem 1 additionally requires NIARA and LCDF instead of Rogerson's (1985) CDF condition. As argued above, NIARA is unproblematic, because it is satisfied by most common utility functions and confirmed by many empirical and experimental studies. The following examples clarify LCDF.

Example 1 (Rogerson 1985)**.** Rogerson's paper contains the following distribution function that is convex in effort and satisfies MLR:

$$F(x, e) = \left(\frac{x}{\overline{x}}\right)^{e-\underline{e}}, \quad x \in [0, \overline{x}],\ e \in (\underline{e}, \infty). \tag{9}$$

This distribution function is not only convex in e, but even satisfies LCDF. Note

$$\log(F(x, e)) = (e - \underline{e}) \log\left(\frac{x}{\overline{x}}\right), \tag{10}$$

which shows that $F(x, e)$ is log-linear in e for all i.

Example 2 (Log-logistic distribution)**.** Let $0 < b \leq 1$. Consider the following distribution function:

$$F(x,e) = \frac{1}{1+(e/x)^b}, \quad x \in [0,\infty),\ e \in (0,\infty). \tag{11}$$

It is not difficult to see that MLR is satisfied. Moreover, note

$$\log(F(x,e)) = -\log\left(1+(e/x)^b\right). \tag{12}$$

Since $b \leq 1$, the expression $(e/x)^b$ is concave in e. Since the logarithm is increasing and concave, equation (12) shows that $\log(F(x,e))$ is convex in e. Thus, LCDF is satisfied.

The following examples apply to discrete output spaces $X = \{x_1,\ldots,x_n\}$, $x_i < x_j$ for $i < j$. In this setup, wages are vectors $(w_1,\ldots,w_n) \in \mathbb{R}^n$, and probability weights $(p_1(e),\ldots,p_n(e))$ replace the density function $f(x,e)$. The previous results extend to the discrete setup without difficulty.

Example 3 (Two outputs)**.** Consider the case with two possible outputs, $x_L < x_H$, and associated probabilities $p_L(e) = 1 - p(e)$, $p_H(e) = p(e)$, for some increasing function p with $0 \leq p(e) \leq 1$. Since p is increasing, MLR is satisfied. LCDF is equivalent to the log-convexity of $1 - p(e)$. One example that satisfies this condition is the function $p(e) = 1 - \exp(-f(e))$, where $f : I \to (0,\infty)$ is increasing and concave.

Example 4 (Spanning condition)**.** Let $(\pi_{1h},\ldots,\pi_{nh})$, $(\pi_{1l},\ldots,\pi_{nl})$ be two probability distributions on $\{x_1,\ldots,x_n\}$ such that π_{ih}/π_{il} is nondecreasing in i. (This implies that π_h first-order stochastically dominates π_l.) Let

$$p_i(e) = \Gamma(e)\pi_{ih} + (1-\Gamma(e))\pi_{il} \tag{13}$$

for some increasing function Γ, with $0 \leq \Gamma(e) \leq 1$. Monotonicity of Γ, combined with the fact that π_{ih}/π_{il} is nondecreasing, yields MLR. Note

$$F_i(e) = F(x_i,e) = \sum_{j=1}^{i} p_j(e) = (1-\Gamma(e))\sum_{j=1}^{i}(\pi_{il}-\pi_{ih}) + \sum_{j=1}^{i}\pi_{ih}. \tag{14}$$

First-order stochastic dominance implies $\sum_{j=1}^{i}(\pi_{il}-\pi_{ih}) \geq 0$. Therefore, LCDF holds if $1-\Gamma(e)$

is log-convex. This requirement is equivalent to

$$\frac{(\Gamma'(e))^2}{-\Gamma''(e)(1-\Gamma(e))} \leq 1, \tag{15}$$

which is exactly the condition under which Abraham and Pavoni (2009) validate the first-order approach for the spanning condition and NIARA utility. Their proof relies heavily on the spanning condition and there is no obvious way how it generalizes to the setting considered in this paper. Moreover, Abraham and Pavoni's reading of the property in (15) is that the Frisch elasticity of leisure must not be larger than one (Abraham and Pavoni 2009, p. 16). This does not capture the precise sense in which (15) tightens the CDF condition from Mirrlees (1979) and Rogerson (1985), in contrast to the interpretation provided here.

4 Alternative sufficient conditions for concavity

In this section, I modify the proof of Theorem 1 to obtain alternative sufficient conditions for concavity of the agent's problem. The first result restricts the class of preferences. The second result uses properties of the primitive of the distribution function. Finally, I rewrite the agent's problem in terms of the so-called state space representation. This leads to a third set of sufficient conditions.

4.1 CRRA utility instead of NIARA

Recall that the two crucial assumptions from Theorem 1, LCDF and NIARA, are equivalent to $F_{ee}F/(F_e)^2 \geq 1$ and $u'''u'/(u'')^2 \geq 1$, respectively. If the latter expression is bounded away from 1, then Lemma 2 can be used to relax LCDF. This yields the following result.

Proposition 3. *Let* $(w^*(\cdot), e^*, s^*)$ *be a solution to* (P2). *Suppose MLR and NIARA. Suppose there exists a number* $\eta > 1$ *such that for all* c

$$\frac{u'''(c)u'(c)}{(u''(c))^2} \geq \eta, \tag{16}$$

and for all $e \in I$, $x \in [\underline{x}, \overline{x}]$,

$$\frac{F_{ee}(x,e)F(x,e)}{(F_e(x,e))^2} \geq \frac{1}{\eta}. \tag{17}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves (P1).

Note that (17) implies that $F_{ee}(x,e)$ is nonnegative. Hence, while (17) is weaker than LCDF, it still requires that the distribution function is convex in effort.

As an important application of Proposition 3, consider CRRA utility: $u(c) = c^{1-\gamma}/(1-\gamma)$. Then we have

$$\frac{u'''(c)u'(c)}{(u''(c))^2} = 1 + \frac{1}{\gamma}. \tag{18}$$

Hence, using Proposition 3, we conclude that the first-order approach is valid if for all $e \in I$, $x \in [\underline{x}, \overline{x}]$,

$$\frac{F_{ee}(x,e)F(x,e)}{(F_e(x,e))^2} \geq \frac{\gamma}{1+\gamma}. \tag{19}$$

Under the spanning condition from Example 4, for instance, this property is equivalent to

$$\frac{(\Gamma'(e))^2}{-\Gamma''(e)(1-\Gamma(e))} \leq 1 + \frac{1}{\gamma} \quad \text{for all } e \in I. \tag{20}$$

This relaxes condition (15).

4.2 Using the primitive of the distribution function

Instead of making assumptions on the distribution function, the present subsection uses curvature assumptions on the primitive of the distribution function. To facilitate the argument, I suppose that the wage scheme is twice continuously differentiable.[7]

The main result is as follows.

Proposition 4. *Let $(w^*(\cdot), e^*, s^*)$ be a solution to* (P2)*. Suppose MLR and NIARA. Set $c^*(x) = w^*(x) + s^*$ and suppose that for all output levels x*

$$-\frac{d^2(u(c^*(x)+s))}{dx^2} \quad \textit{is positive and log-convex in saving } s, \tag{C1}$$

$$\widetilde{F}(x,e) = \int_{\underline{x}}^{x} F(z,e)\,dz \quad \textit{is log-convex in effort } e. \tag{LCP}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves

[7] As the Kuhn-Tucker condition (3) shows, the wage scheme $w^*(x) = c^*(x) - s^*$ will be C^2 in x if $f_e(x,e)/f(x,e)$ is C^2 in x and $u'(c), \alpha(c)$ are C^2 in c.

(P1).

Unfortunately, there is no simple way of expressing condition (C1) in terms of the fundamentals of the model. However, it is easy to see that (C1) is a concavity property of the contract: A's (ex-post) consumption utility, $u(c^*(x) + s))$, depends on output x in a concave way. In addition, the curvature between utility and output changes with saving s in a log-convex way.

The next result shows that (C1) is satisfied if the consumption scheme $c^*(x)$ is concave in output x. Hence, (C1) is guaranteed under an appropriate concavity property of the likelihood ratio function $f_e(x, e)/f(x, e)$; see the appendix for details.

Lemma 5. *Suppose NIARA and suppose $-u'''(c)/u''(c)$ is nonincreasing in c. Then condition* (C1) *is necessary but not sufficient for $c^*(x)$ to be concave in x.*

The assumption that $-u'''(c)/u''(c)$ is nonincreasing in c (nonincreasing absolute prudence) is innocuous. For instance, it is satisfied for all utility functions with hyperbolic absolute risk aversion (HARA).

To capture the second condition in Proposition 4, it is important to note that log-convexity is preserved under integration (Boyd and Vandenberghe 2004, p. 106). Therefore, log-convexity of the primitive, LCP, is a weaker assumption than log-convexity of the distribution function, LCDF. Intuitively, the primitive $\widetilde{F}(x, e)$ will be log-convex in e if the distribution function $F(x, e)$ is log-convex in e for small values of x and "not too misbehaved" for large values of x. In fact, $F(x, e)$ does not even have to be convex in e as the following example shows.

Example 5 (Beta Prime distribution)**.** Consider the Beta Prime distribution with parameter $b = 2$:

$$f(x, e) = \frac{x^{e-1}(1+x)^{-e-2}}{B(e, 2)}, \quad x \in [0, \infty),\ e \in (0, \infty), \tag{21}$$

where $B(e, b)$ represents the Beta function. The likelihood ratio function $f_e(x, e)/f(x, e)$ is nondecreasing concave in x, hence the class of preferences satisfying (C1) is nonempty. The distribution function is

$$F(x, e) = (1 + e + x)x^e(1 + x)^{-e-1}. \tag{22}$$

It is easy to see that $F(x, e)$ is not convex in e for all x. However, the primitive of the distribution function,

$$\widetilde{F}(x, e) = x\left(\frac{x}{1+x}\right)^e, \tag{23}$$

is log-linear in e. Therefore, LCP is satisfied.

Given this example, Proposition 4 even validates the first-order approach for a class of setups where the distribution function is not convex in effort.

Finally, it may be helpful to relate Proposition 4 to the model without hidden saving. In that case, Jewitt's (1988) Theorem 1 validates the first-order approach when $u(c^*(x))$ is concave in x and $\widetilde{F}(x,e)$ is convex in e. Both of these conditions are weaker than their respective counterparts in Proposition 4.

4.3 The state space formulation

So far, I have formulated the output technology in the notation of Mirrlees (1974, 1976), using effort-dependent probability distributions. There is also an older formulation, due to Spence and Zeckhauser (1971) and Ross (1973), which explicitly specifies a stochastic production function. In that setup, output is given by

$$x = \varphi(e,z), \tag{24}$$

where e denotes effort and z denotes the stochastic state of nature. It is natural to assume nonincreasing marginal returns to effort in each state of nature, i.e., concavity of $\varphi(e,z)$ in e. In terms of the Mirrlees representation, this condition is equivalent to the distribution function $F(x,e)$ being quasiconvex in (x,e) (Jewitt 1988, Lemma 2).

Using the state space formulation, A's second-period consumption utility can be written as

$$\int_{\underline{x}}^{\overline{x}} u(c^*(x)+s) f(x,e)\, dx = \mathbb{E}[u(c^*(\varphi(e,z))+s)], \tag{25}$$

where $\mathbb{E}[\cdot]$ denotes expectations with respect to the state of nature z. To establish concavity of this expression in A's decision variables (e,s), notice that concavity is preserved under summation and under nondecreasing concave transformations. Hence, since u is nondecreasing concave, the agent's decision problem will be concave in (e,s) if $c^*(\varphi(e,z))$ is concave in e for all z.

This generates the following result.

Proposition 6. *Let $(w^*(\cdot), e^*, s^*)$ be a solution to* (P2). *Suppose that the following conditions* *hold:*

$$F(x,e) \text{ is quasiconvex in } (x,e), \tag{26}$$

$$f_e(x,e)/f(x,e) \text{ is nondecreasing and concave in } x \text{ for all } e, \tag{27}$$

$$g(c) := \left(\frac{1}{R\beta u'(c)} - \xi\alpha(c)\right) \text{ is increasing and convex in } c. \tag{28}$$

Then, given this contract, the agent's decision problem is concave. Hence, the contract solves (P1).

As an important application of Proposition 6, consider CARA utility: $u(c) = -\exp(-\alpha c)/\alpha$. Then we have $g(c) = (R\beta)^{-1}\exp(\alpha c) - \xi\alpha$. Obviously, this function is increasing and convex in c. Therefore, Proposition 6 validates the first-order approach for CARA utility when the distribution function $F(x,e)$ is quasiconvex in (x,e) and the likelihood ratio function $f_e(x,e)/f(x,e)$ is nondecreasing concave in x.[8]

There are other examples, such as CRRA utility, for which g is not convex, however. In that case, the concavity property of the likelihood ratio function formulated in (27) has to be strengthened to conclude that the first-order approach is valid. The details can be found in the appendix.

5 Proofs

Proof of Lemma 1. A's objective function is

$$(e,s) \mapsto u(c_0^* - \frac{s}{R}) + \beta \int_{\underline{x}}^{\overline{x}} u(c^*(x)+s) f(x,e)\, dx - \beta v(e). \tag{29}$$

Since u is concave, the first summand is concave in (e,s). Since v is convex, the third summand is concave in (e,s). □

Proof of Lemma 2. Using partial integration, A's second-period consumption utility can be

[8]In the present paper, preferences over consumption and effort are additively separable. Notice that for CARA utility and multiplicatively separable preferences, the validation of the first-order approach becomes much simpler, since the agent's effort choice will be independent of his wealth level. For most applications, this appears to be an unrealistic predicition, however.

rewritten as

$$\int_{\underline{x}}^{\overline{x}} u(c^*(x)+s)f(x,e)\,dx = u(c^*(\overline{x})+s) - \int_{\underline{x}}^{\overline{x}} (c^*)'(x)u'(c^*(x)+s)F(x,e)\,dx. \tag{30}$$

Hence, A's second-period consumption utility is concave in (e,s) if the function

$$(e,s) \mapsto -\int_{\underline{x}}^{\overline{x}} (c^*)'(x)u'(c^*(x)+s)F(x,e)\,dx \tag{31}$$

is concave, or equivalently if the function

$$(e,s) \mapsto \int_{\underline{x}}^{\overline{x}} (c^*)'(x)u'(c^*(x)+s)F(x,e)\,dx \tag{32}$$

is convex. We want to show that

$$g(e,s;x) = u'(c^*(x)+s)F(x,e) \tag{33}$$

is convex in (e,s) for all x. Since $(c^*)'(x) \geq 0$ by assumption, and since convexity is preserved under integration, this will imply convexity of (32).

The function $g(e,s;x)$ is convex in (e,s) if and only if its Hessian has a nonnegative diagonal and a nonnegative determinant. Omitting all arguments, the Hessian equals

$$H = \begin{pmatrix} F_{ee}u' & F_e u'' \\ F_e u'' & F u''' \end{pmatrix}. \tag{34}$$

The first diagonal entry is nonnegative by assumption. Condition (6) is equivalent to the statement that the determinant of H is nonnegative. In that case, the second diagonal entry of H must also be nonnegative. □

Proof of Theorem 1. By Lemma 1, it is sufficient to establish concavity of A's second-period consumption utility. Due to MLR and NIARA, the Kuhn-Tucker condition (3) implies that consumption $c^*(x)$ is continuously differentiable and nondecreasing in output x. Moreover, LCDF and NIARA imply that condition (6) from Lemma 2 is satisfied. Hence, A's second-period consumption utility is concave. □

Proof of Proposition 3. Direct consequence of Lemma 1 and Lemma 2. □

Proof of Proposition 4. As Lemma 1 shows, it is sufficient to establish concavity of

$$(e, s) \mapsto \int_{\underline{x}}^{\overline{x}} u(c^*(x) + s) f(x, e)\, dx. \tag{35}$$

This is equivalent to establishing convexity of

$$(e, s) \mapsto - \int_{\underline{x}}^{\overline{x}} u(c^*(x) + s) f(x, e)\, dx. \tag{36}$$

Using two steps of partial integration (see Conlon 2009), the latter function can be rewritten as

$$-u(c^*(\overline{x}) + s) + (c^*)'(\overline{x}) u'(c^*(\overline{x}) + s))\widetilde{F}(\overline{x}, e) + \int_{\underline{x}}^{\overline{x}} \left(-\frac{d^2(u(c^*(x) + s))}{dx^2}\right) \widetilde{F}(x, e)\, dx. \tag{37}$$

First, note that the expression $-u(c^*(\overline{x}) + s)$ is convex in (e, s) due to the concavity of u. Moreover, the expression

$$(c^*)'(\overline{x}) u'(c^*(\overline{x}) + s))\widetilde{F}(\overline{x}, e) \tag{38}$$

is convex in (e, s) by an argument similar to Lemma 2. For the third term in (37), note that

$$-\frac{d^2(u(c^*(x) + s))}{dx^2}\widetilde{F}(x, e) \tag{39}$$

is the product of a function that is log-convex in s and a function that is log-convex in e. Such products are convex in (e, s) as one easily verifies. Since convexity is preserved under integration, the third term in (37) is thus convex as well. This completes the proof. □

Proof of Lemma 5. Suppose $c^*(x)$ is concave in x. The function studied in (C1) can be represented as

$$-\frac{d^2(u(c^*(x) + s))}{dx^2} = \left(-(c^*)''(x)\right) u'(c^*(x) + s) + \left((c^*)'(x)\right)^2 \left(-u''(c^*(x) + s)\right). \tag{40}$$

The first summand in (40) is log-convex in s, since $-(c^*)''(x) \geq 0$ and since u' is log-convex due to NIARA. The second summand is log-convex in s, since $((c^*)'(x))^2 \geq 0$ and since $-u''$ is log-convex when $-u'''/u''$ is nonincreasing. Since log-convexity is preserved under summation

(Boyd and Vandenberghe 2004, p. 105), the function studied in (C1) is therefore log-convex in the variable s.

On the other hand, suppose that the function studied in (C1) is log-convex in s. As (40) shows, this does not imply that $(c^*)''(x)$ is nonpositive in general. □

Proof of Proposition 6. By Lemma 1, it is sufficient to consider A's second-period consumption utility. Moreover, due to quasiconvexity of the distribution function, the output technology can be represented by a production function $x = \varphi(e, z)$, with $\varphi(e, z)$ nondecreasing concave in effort e and nondecreasing in the stochastic state of nature z (Jewitt 1988, Lemma 2). Using this representation, we can write A's second-period consumption utility as

$$\int_{\underline{x}}^{\overline{x}} u(c^*(x) + s) f(x, e)\, dx = \mathbb{E}[u(c^*(\varphi(e, z)) + s)], \tag{41}$$

where $\mathbb{E}[\cdot]$ denotes expectations with respect to the state of nature z.

We claim that c^* is nondecreasing concave. Recall from the Kuhn-Tucker condition (3) that solutions to (P2) are characterized by

$$c^*(x) = g^{-1}\left(\lambda + \mu \frac{f_e(x, e^*)}{f(x, e^*)}\right), \tag{42}$$

with $g(c) = 1/(R\beta u'(c)) - \xi\alpha(c)$. By assumption, g is increasing and convex. Equivalently, g^{-1} is increasing and concave. Since $f_e(x, e^*)/f(x, e^*)$ is nondecreasing concave in x by assumption, this implies that $c^*(x)$ is nondecreasing concave in x.

Now, since $\varphi(e, z)$ is concave in e and $c^*(x)$ is nondecreasing concave in x, the composition $c^*(\varphi(e, z))$ is concave in e. Hence, the function $c^*(\varphi(e, z)) + s$ is concave in (e, s). Since u is nondecreasing concave, and since concavity is preserved under taking expectations, this completes the proof. □

6 Concluding remarks

This paper proposes a general method to validate the first-order approach for two-period moral hazard problems with hidden saving. Compared to the model without hidden saving, I additionally impose an assumption on the convexity of the agent's marginal utility of consumption and a restriction of Rogerson's (1985) CDF condition. I obtain alternative sets of sufficient conditions by relaxing the latter property and including conditions on the curvature of the wage scheme.

Given suitable properties of the value function, the present results obviously extend to multi-period versions of the problem. A characterization of the value function is beyond the scope of the present paper, however. Notice that, due to the hidden state and the endogenous probability distribution, this task is not straightforward.

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Appendix

A Concave consumption schemes

This section characterizes when the consumption scheme $c^*(x)$ solving the first-order problem (P2) is concave in output x. Due to equation (3), the consumption scheme is characterized by

$$c^*(x) = g^{-1}\left(\lambda + \mu L(x)\right), \tag{43}$$

with $g(c) = 1/(R\beta u'(c)) - \xi\alpha(c)$, $L(x) = f_e(x, e^*)/f(x, e^*)$. The first derivative of g equals

$$g'(c) = \frac{\xi u'''(c)u'(c) - ((R\beta)^{-1} + \xi u''(c))u''(c)}{u'(c)^2}, \tag{44}$$

which yields

$$(c^*)'(x) = \frac{\mu L'(x)u'(c^*(x))^2}{\xi u'''(c^*(x))u'(c^*(x)) - ((R\beta)^{-1} + \xi u''(c^*(x)))u''(c^*(x))}. \tag{45}$$

Omitting the arguments x and $c^*(x)$, the latter implies

$$\begin{aligned}(c^*)''(x) =& \frac{\mu}{(\dots)^2}\Big[\left(L''(u')^2 + 2L'u''u'(c^*)'\right) \left(\xi u'''u' - ((R\beta)^{-1} + \xi u'')u''\right) \\ &\qquad - L'(u')^2(c^*)' \left(\xi u^{(4)}u' - ((R\beta)^{-1} + \xi u'')u'''\right) \Big].\end{aligned} \tag{46}$$

Hence, given the assumption

$$(u')^2[\xi(u'''u' - (u'')^2) - u''(R\beta)^{-1}] > 0, \tag{47}$$

which is obviously true under NIARA, $c^*(x)$ is concave in x if and only if the likelihood ratio function satisfies the following concavity condition:

$$\begin{aligned}L'' \leq \frac{L'(c^*)'}{(u')^2[\xi(u'''u' - (u'')^2) - u''(R\beta)^{-1}]} &\Big[2\xi u'(-u'')(u'''u' - (u'')^2) + 2u'(u'')^2(R\beta)^{-1} \\ &+ \xi(u')^2(-u'')u''' + \xi(u')^3u^{(4)} - (u')^2u'''(R\beta)^{-1}\Big].\end{aligned} \tag{48}$$