A Quest for Knowledge*

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Abstract

Is more novel research always desirable? We develop a model in which knowledge shapes society's policies and guides the search for discoveries. Researchers select a question and how intensely to study it. The novelty of a question determines both the value and the difficulty of discovering its answer. We show that the benefits of discoveries are non-monotone in novelty. Through a dynamic externality, moonshots—research on questions more novel than myopically optimal—can improve the evolution of knowledge. Incentivizing moonshots requires promising ex-post rewards. However, even a myopic funder combines rewards with ex-ante cost reductions to increase research effort.

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[Evolution] comes through asking the right questions, because the answer pre-exists. ... You don't invent the answer. You reveal the answer.

Jonas Salk, discoverer of the polio vaccine

1 Introduction

In a letter to Franklin D. Roosevelt, Vannevar Bush (1945) pleads with the president to preserve freedom of inquiry by federally funding basic research—the "pacemaker of technological progress." That letter paved the way for the creation of the National Science Foundation (NSF) in 1950. The NSF today, like the vast majority of governments and scientific institutions, cherishes scientific freedom and allows academic researchers to select research projects independently.

With scientific freedom comes the responsibility for "asking the right questions" that Jonas Salk refers to in the epigraph. However, what are the right questions? Biologist and Nobel laureate Peter Medawar (1967) famously notes that "research is surely the art of the soluble. ... Good scientists study the most important problems they think they can solve." Finding the most important yet soluble question is nontrivial. One reason is that both importance and solubility depend on the current state of knowledge (see, for example, Iaria, Schwarz, and Waldinger, 2018).

In this paper, we develop a microfounded model of knowledge creation through research. Our model captures (i) the role of existing knowledge in determining the benefits and cost of research, (ii) the spillovers a discovery creates on conjectures about similar questions, and (iii) the researcher's freedom to choose what question to study and how intensely to study it. We characterize the researcher's choices and address classical questions of science funding: Should we incentivize researchers to study questions far beyond the current knowledge frontier? Do such moonshots improve the evolution of knowledge? When and how should a budget-constrained funder incentivize innovative research?

We model the value of knowledge as the extent to which it improves decision making.[1] We represent society by a single decision maker who faces problems that correspond to questions. In her response to these problems, she uses the public good of knowledge. Knowledge is the set of questions to which the answer has already been discovered. Because answers to similar questions are correlated, knowledge also provides the decision maker with conjectures regarding questions to which the answer is undiscovered. The precision of a conjecture depends on the question's location relative to existing knowledge.[2] We conceptualize the correlation by assuming that answers to questions follow the realization of a Brownian path. Figure 1 depicts that idea. Questions are on the horizontal axis, and the gray line represents the answers to all questions. Dots (●) represent existing knowledge. Because of the assumption of a Brownian path, all conjectures follow a normal distribution. The mean and the variance depend on existing knowledge. The solid black lines in Figure 1 represent the mean; the dashed lines provide the band of the 95 percent prediction interval.[3]

[1] Following, for example, Jacob Marschak (1974)'s "Knowledge is useful if it helps to make the best decisions". Hjort et al. (2021) provide evidence on how knowledge creation improves decision making.

[2] At the end of the introduction, we describe how our model applies to the protein folding problem in structural biology. An example of the spillovers we have in mind is the COVID-19 vaccine development by Moderna, which "took all of one weekend." The speed was a direct consequence of similarities of relevant proteins on the virus, SARS-CoV-2, with already discovered proteins (see, for instance, in This American Life (2020)).

[3] The 95 percent prediction intervals depend on existing knowledge and describe the following relation: for each question, with a probability of 95 percent, the answer lies between the respective dashed lines

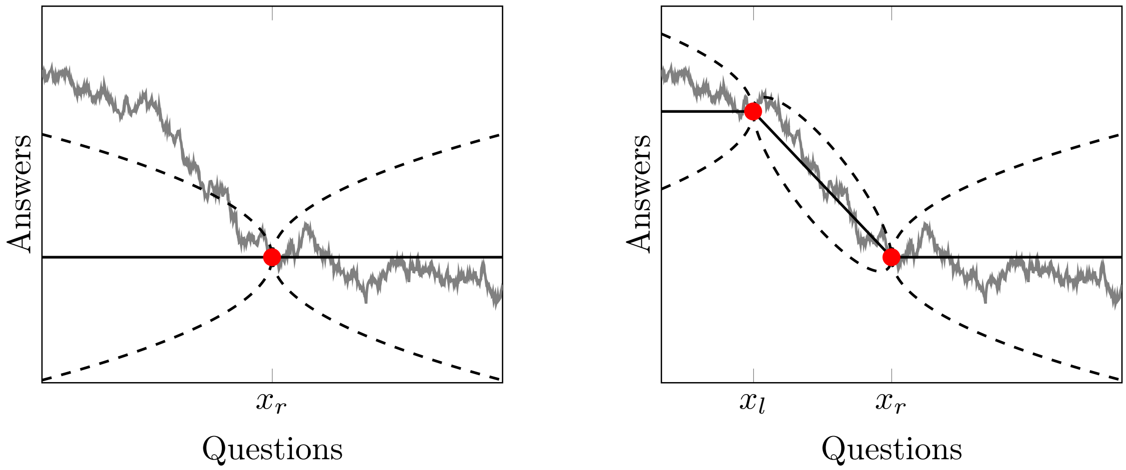


Figure 1: *Existing knowledge and conjectures.*

Our first contribution is to characterize the benefits of a discovery. To gain intuition, consider the left panel of Figure 1. Only the answer to question x_r is known. Assume that researchers discover the answer to question x_l. We now move to the right panel. Decision making improves in three ways. First, the decision maker has precise knowledge about the answer to x_l. Second, her conjectures about all questions to the left of x_r improve. Third, her conjectures improve the most in the newly created *area of questions* $[x_l, x_r]$ in which the decision maker now has two pieces of knowledge that help her form conjectures.

The benefit of discovering the answer to x_l depends on the question's distance from x_r. The effect of an increase in the distance between x_l and x_r is similar to the effect of output expansion by a monopolist. Consider first a discovery close to existing knowledge which implies a narrow area $[x_l, x_r]$. There are only few questions in the area, but the conjectures about them are precise; that is, the variance of the conjectures is low. As the distance of x_l from x_r increases, more questions lie inside the area—a marginal gain. At the same time, the conjectures become less precise—an inframarginal loss. The benefit of a discovery is maximized at an intermediate distance.

If both x_l and x_r are known initially, discoveries *advance knowledge* beyond the frontier if the discoveries concern questions $x \notin [x_l, x_r]$. Advancing beyond the frontier works in the manner described in the paragraph above. Alternatively, discoveries *deepen knowledge* if they concern questions $x \in [x_l, x_r]$. Depending on the distance between x_l and x_r, advancing knowledge or deepening knowledge may be optimal. If x_l and x_r are close, knowledge is dense: the conjecture about any question in $[x_l, x_r]$ is already precise. In this case, advancing knowledge beyond the frontier provides larger benefits than deepening knowledge does. If x_l and x_r are far apart, knowledge is sparse: conjectures about questions in $[x_l, x_r]$ are imprecise. Obtaining an answer to a question $x \in [x_l, x_r]$ divides this single area of imprecise conjectures into two areas with precise conjectures. In this case, deepening knowledge provides larger benefits than advancing knowledge beyond the frontier does.

Overall, the largest benefit comes from deepening knowledge between distant, yet not too distant, pieces of knowledge. Advancing knowledge beyond the frontier beats deepening knowledge in an existing area only if all available areas are short.

Our second contribution is to characterize a researcher's optimal choice of which

given existing knowledge.

question to tackle and how much effort to invest in studying that question. We assume that the researcher's benefits of a finding are proportional to the benefits of a discovery discussed above. In addition, we conceptualize the research process as the search for an answer. We assume that it requires effort to search for an answer and that the cost of effort is increasing and convex. We propose a cost function that derives from this idea and provide a microfoundation. The cost function relates the cost of research to *novelty* (the distance of the question to existing knowledge) and *output* (the probability that search results in discovery). The link originates in the initial conjecture, which depends on the novelty of the question. The more precise that conjecture, the higher the output for any given level of effort.

When characterizing the researcher's choices, we show that novelty and output are non-trivially related. The relation depends on the structure of existing knowledge. In particular, novelty and output can be substitutes or complements. As a consequence, more novelty does not always come at the cost of higher risk: whenever novelty and output are complements, the additional benefit from higher novelty induces the researcher to exert substantially more effort. If the question lies in a short area, novelty and output are complements. In this case, the more novel the question, the higher the probability that the researcher will discover the answer. The benefits of discovery increase in novelty, yet the cost remains low. If the question lies within a larger area, novelty and output are substitutes for small levels of novelty but complements for intermediate levels. In this case, at the boundaries of the area, x_l and x_r, the marginal cost of output increases relatively fast in novelty, causing the researcher to reduce output in response to an increase in novelty. However, moving away from one end of the area means moving toward the other end. While moving away from one end reduces the precision of conjectures, moving toward the other attenuates that reduction. As novelty increases, the attenuation becomes stronger, mitigating the marginal-cost effect. Eventually, the marginal-benefit increase dominates: novelty and output become complements. As the area length increases further, the region in which novelty complements output shrinks. If the researcher advances knowledge beyond the frontier, novelty and output are substitutes throughout.

In general, output is higher when the researcher deepens knowledge than when she advances beyond the frontier. Output peaks in areas of intermediate length. In such areas, the researcher pursues the question at the midpoint of the area.

Our third contribution is to apply our previous insights to the design of research incentives. We consider a budget-constrained funder facing a sequence of researchers. We assume that the funder has access to two instruments to affect the researcher's choices: (i) reducing the researcher's cost ex ante and (ii) rewarding the researcher for discoveries ex post. We characterize the set of novelty-output pairs that the funder can implement and show that there can be complementarities between output and novelty. In particular, a cost-neutral increase in induced novelty may increase induced output. Therefore, a funder who maximizes the myopic benefits to society may find it optimal to incentivize excessive novelty, even if only to increase the output.

To address the incentives of a forward-looking funder, we first determine under what circumstances moonshot discoveries—discoveries far from existing knowledge—are desirable. Moonshot discoveries are suboptimal in the short run. They create knowledge that is too disconnected from existing knowledge and therefore provide little immediate benefit. However, moonshots guide future researchers aiming at questions between the moonshot and previously existing knowledge. As a result of the moonshot, future researchers increase their output, and knowledge created over time becomes more valuable than otherwise. If the funder is patient and the cost of research is intermediate, the positive dynamic externality of moonshots dominates the implied myopic loss. We show that a forward-looking funder must provide ex post rewards to incentivize moonshots—even when she could eliminate the cost entirely.

To summarize, we make three contributions. First, we offer a framework that endogenously links typical measures of research (novelty and output) to typical premises about the research process (selection from a large pool of questions, conjectures determined by existing knowledge, and the need for costly effort to obtain a discovery). We generate this link by conceptualizing the discovery process of a Brownian path as a search for realized values guided by conjectures that build on known realizations. Second, we shed light on the nontrivial relation between the novelty of the research and the expected research output. We show that whether the two are complements or substitutes crucially depends on the structure of existing knowledge. To do so, we characterize the researcher's optimal policy as a function of existing knowledge. Third, we provide new insights into two classical questions in the science of science funding: (i) Should society incentivize research far beyond the frontier even if the immediate benefits of such a discovery are low? Yes, if the cost of research is intermediate and society is patient. (ii) Which mode of funding provides larger expected benefits: ex ante cost reductions or ex post rewards? It depends. A funder that aims to maximize the benefits of research may strictly prefer to combine both modes. A funder that aims at incentivizing research far beyond the frontier has to offer ex post rewards.

Illustration: The Protein Folding Problem. To illustrate our model idea, consider the protein folding problem in structural biology. Besides being an important research field in itself, structural biology has received increasing attention from economists. See, for example, Hill and Stein (2020, 2021) who also provide an excellent overview and further details of this research field.

A protein's three-dimensional structure derives from its underlying amino-acid sequence. However, how a protein's amino-acid sequence translates into its structure is a complex problem. Structural biologists work on experimentally discovering such structures, which is a difficult time- and resource-intensive process. Knowledge of a protein's structure provides valuable information about the protein's functionality and, for example, possible treatments of a disease linked to the protein. The accumulated knowledge of protein structures is accessible via the Protein Data Bank (PDB), which contains more than 150,000 experimentally determined protein structures.[4]

Importantly, knowing a particular protein's structure is valuable beyond its immediate applications. Proteins with similar amino-acid sequences are likely to have similar structures as well (see, for example Berman, 2008). Hence, the knowledge of known mappings from amino-acid structures feeds into predicting the structure of proteins with similar amino-acid sequences (Deep Mind's AlphaFold being one particularly successful prediction tool, see Senior et al., 2020). Such predictions are used both to guide experimental verification of structures and to search for treatments and vaccines.[5] Providing structure-predicting researchers with the best possible data to quickly predict the structure of essential proteins is therefore highly valuable.

Suppose that we can construct a one-dimensional real-valued amino-acid similarity index for a protein family. Then, each protein can be considered as a question in our model. The answer for each protein is its three-dimensional structure. As in our model,

[4]Notably, the PDB is open-source and academic journals often require the publication of identified protein structures in the PDB to make it accessible to other researchers worldwide.

[5]See the discussion in Oxford Protein Informatics Group (2021).

proteins with similar amino-acid sequences are expected to have similar structures. The existing knowledge, i.e., those proteins with known structures, is stored in the PDB. It is used to develop treatments against diseases (a value to society) and to guide research about similar proteins with still unknown structures (by forming conjectures).

1.1 Related Literature

Ample empirical literature in the science of science has documented the importance of novelty and output for progress in science. Fortunato et al. (2018) provide an extensive summary of it. The importance of (accessible) pre-existing knowledge for research purposes is documented, for example, in Iaria, Schwarz, and Waldinger (2018). We aim to complement the (quasi-)experimental approach in these papers by providing a simple yet flexible formal model based on few parameters to make it identifiable and testable.

Our model relies on the expanding literature following Callander (2011a) that uses Gaussian processes to model the search for answers. We differ in our notion of both benefits and costs, which leads to novel incentives and results. Similar to Garfagnini and Strulovici (2016), Callander and Clark (2017), and Bardhi (2019) payoffs are determined by the realization of an entire Brownian path. However, the value of incomplete knowledge differs, and thus the value of learning an additional point also differs. In our model, the value of answering one additional question is how much it reduces the uncertainty about other, unknown answers. Hence, the added benefit of answering a question is detached from the answer itself leading—in contrast to their models—to continuation strategies independent of the realized answer. That independence is key to gaining tractability, especially when we address the interplay between funding and the evolution of knowledge. Moreover, our notion of cost leads to an endogenous success probability that depends on the question chosen and the existing knowledge.

Similar to Garfagnini and Strulovici (2016), the researcher's cost to discover a certain point depends on the relation of that point to the existing knowledge.[6] However, the cost-generating process differs significantly. In Garfagnini and Strulovici (2016) cost depend only on whether the search occurs inside or outside the knowledge frontier. In our model, the agent decides on the intensity of her search for an answer. Her intensity choice affects both benefits and cost.

The differences in benefits and cost allow us to answer novel questions. How does knowledge accumulate over time? What is the probability that a researcher succeeds to answer their self-chosen research questions given their self-chosen intensity?

Our findings relate to other theoretical models in the science of science literature that consider particular aspects of the scientific process we have in mind.[7] Aghion, Dewatripont, and Stein (2008) consider a setting in which progress has a predefined step-by-step sequential structure. To advance to the next question, a particular prior question has to be answered. We offer greater flexibility in that we posit that any question can—in principle—be addressed at any time. However, the benefits from a discovery and

[6] Jovanovic and Rob (1990), too, study the choice between expanding and deepening research. In Jovanovic and Rob (1990), expanding implies an i.i.d. draw at a fixed cost while deepening is costless. In our model, by contrast, all questions are connected and cost depends on both existing knowledge and the degree of novelty. Moreover, see Callander and Hummel (2014), Callander, Lambert, and Matouschek (2018), Callander and Matouschek (2019), and Bardhi and Bobkova (2021) for applications different from ours in a related framework. Some of the results in Section 5 are reminiscent of Callander (2011b) in a different context.

[7] There is a literature orthogonal to ours that views science as establishing links between known answers (for example, Rzhetsky et al., 2015). Our model is complementary. We consider research as the search for answers given a reasonable estimate of the link.

the effort needed for the discovery depend on previous work.[8] Bramoullé and Saint-Paul (2010) model the decision of a researcher to deepen knowledge in a given area or to advance the knowledge frontier. The main driver in their model is the assumption that as an area gets increasingly crowded, the reputation a researcher gains from new developments in that area declines.[9] We offer a decision-based microfoundation that provides a measure of uncertainty, in line with Frankel and Kamenica (2019). It reaches a similar conclusion: as the opportunities in the area become increasingly narrow, the informational content of an additional finding decreases, and hence its value does too. However, unlike in Bramoullé and Saint-Paul (2010), the researcher in our model has more discretion, as she chooses not only the area but also the degree of novelty and the level of research intensity, which directly determines the probability of success. Both choices are continuous, and shrinking the research area may be beneficial if it leads to better conjectures by closing the gap between existing pieces of knowledge.[10]

Prendergast (2019) studies agency concerns when incentivizing creative innovation. His model relates to ours on a high level, yet the mapping is not straightforward. Neither model nests the other. Under a particular narrow structure of existing knowledge in our model and a specific assumption on the agent's reduced-form cost in his, the two models overlap. Beyond this intersection, however, the scope, the modeling choices, and the focus differ. Unsurprisingly, results differ too.

1.2 Roadmap

Our model of research builds on several fundamental ingredients that lead us to the researcher's objective function via intermediate results. Therefore, we chose a step-by-step approach in setting up the model of a researcher to emphasize the role of each model ingredient. With each additional model feature, we analyze the respective consequences. In Section 2, we provide the basic model of knowledge and society's decision making. In Section 3, we obtain our first result—a characterization of the benefits of a discovery to society and of the benefit-maximizing research questions given any existing knowledge. In Section 4, we introduce a researcher and construct her objective based on the benefits of discoveries to society and a microfounded cost function. Building on this objective, we obtain our second result—a researcher's optimal choices and the properties thereof. In Section 5, we analyze the interaction between a long-lived decision maker who incentivizes a series of short-lived researchers—our third result. Section 6 summarizes and provides an outlook on other applications.

[8] To (ab)use Newton's metaphor: Any researcher can build a ladder to see farther, but the effort required depends on the existing giants' shoulders. Related ideas appear in Scotchmer (1991), Aghion et al. (2001), Bessen and Maskin (2009), and Bryan and Lemus (2017).

[9] Similar to Bramoullé and Saint-Paul (2010), we model innovation as a public good. That differentiates us from most models of R&D competition. Yet, similarly to, for example, Letina (2016), Letina, Schmutzler, and Seibel (2020), and Hopenhayn and Squintani (2021), we assume that progress corresponds to successful search in an ocean of possibilities. Unlike in those approaches, in our setting, benefit and cost depend on the question's relation to existing knowledge.

[10] Other recent theoretical work studies frictions in the scientific process that we abstract from. Bobtcheff, Bolte, and Mariotti (2017), Akerlof and Michaillat (2018), and Andrews and Kasy (2019) study inefficiencies due to the publication process, career concerns, or homophily. Hill and Stein (2020, 2021) provide empirical counterparts. Frankel and Kasy (2021) provide a normative justification. Similar to us, Liang and Mu (2020) look at (a sequence of) myopic researchers aiming to discover the truth. Unlike us, they focus on the choice of the learning technology and show that depending on the complementarities between technologies, researchers may persistently select an inefficient technology.

2 A Model of Knowledge

We set up a model of knowledge with the following desired properties:

(i.) Knowing the answer to a question informs conjectures about other questions.

(ii.) The distance between two questions determines the impact that answering one question has on the conjecture about the other question.

(iii.) The set of available questions is unbounded.

(iv.) Knowledge informs decision making.

We first set up a formal model of knowledge and then introduce society as a decision maker that applies knowledge in its decision-making process.

Questions and answers. We represent the universe of questions by the real line, $\mathbb{R}$.[11] A specific *question* is an element $x \in \mathbb{R}$. Each question x has precisely one *answer*, $y(x) \in \mathbb{R}$. A question-answer pair $(x, y(x))$ is thus a point in the two-dimensional Euclidean space.

The answer $y(x)$ to question x is determined by the realization of a random variable, $Y(x) : \mathbb{R} \to \mathbb{R}$. We provide more structure for $Y(x)$ below.

Truth and knowledge. *Truth* is the collection of all question-answer pairs. It is the graph of the realization of the random variable $Y(x)$, over its domain $\mathbb{R}$. *Knowledge* is the finite collection of known question-answer pairs. We denote it by $\mathcal{F}_k = \{(x_i, y(x_i))\}_{i=1}^k$. For notational convenience, we assume that $\mathcal{F}_k$ is ordered such that $x_i < x_{i+1}$. We refer to x_1 and x_k as the *frontier* of current knowledge.

The key assumption of our model of knowledge concerns the truth-generating process $Y(x)$. We assume that $Y(x)$ follows a standard Brownian motion defined over the entire real line.[12] This assumption captures the notion that the answer to question x is likely to be similar to the answer to a close-by question x'. As the distance between x and x', $|x - x'|$, increases, the uncertainty increases. Yet a correlation remains.

Knowledge $\mathcal{F}_k$ implicitly determines a partition of the real line consisting of $k + 1$ elements

$$\mathcal{X}_k := \{(-\infty, x_1), [x_1, x_2), \cdots, [x_{k-1}, x_k), [x_k, \infty)\}.$$

In what follows, we make frequent use of the interval length X_i of an element of the partition that a particular question $x \in [x_i, x_{i+1})$ is part of.

We introduce the following terminology. We refer to each element of the partition $\mathcal{X}_k$ as an *area*. We call $(-\infty, x_1)$ area 0, $[x_1, x_2)$ area 1, and so on until area k, which is $[x_k, \infty)$. The length of area $i \in \{1, .., k-1\}$ is $X_i := x_{i+1} - x_i$, and $X_0 = X_k = \infty$.

Conjectures. A *conjecture* is the cumulative distribution function $G_x(y|\mathcal{F}_k)$ of the answer $y(x)$ to question x given knowledge $\mathcal{F}_k$. Conjectures about questions to which

[11] Our assumption implies that the relation between any two questions can be represented in a single dimension. We think of our universe of questions as being within one specific and mature discipline such as the protein-folding problem from the Introduction. In Appendix G, we discuss how our results generalize and present a model that includes *seminal discoveries*—discoveries that open up a new field of research.

[12] As in Callander (2011a), the realized truth Y is a random draw from the space of all possible paths $\mathcal{Y}$ generated by a standard Brownian motion going through some initial knowledge point $(x_0, y(x_0))$. While the process has been fully realized at the beginning of time, knowledge is the filtration known to the observer $\mathcal{F}_k$. We choose a standard Brownian path with 0 drift and variance of 1 for convenience only. Our model extends naturally to other Gaussian processes. The x dimension should not be confused with a sequential structure of finding answers. Any question-answer pair $(x, y(x))$ is discoverable.

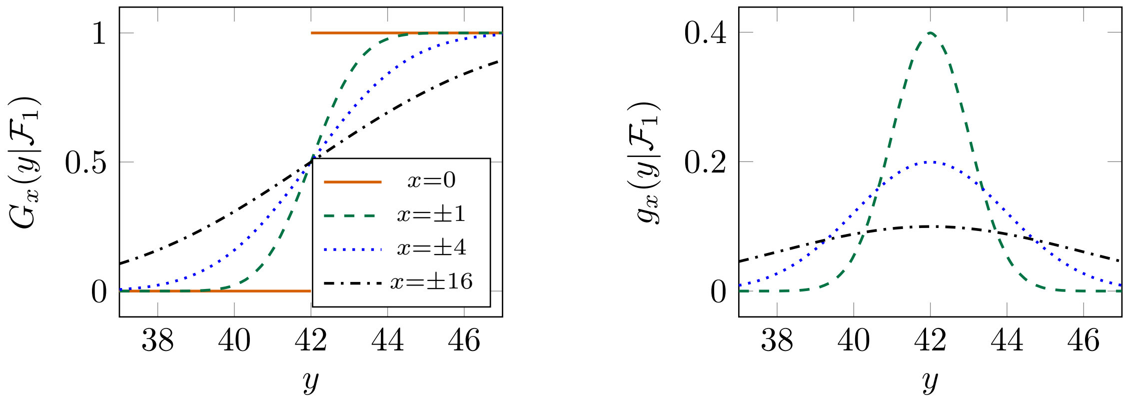


Figure 2: *Distributions of answers for different distances to knowledge when* $\mathcal{F}_1 = (0, 42)$. Given that the only question to which the answer is known is $x = 0$, we can determine knowledge about questions of distances 1, 4, and 16 from $x = 0$. All answers have the same mean (42), but the variance and thus the precision of the conjecture differ. For $x = 0$, the answer is known and $G_0(y|\mathcal{F}_1)$ is a step function. Questions with longer distances have larger variances. The left panel depicts the respective distribution functions; the right panel depicts the densities.

the answer is known are trivial; if $(x_i, y(x_i)) \in \mathcal{F}_k$, then $G_{x_i}(y|\mathcal{F}_k) = \mathbf{1}_{y \geq y(x_i)}$, a right-continuous step function jumping to 1 at $y = y(x_i)$. The conjecture for a yet-to-be-discovered $y(x)$, $G_x(y|\mathcal{F}_k)$, is also well defined. Because $Y(x)$ is determined by a Brownian motion, $G_x(y|\mathcal{F}_k)$ is a cumulative distribution function of a normal distribution with mean $\mu_x(Y|\mathcal{F}_k)$ and variance $\sigma_x^2(Y|\mathcal{F}_k)$. Both μ_x and σ_x^2 follow immediately from the properties of the Brownian motion.

Property 1 (Expected Value)**.** Given knowledge $\mathcal{F}_k$, the conjecture $G_x(y|\mathcal{F}_k)$ has the following mean:

$$\mu_x(Y|\mathcal{F}_k) = \begin{cases} y(x_1) & \text{if } x < x_1 \\ y(x_i) + \frac{x - x_i}{X_i}(y(x_{i+1}) - y(x_i)) & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ y(x_k) & \text{if } x \geq x_k. \end{cases}$$

Property 2 (Variance)**.** Given knowledge $\mathcal{F}_k$, the conjecture $G_x(y|\mathcal{F}_k)$ has the following variance:

$$\sigma_x^2(Y|\mathcal{F}_k) = \begin{cases} x_1 - x & \text{if } x < x_1 \\ \frac{(x_{i+1} - x)(x - x_i)}{X_i} & \text{if } x \in [x_i, x_{i+1}), i \in \{1, ..., k-1\} \\ x - x_k & \text{if } x \geq x_k. \end{cases}$$

Figure 2 illustrates the distributions for different distances from existing knowledge assuming that knowledge is $\mathcal{F}_1 = (0, 42)$. In Appendix C, we provide a graphical example that highlights the ingredients of our model of knowledge and how adding additional question-answer pairs to knowledge influences conjectures.

2.1 Society and Decision Making

We represent society by a single decision maker. That decision maker observes knowledge $\mathcal{F}_k$ and takes a continuum of actions—one for each question $x \in \mathbb{R}$. For each question x, she can either take a *proactive* action $a(x) \in \mathbb{R}$ or select an outside option $a(x) = \varnothing$—for example, the act of doing nothing. The decision maker's choice is thus represented by a function $a : \mathbb{R} \to \mathbb{R} \cup \varnothing$.

The expected payoff of selecting the outside option $a(x) = \varnothing$ is finite, safe (that is, independent of $y(x)$), and question-invariant. We normalize it to 0. The choice of $\varnothing$ reflects the idea that for a subject on which very little is known, it is not wise to take proactive actions. One interpretation of preferring to choose $\varnothing$ is what is known as the precautionary principle: if uncertainty is large, prudence is preferred over risking a poor proactive choice.[13] The payoff of addressing a question x proactively is represented by a monotone transformation of the quadratic loss around the true answer to question x, $y(x)$. The decision maker's payoff on a particular question x from action $a(x)$ is

$$u(a(x); x) = \begin{cases} 1 - \frac{(a(x)-y(x))^2}{q} & \text{if } a(x) \in \mathbb{R} \\ 0 & \text{if } a(x) = \varnothing, \end{cases}$$

for a given $q > 0$. The scaling parameter q measures the error tolerance of the decision maker: if the proactive choice $a(x)$ is less than $\sqrt{q}$ away from the optimal choice—the true answer $y(x)$—the decision maker prefers the proactive choice over the outside option.

To keep the analysis focused, we abstract from any prioritization the decision maker might have among different questions; that is, we assume that the decision maker values all questions equally. If $a(x)$ is such that $u(a(x); x)$ is (locally) integrable, then total payoffs to the decision maker are given by

$$\int_{-\infty}^{\infty} u(a(x); x) dx.$$

Technically, it is the outside option $\varnothing$ and the finiteness of knowledge that ensure a bounded payoff at the optimum.[14] Thus, the outside option guarantees that knowledge contributes in a quantifiable way to the decision maker's total payoff.

3 The Benefits of Discovery

Discovery occurs whenever an answer is found and the new question-answer pair is added to existing knowledge $\mathcal{F}_k$. In this section, we formulate a measure of the benefits of discovery for the decision maker.

3.1 The Value of Knowledge

Knowledge informs decision making. For each question x, the decision maker uses the conjecture $G_x(y|\mathcal{F}_k)$ to decide on $a(x)$. Suppose the decision maker addresses a question x proactively, i.e., $a(x) \neq \varnothing$. Her expected payoff for that question is

[13] What we have in mind as outside options are longstanding policies with a finite expected payoff where the decision maker decides whether to revise her policy. Take the discussion about how to respond to climate change: Since the Kyoto Protocol, decision makers have reevaluated policies on several issues (for example, transportation, energy, protection of nature). For each issue, they use current knowledge and decide whether to continue with business as usual or to change policy.

[14] The decision maker's problem is straightforwardly solved pointwise resulting in a sufficiently well-behaved per-question payoff $u(\cdot)$ to ensure integrability. An alternative assumption to facing all questions is that the decision maker faces a single question at random. However, in the case of a uniform distribution—which would resemble equal weighting of questions—we need to restrict attention to draws from a large subset, $[\underline{x}, \overline{x}]$, of the set of all questions $\mathbb{R}$. If the subset from which the questions are drawn is large enough, the two assumptions are equivalent when the decision maker acts optimally. Other weighting functions on questions are straightforward to incorporate; yet, they come at a significant cost of clarity in the analysis.

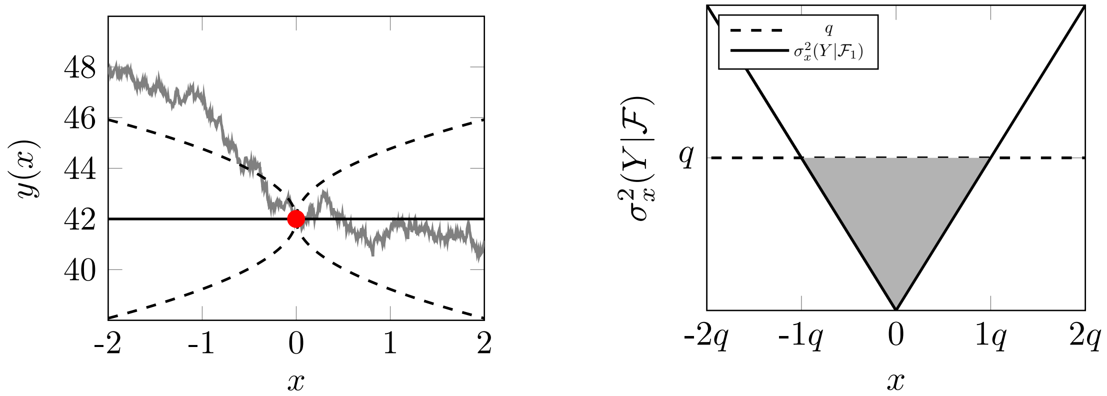


Figure 3: *The value of knowing $\mathcal{F}_1$.* The left panel depicts the same situation as the left panel in Figure 1. Only the answer to question 0 is known. The right panel depicts the the value of knowledge $v(\mathcal{F}_1)$. It is proportional to the shaded part.[a] The variance $\sigma_x^2(Y|\mathcal{F}_k) = |x|$ is the Euclidean distance from 0. The expected payoff from taking an action equal to the mean of the conjecture, $a = \mu_x = 42$, is the vertical distance between the action and the dashed line. For $|x| \leq q$, $a = \mu_x = 42$ is preferred to $a = \varnothing$.

[a]The shaded part is proportional as we have to multiply it with the normalizing factor $1/q$. This holds throughout this section.

$$Eu(a \neq \varnothing; x|\mathcal{F}_k) = \int 1 - \frac{(a - y(x))^2}{q} \mathrm{d}G_x(y|\mathcal{F}_k).$$

Because of the quadratic loss, the optimal action in this case corresponds to the mean of the distribution, $\mu_x(Y|\mathcal{F}_k)$ with payoff

$$\begin{aligned} Eu(\mu_x(Y|\mathcal{F}_k); x|\mathcal{F}_k) &= \int 1 - \frac{(\mu_x(Y|\mathcal{F}_k) - y(x))^2}{q} \mathrm{d}G_x(y|\mathcal{F}_k) \\ &= 1 - \frac{\sigma_x^2(Y|\mathcal{F}_k)}{q}. \end{aligned}$$

Addressing the question proactively is therefore optimal only if $\sigma_x^2(Y|\mathcal{F}_k) \leq q$, that is, only if the decision maker's conjecture is sufficiently precise. Otherwise, the decision maker prefers the outside option, $a(x) = \varnothing$, with payoff 0.

The decision maker's optimal policy is thus

$$a^*(x) = \begin{cases} \mu_x(Y|\mathcal{F}_k), & \text{if } \sigma_x^2(Y|\mathcal{F}_k) \leq q \\ \varnothing, & \text{if } \sigma_x^2(Y|\mathcal{F}_k) > q \end{cases}.$$

This implies a total expected payoff to the decision maker given existing knowledge $\mathcal{F}_k$ of

$$v(\mathcal{F}_k) := \int_{-\infty}^{\infty} Eu(a^*(x); x|\mathcal{F}_k)\mathrm{d}x = \int_{-\infty}^{\infty} \max\left\{\frac{q - \sigma_x^2(Y|\mathcal{F}_k)}{q}, 0\right\} \mathrm{d}x.$$

We refer to $v(\mathcal{F}_k)$ as the *value of knowing* $\mathcal{F}_k$; that is, $v(\mathcal{F}_k)$ is the decision maker's gain from following the optimal policy given knowledge $\mathcal{F}_k$ compared with refraining from any proactive choices—that is, $\forall x \in \mathbb{R}\ a(x) = \varnothing$.

The right panel of Figure 3 provides a graphical representation of $v(\mathcal{F}_1)$. The left panel of Figure 4 represents $v(\mathcal{F}_2)$, the right panel of Figure 4 represents $v(\mathcal{F}_4)$.

3.2 The Benefits of Discovery

The benefits of a discovery come in the form of an enhanced value of knowledge. Formally, adding $(x, y(x))$ to $\mathcal{F}_k$ provides the benefit

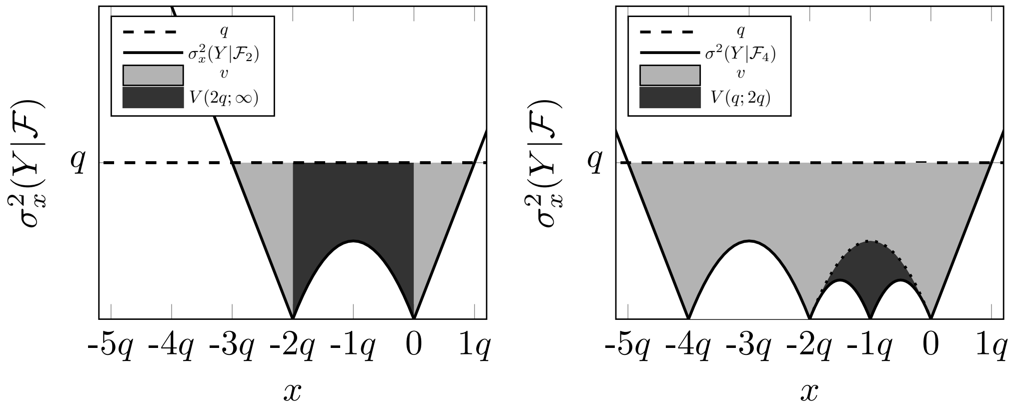


Figure 4: *The benefits of discovery.*
LEFT PANEL: BENEFIT OF A KNOWLEDGE-EXPANDING DISCOVERY. Benefits of discovering the answer to question $-2q$ when initial knowledge was $\mathcal{F}_1 = (0, y(0))$. Outside the frontier—where $x \notin [-2q, 0]$—the variance is $\sigma^2_x(Y|\mathcal{F}_k) = d(x)$. Inside, it is smaller with $\sigma^2_x(Y|\mathcal{F}_k) = d(x)(X - d(x))/X$, where $X = 2q$ is the length of the interval $[-2q, 0]$. The value of $\mathcal{F}_2$ is proportional to the entire shaded part. The net benefit of discovering the answer to question $x = -2q$ is proportional to the dark-shaded part.
RIGHT PANEL: BENEFIT OF KNOWLEDGE-DEEPENING DISCOVERY. Value of knowledge and the benefit of discovery when the research deepens knowledge by discovering the answer to question $x = -q$. The total value of $\mathcal{F}_4$ is proportional to the entire shaded part. The net benefit of discovering the answer to question $x = -q$ is proportional to the dark-shaded part (relative to pre-existing knowledge $\mathcal{F}_3$).

$$V(x; \mathcal{F}_k) := v(\mathcal{F}_k \cup (x, y(x))) - v(\mathcal{F}_k).$$

The value of the discovery depends on the question being answered, x, and on existing knowledge $\mathcal{F}_k$. We distinguish two scenarios: *expanding* knowledge beyond the frontier and *deepening* knowledge in an area. A discovery $y(x)$ expands knowledge if $x \notin [x_1, x_k]$. A discovery $y(x)$ deepens knowledge in area i if $x \in [x_i, x_{i+1}]$.

We first state the benefits-of-discovery function. Three corollaries to that statement contain our first contribution. The corollaries characterize the properties of the benefits of discovery including a characterization of the benefits-maximizing discoveries as a function of existing knowledge. The two main factors determining the benefits are the *distance from knowledge*, which we formally define below, and the *research area*. Area length X is a sufficient statistic for the research area.

Definition 1 (Distance)**.** The distance of question x from knowledge $\mathcal{F}_k$ is the minimal Euclidean distance to a question to which the answer is known:

$$d(x) := \min_{\xi \in \{x_1, x_2, \dots x_k\}} |x - \xi|$$

Definition 2 (Variance)**.** The variance of a question with distance d in an area of length X is

$$\sigma^2(d; X) := d(X - d)/X.$$

Note that $\sigma^2(d; X) = \sigma_x(Y|\mathcal{F}_k)$ whenever $d(x) = d$ and x is in an area of length X. This allows us to simplify notation and to focus on the variables d and X exclusively rather than keeping track of the exact question x and its research area. We abuse notation by stating the benefits of discovery as $V(d; X)$.

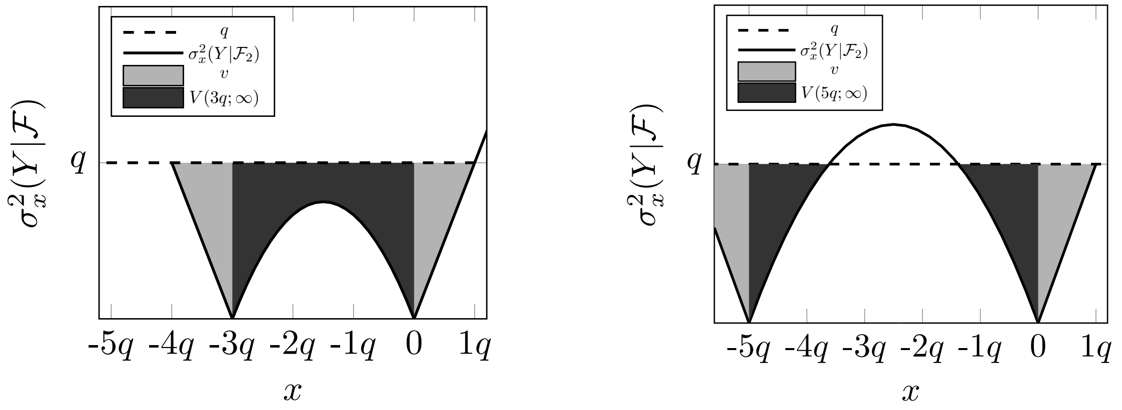


Figure 5: *Benefit-maximizing (left) and too large (right) distance of x given $\mathcal{F}_1$.* Given that the value of doing nothing is $q = 1$, the benefit-maximizing distance when expanding knowledge is $d = 3q$. The left panel depicts the benefit-maximizing choice, given $\mathcal{F}_1$, when expanding to the negative domain and $d = 3q$ and thus $x = -3$; the right panel shows the effect of a choice that is too far away ($x = -5, d = 5q$). The gain in knowledge $V(d;\infty)$ is proportional to the dark-shaded part. It is larger in the left panel than in the right panel.

Proposition 1. *Consider discovery $(x, y(x))$ in an area of length $X \leq \infty$ with distance $d = d(x)$. The benefit of the discovery is*

$$V(d;X) = \frac{1}{6q}\Big(2X\sigma^2(d;X) + \mathbf{1}_{d>4q}\sqrt{d}(d-4q)^{3/2} + \mathbf{1}_{X-d>4q}\sqrt{X-d}\,(X-d-4q)^{3/2}. - \mathbf{1}_{X>4q}\sqrt{X}(X-4q)^{3/2}\Big)$$

Expanding knowledge beyond the frontier has a benefit of $V(d;\infty) := \lim_{X\to\infty} V(d;X)$.

Proposition 1 states that expanding knowledge is equivalent to the limiting case of deepening knowledge.

The terms in $V(d;X)$ without an indicator function measure the direct reduction in the variance due to a discovery and hence the effect on decision making conditional on a proactive action $a \neq \varnothing$. The terms with an indicator function, $\mathbf{1}$, become active whenever the corresponding area contains questions with too imprecise conjectures (see, for example, the right panel of Figure 5). Such conjectures induce the decision maker to select the outside option $\varnothing$ rather than making a proactive decision to limit losses to 0. The indicator-function terms that enter positively correspond to choices in the newly created areas. The indicator-function term that enters negatively corresponds to choices in an old area that gets replaced. Therefore, the presence of the outside option protects the decision maker from risky proactive actions when there is high uncertainty about the question's optimal answer.

Figure 4 illustrates the benefits of discovery for expanding knowledge (left panel) and deepening knowledge (right panel). The right panel of Figure 6 on page 15 illustrates the functions for different area lengths X. To gain intuition, it is useful to discuss expanding knowledge and deepening knowledge separately.

Expanding knowledge. Our first corollary states the closed form of $V(d;\infty)$.

Corollary 1. $V(d;\infty) = \frac{1}{6q}\left(6qd - d^2 + \mathbf{1}_{d>4q}\sqrt{d}(d-4q)^{3/2}\right)$

We focus on discovering the answer to $x < x_1$, which occurs when the decision maker moves from Figure 3 to the left panel of Figure 4. The case of $x > x_k$ is analogous.

The benefit of expanding knowledge comes from the new area $[x, x_1)$ it creates. The discovery of $y(x)$ pushes the knowledge frontier to the left and creates new research area $[x, x_1)$. The benefit of the discovery is the value of that new area (the dark-shaded part in Figure 4, left panel).[15]

The value of adding an area depends on (i) the amount of questions in that area and (ii) the degree of improvement in decision making relative to the outside option $a = \varnothing$. The benefits-maximizing question resolves a classic marginal-inframarginal trade-off similar to that in a monopoly-pricing decision: Increasing the area length of the newly created area has two opposing effects on the value of discovery. The marginal gain is the increase in the amount of questions that the conjecture improves. However, it comes at a cost because it decreases the precision of conjectures about all inframarginal questions in the area.

Figure 4 and Figure 5 illustrate the benefits of discovery from creating too short (left panel of Figure 4), ideal (left panel of Figure 5), and too large (right panel of Figure 5) areas. The largest benefits come at an intermediate level at which all conjectures have a variance strictly smaller than q, as the next corollary shows. The inframarginal losses outweigh the marginal gains at that point. The decision maker refrains from using the outside option for all questions inside the new area. We define the benefits-maximizing distance in area X as

$$d^0(X) := \max_d V(d; X).$$

Corollary 2. *The benefits of expanding knowledge are single peaked in d. The benefit-maximizing distance $d^0(\infty) = 3q$. The maximum benefits of expanding knowledge are $V^\infty := V(3q; \infty) = \frac{3}{2}q$.*[16]

Deepening knowledge is the process of discovering answer $y(x)$ to question x in area i with two bounds, x_i and x_{i+1}. The answers $y(x_i)$ and $y(x_{i+1})$ are known. We illustrate the process in the right panel of Figure 4. The difference from expanding knowledge is that instead of creating a new area, deepening knowledge replaces the old area, $[x_i, x_{i+1})$, with two new areas, $[x_i, x)$ and $[x, x_{i+1})$.

The benefits of a discovery depends on the combination of improved decision making in either of the areas. We know from Corollary 2 that the largest benefits in a single area come from an area of length $3q$. Thus, if the old area, i, had length $X_i = 6q$, a discovery at the midpoint would provide the largest benefits. However, if $X_i \neq 6q$, at least one of the two areas would have a length different from $3q$.

If $X_i \neq 6q$, two forces are at play. First, there is a benefit to replacing the old area with two symmetric new areas such that each is half the length of the old area. The intuition echoes that of expanding knowledge: the inframarginal loss increases when an area becomes too large. Thus, choosing two areas with the same length reduces the inframarginal losses compared with the case of one large and one small area. Inspection of the right panel of Figure 4 provides graphical intuition.

Second, benefits decline if area length is larger than $3q$, as conjectures inside the area become increasingly imprecise. Maintaining symmetry implies that newly created areas are larger than $3q$ if $X_i > 6q$ and thus too large to maximize the benefits.

[15] More precisely, the conjectures about questions to the left of the old frontier are replaced by conjectures inside the new research area, and conjectures to the left of the new frontier also become more precise. However, as can be seen in the left panel of Figure 4, the variance reduction to the left of the frontier is always the same. Hence the benefits are the same as if only the new area was added.

[16] The results of this and the next corollary follow directly from an analysis of $V(\cdot;\cdot)$ derived in Proposition 1. However, deriving them is not entirely straightforward, so we do so in the Appendix.

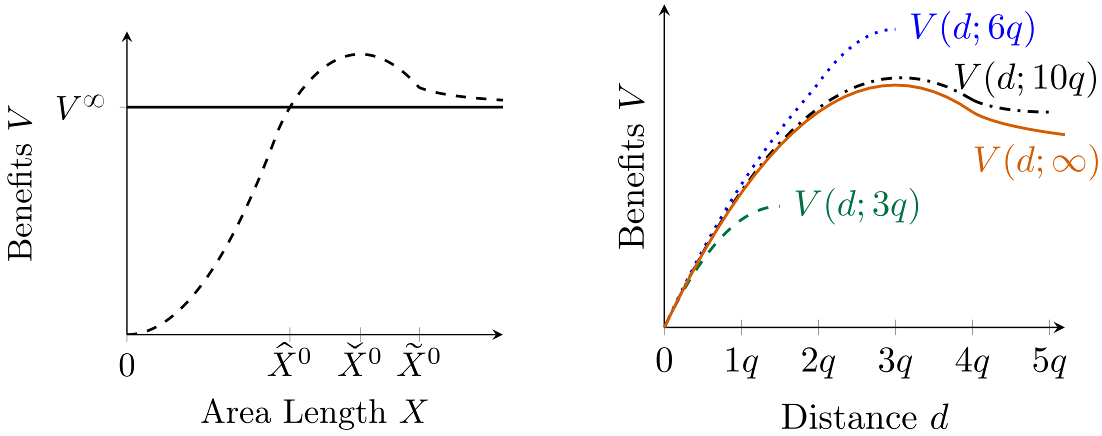


Figure 6: *The benefit of discovery.*
Left panel: Benefits of discovery as a function of area length X. The graph plots the benefits of discovery $V(d^0(X);X)$ for areas of length $X < \infty$ (dashed line). The solid line is the maximum benefits of discovery when expanding knowledge V^∞. Deepening beats expanding knowledge if $X > \widehat{X}^0 \approx 4.3q$. $V(d^0(X);X)$ is maximal at $\check{X}^0 \approx 6.2q$; $d^0(X) < X/2$ if $X > \widetilde{X}^0$.
Right panel: Benefits of discovery given X as a function of distance d. The graph depicts expanding knowledge (solid line) and deepening knowledge (dashed and dotted lines) for area lengths $X \in \{3q, 6q, 10q, \infty\}$.
Note: Plots for deepening knowledge end at the maximum distance in each area, $d = X/2$.

If the initial area length X_i was small, the first force would dominate. It would be better to divide the area evenly even if each new area was less than $3q$ long. However, if X_i was large, the trade-off would be resolved in favor of creating one high-value area at the cost of having imprecise conjectures in the other, larger, area. A cutoff $\widetilde{X}^0 \in [6q, 8q]$ exists such that it is benefits maximizing to create two symmetric areas if and only if $X_i < \widetilde{X}^0$.

For what initial area length X_i does transforming the area into two new ones provide the largest benefits? As explained above, two areas of length $3q$ provide the largest value. However, we have to take into account that the two new areas replace an old area. The larger the area that gets replaced, the less its initial value. On the other hand, the larger the old area (beyond $6q$), the lower the value of the two new areas. The initial area length that provides the largest benefits when the area is replaced is $\check{X}^0 \approx 6.2q$. That is, it is more than $6q$.

Expanding versus deepening knowledge. On the one hand, creating new areas means no knowledge needs to be replaced, as all old areas remain. On the other hand, deepening knowledge means creating two areas with relatively precise conjectures. If an area is small, deepening knowledge provides only a small benefit. Conjectures are already precise. If an area is large, conjectures are imprecise and deepening knowledge is more beneficial. Overall, there is a cutoff $\widehat{X}^0 \approx 4.3q$ such that deepening knowledge in X_i is more beneficial than expanding knowledge if and only if $X_i > \widehat{X}^0$.

Our next corollary and the associated figure (Figure 6) summarize the discussion.

Corollary 3. *There are three cutoff area lengths,* $4q < \widehat{X}^0 < 6q < \check{X}^0 < \widetilde{X}^0 < 8q$*, such that the following propositions hold:*

- *The benefits of expanding knowledge by* $3q$ *dominate the benefits of deepening knowledge in area* i *if and only if* $X_i < \widehat{X}^0$*.*
- *The maximum benefits of deepening knowledge in area* i *are increasing in area length if* $X_i < \check{X}^0$*; they are decreasing if* $X_i > \check{X}^0$*.*
- *The distance* $d^0(X_i)$ *of the benefits-maximizing discovery is increasing in* X_i *for* $X_i < \widetilde{X}^0$ *and decreasing for* $X_i > \widetilde{X}^0$*. If* $X_i < \widehat{X}^0$*,* $d^0(X_i) = X_i/2$*. Otherwise* $d^0(X_i) \in (3q, \min\{X_i/2, 4q\}]$*. As* $X \to \infty$*,* $d^0(X) \to d^0(\infty)$ *and* $V(d;X) \to V(d;\infty)$ *uniformly.*

4 The Researcher

4.1 The Researcher's Objective

In this section, we introduce a researcher to our model. We assume that the researcher chooses both her research question x and a research effort. Conditional on the question x, we assume a one-to-one relationship between effort and *output* ρ, the probability that the researcher finds the answer. Further, we assume that the researcher's benefits of discovering the answer $y(x)$ are proportional to the benefits of the discovery to the decision maker. That is, the researcher receives an expected benefit of $\rho V(d;X)$ if she chooses the probability ρ to discover the answer to a question in area X with distance d.[17] We parametrize the *researcher's cost* as

$$c(\rho, d; X) = \tilde{c}(\rho)\sigma^2(d; X),$$

with $\tilde{c}(\rho) := (erf^{-1}(\rho))^2$, and erf^{-1} being the inverse error function of the normal distribution. Thus, we assume a cost function, which is (i) multiplicatively separable in ρ and $(d;X)$, (ii) increasing in d and X, (iii) and concave in d; the concavity decreases in X with the limiting case in which the cost function is linear in d as $X \to \infty$ (see left panel of Figure 7).

The cost function links output and novelty: for a given level of effort, the probability of a successful search depends on the precision of the conjecture about a question. Research on a more novel question inside the same research area with the same level of effort entails higher risk.

The use of the variance, $\sigma^2(d;X)$, to measure the cost of uncertainty about a question is natural. While assuming an increasing and convex function in the success probability ρ is natural as well, the use of the squared inverse error function, $(erf^{-1}(\rho))^2$, requires further motivation. In Appendix D, we provide a search model to microfound the cost function we impose. There, we assume that the researcher can—at a cost—decide on a sampling-interval $[a(x), b(x)]$ in the y-dimension. She discovers the answer if and only if $y(x) \in [a(x), b(x)]$. Any (i) homogeneous, (ii) increasing, and (iii) convex sampling

[17] One rationale for discarding nonfindings is a moral hazard concern: science is complex, and it is impossible to distinguish the absence of a finding from the absence of proper search. Our model can easily account for the possibility of publishing nonfindings; unsurprisingly, these increase the value of knowledge as well. The difficulty of publishing the absence of evidence, however, has long been recognized in the literature. See, for example, Sterling (1959). In principle, it is relatively straightforward to compute updated answer distributions based on null results in our setting. Including this in our researcher model, however, is beyond the scope of this paper.

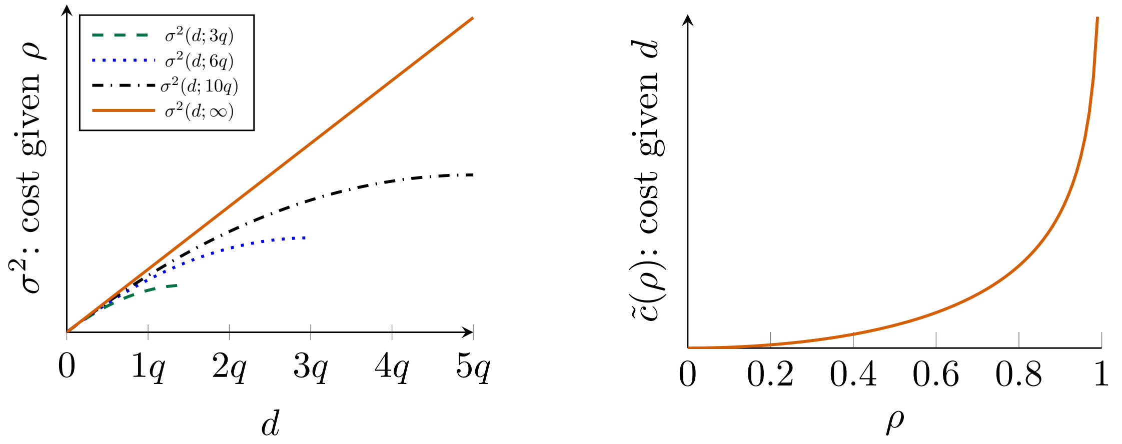


Figure 7: *Cost of research as a function of distance from knowledge (left) and probability of discovery (right).* As area length X decreases, cost diminishes for a given (d, ρ). The cost is linear in d when expanding knowledge but strictly concave when deepening knowledge. The cost function is convex in ρ. The left panel plots $\sigma^2(d; X)$ for different X's; the right panel depicts $\tilde{c}(\rho)$. In the left panel, plots for area length $X < 10q$ end at the maximum possible distance ($d = X/2$).

cost function over the interval length $|a(x) - b(x)|$ would imply a reduced-form cost function similar to the one we impose and therefore not alter our results qualitatively. The reduced-form cost function we impose above corresponds to a quadratic sampling cost $(a(x) - b(x))^2$. A general increasing and convex $\tilde{c}(\rho)$ not relying on erf^{-1} is possible, yet harder to microfound. Moreover, erf^{-1} has a set of desired properties. In Appendix A, we provide a discussion.

The researcher's payoff is

$$u_R(d, \rho; X) := \rho V(d; X) - \eta c(\rho, d; X)$$

where $\eta > 0$ determines the relative weights of cost and benefits, respectively.

We chose to abstract from any motivations other than the researcher's desire to increase the value of knowledge. That is, we implicitly assume that the market for research is frictionless and rewards researchers for their direct contributions to the value of knowledge. This assumption allows us to provide a clean analysis of the researcher's trade-offs in the absence of any other exogenous effects. We address additional frictions in our Final Remarks.

It follows from the previous discussion that the researcher resolves the tension between the cost and benefits of research by choosing a research question together with an appropriate level of effort. A short distance to existing knowledge allows her to find answers with a high probability at a relatively low cost. However, such answers provide little benefits to society. By increasing the distance, the researcher increases her benefits conditional on discovery. At the same time, either the cost increases or she has to accept a lower success probability. This trade-off is at the heart of many discussions of research funding.[18]

In the following, we characterize the researcher's optimal choice and elaborate on the resolution of the novelty-output trade-off. While this characterization is a central contribution of this paper, it also serves as a building block for the applications that follow.

[18] See, for example, the emphasis on high-risk/high-reward research by the European Research Council (`https://ec.europa.eu/research/participants/data/ref/h2020/call_ptef/ef/h2020-call-ef-erc-stg-cog-2015_en.pdf`) and the National Institutes of Health (NIH) (`https://commonfund.nih.gov/highrisk`).

4.2 The Researcher's Decision

The researcher solves

$$\max_{X \in \{X_0,\dots,X_k\}} \underbrace{\max_{\substack{d\in[0,X/2],\\ \rho\in[0,1]}} \rho V(d;X) - \eta c(\rho,d;X)}_{=:U_R(X)} .$$

If there is no cost ($\eta = 0$), we can apply Proposition 1 to derive the optimum. For any X, the researcher selects $\rho = 1$. For $X < \widetilde{X}^0$ she selects $d = X/2$; and for $\widetilde{X}^0 < X < \infty$ she selects $d \in (3q, X/2)$. If $X = \infty$, the researcher chooses $d = 3q$. She prefers to expand knowledge if and only if $X_i \leq \widehat{X}^0$ for any area $X_i < \infty$ defined by $\mathcal{F}_k$.

However, for $\eta > 0$, the researcher's decision about effort is nontrivial and linked with her decision about the research question. Choosing a question with a small distance allows for a high probability of discovery at a low cost. Her initial conjecture about the answer is already precise. Nevertheless, her payoff is low, as such a discovery provides little benefit.

By increasing the distance, the researcher increases the benefits of discovery but also increases the cost, ceteris paribus. The effect on the optimal probability of discovery is ambiguous: depending on which effect dominates, the distance and the probability of discovery are substitutes or complements.

Definition 3. Let $\rho^*(d;X)$ be the probability of success ρ that maximizes the researcher's payoff given d and X. Output ρ and novelty d are complements (substitutes) if $\rho^*(d;X)$ is strictly increasing (decreasing) in d.

It turns out that how output and novelty behave jointly depends crucially on both the length of the research area and the level of novelty. If novelty is too high, the benefits of discovery decrease in novelty (see Proposition 1). In this case, novelty and output are substitutes. Reducing novelty increases the marginal benefits and reduces the marginal cost of increasing output.

Optimal choice within a research area. First, we consider how the researcher's choice of distance and probability of discovery interact within an area of length X. The following proposition summarizes the joint behavior of d and ρ.

Proposition 2. *Suppose $\eta > 0$.*

1. *When the researcher expands knowledge,*
   - *i.) distance d and probability of discovery ρ are substitutes, and*
   - *ii.) the optimal choice of $d \in (2q, 3q)$.*
2. *When the researcher deepens knowledge in an area of length X,*
   - *i.) d and ρ are*
     - *a.) independent if $X \leq 4q$,*
     - *b.) complements if $X \in (4q, (4+\sqrt{6})q)$,*
     - *c.) substitutes for $d \in (0, \hat{d}(X))$[19] and complements for $d \in (\hat{d}(X), \frac{X}{2})$ if $X \in ((4+\sqrt{6})q, 8q)$, and*
     - *d.) substitutes if $X > 8q$.*
   - *ii.) the researcher's optimal choice of d is at d's maximum value $d = \frac{X}{2}$ if $X \leq \widetilde{X}$ and at the interior value $d < \frac{X}{2}$ if $X > \widetilde{X}$.*

[19] $\hat{d}$ is defined in the proof of the proposition.

Whenever the researcher expands knowledge, an increase in distance increases the marginal cost of the success probability more than the marginal benefit. Thus, any increase in distance comes at the cost of reducing the success probability. The optimal distance is intermediate: a short distance provides only small benefits. However, the benefits-maximizing distance is $3q$ and the researcher will never select a question farther than $3q$ beyond the frontier. As research comes with the risk of failure and at a cost, the optimal distance is shorter than the benefits-maximizing distance.

Whenever the researcher deepens knowledge, the effect that distance has on the optimal choice of ρ changes in contrast to the case of expanding knowledge. The variance of the conjecture is concave in distance whenever $X < \infty$ because the researcher uses information from two answered questions to form a conjecture. Thus, the negative effect of moving farther from one question in the set of existing knowledge is mitigated by moving closer to another question in the set. Therefore, increasing the distance from existing knowledge is less costly in terms of success probability when deepening knowledge than when expanding knowledge. Distance and probability of discovery can become complements.

For short research areas—areas for which the decision maker addresses all questions inside this area proactively—, the benefits of discovery are proportional to the variance of the question (see Proposition 1): the probability of discovery is independent of the choice of distance. If instead the initial area is larger, then the decision maker's losses prior to the new discovery were limited because she chose the outside option for some questions. A more distant new discovery therefore leads to a larger increase in the benefit than in the variance; d and ρ become complements.

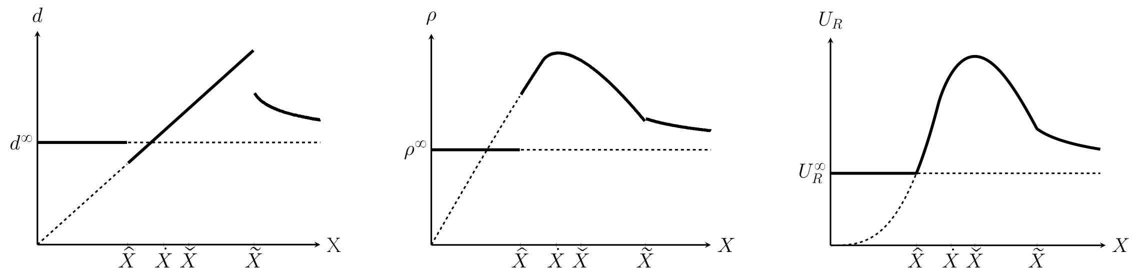
To see this, note that the marginal cost of the success probability are increasingly flat in novelty once the distance approaches the midpoint of the interval. The marginal benefit of the success probability corresponds to the benefits of a discovery. Whenever these benefits increase steeply in novelty relative to the increase of the marginal cost of success probability, novelty and output are complements. If the research area becomes very large, the region in which novelty and output are complements vanishes. The reason is that once the marginal cost of success probability become flat, the marginal value of success probability is even flatter or already decreasing.

Finally, we show in Proposition 2 that the researcher chooses the largest possible distance whenever the area is not too large. If distance and success probability are complements, she chooses the maximum distance and the largest probability of discovery in this interval. Intuitively, there is no trade-off between novelty and output in such research areas: any increase in novelty is accompanied by an increase in output. However, for large research areas and expanding knowledge, a trade-off arises; novelty and output become substitutes.

Optimal choice among intervals. We now characterize the researcher's choice of research area X. We take the optimal choice inside each area as given. Let $d(X)$ and $\rho(X)$ be the researcher's choices conditional on an area of length X, and let $U_R(X)$ be the associated payoff. The respective objects for expanding research are d^∞, ρ^∞, and U_R^∞. The following proposition summarizes the findings. Figure 8 illustrates the proposition.

Proposition 3. *Suppose $\eta > 0$. There is a set of cutoff values $2q < \widehat{X} < \dot{X} < \check{X} < \widetilde{X} < 8q$ such that the following claims hold:*

1. *The researcher expands knowledge if and only if all available research areas are shorter than $\widehat{X}$.*



(a) Distance d (b) Probability ρ (c) Payoff U_R

Figure 8: *Outcomes of the researcher's choices in areas of different length.* The graphs indicate the researcher's choices conditional on area length X. We compare them with optimal choices when expanding knowledge (the horizontal line in each graph). On the horizontal axis we indicate the cutoffs $\widehat{X}, \check{X}, \dot{X}$, and $\widetilde{X}$ from Proposition 3.
The solid lines plot the optimal choice conditional on X being the best available area. For small areas ($X < \widehat{X}$), the researcher prefers to expand knowledge. For areas of length $X > \widehat{X}$, she prefers deepening knowledge to expanding knowledge. If the area has length $X < \widetilde{X}$, the researcher selects the largest distance possible in area X—that is, $d = X/2$. If $X > \widetilde{X}$, it is optimal to select distance $d < X/2$ closer to one of the end points of the area. For small areas ($X < \dot{X}$), $\rho(X)$ increases in X. For large areas ($X > \dot{X}$), $\rho(X)$ decreases in X (apart from a discontinuous jump at $\widetilde{X}$). The researcher's payoff increases in area length for $X < \check{X}$ and decreases for $X > \check{X}$. The order of the cutoffs is independent of the value of the cost parameter η.

2. *The optimal choices of distance $d(X)$ and probability of discovery $\rho(X)$ are nonmonotone in X. Probability $\rho(X)$ has a maximum at $\dot{X}$; distance $d(X)$ has a maximum at $\widetilde{X}$.*
3. *The researcher's payoffs $U_R(X)$ are single peaked with a maximum at $\check{X}$.*

Proposition 3 shows that the pattern in the choice of distance is qualitatively the same as in Corollary 3. However, the cost adds another dimension to the problem: the optimal probability depends on area length. Consider a short area X. The scope for improvement of the decision maker's policies is small, as conjectures are already precise. Thus, investing in discovery has a small expected payoff. The researcher does not invest much in the search for an answer despite the small cost. Thus, the probability of discovery is small. Now consider a large area. The benefits of deepening research is greater than in the case of the small area. However, because the conjectures are imprecise, the cost is larger too. The researcher does not invest much in discovery, as the probability of discovery is relatively flat in effort. Thus, the probability of discovery is small. In an area of intermediate length, the benefits of discovery are relatively high, yet conjectures are relatively precise and limit the cost. The return on investment is largest and the probability of discovery is highest.

Moreover, the researcher only trades off $d(X)$ against $\rho(X)$ if X is of intermediate length. If X is small, an increase in X increases the benefits of research. Cost is small, and the researcher has an incentive to increase $d(X)$ and $\rho(X)$. As X becomes larger, the marginal increase in the benefits of research decline yet the marginal cost of research increases for both $d(X)$ and $\rho(X)$. Eventually the researcher faces a trade-off: should she lower $\rho(X)$ to maintain $d(X) = X/2$? It turns out that she should. While the researcher wants to remain at a boundary in her choice of $d(X)$, she mitigates the increased cost by lowering $\rho(X)$. As X increases further, she eventually also lowers $d(X)$. After a discrete jump upward of $\rho(X)$ at $\widetilde{X}$ following the jump downward of $d(X)$ from the midpoint $X/2$ to an interior point, d and ρ comove again; both decline in X.

The researcher's preferred area length, $\check{X}$, is in a region in which a trade-off between

$\rho(X)$ and $d(X)$ exists. While the researcher would prefer a larger research area to increase the benefits of research, she would prefer a smaller research area to reduce the cost of finding an answer. Thus, $d(X)$ is increasing and $\rho(X)$ decreasing at the point at which $U_R(X)$ is maximal.

Note that we have thus far only characterized the researcher's decision *conditional* on area length X and compared the resulting payoffs of the researcher. An explicit analytical characterization of the researcher's choice depends on the available research areas; the choice is determined by existing knowledge $\mathcal{F}_k$. Given our characterization, computing the optimal area is straightforward.[20]

5 Moonshots, Funding, and the Evolution of Knowledge

In this section, we apply our model and the previous analysis to classical questions in the economics of science. In particular, we consider a decision maker who faces a sequence of short-lived researchers but may be forward-looking herself. The decision maker can incentivize researchers through funding. We ask the following questions: How do short-run and long-run incentives differ? Is there a value to incentivizing a moonshot—expanding knowledge farther beyond the frontier than myopically optimal? Does it matter which type of funding the decision maker uses?

Answering these questions allows us (i) to evaluate the evolution of knowledge generated by short-lived researchers from the long-run decision maker's perspective, (ii) to determine tools that improve the evolution of knowledge, and (iii) to derive a funder's optimal funding mix to improve the evolution of knowledge.

5.1 Sequential Research

We begin with the benchmark scenario of a sequence of short-lived independent researchers absent outside incentives.

Time is discrete, $t = 1, 2, ...$ and at each time t, a single researcher is born, observes only the current level of knowledge, $\mathcal{F}_t$, and decides on

- a research question $x \in \mathbb{R}$ (which implies d and X), and
- an interval $[a, b] \subseteq \mathbb{R}$ in which to search for the answer (which implies a success probability ρ, see Appendix D for details).

If the answer $y(x) \in [a, b]$, then $(x, y(x))$ is added to the existing knowledge. Otherwise, knowledge remains unchanged. The researcher obtains her payoff u_R, and disappears. Thus, knowledge evolves (up to relabeling) to

$$\mathcal{F}_{t+1} = \begin{cases} \mathcal{F}_t \cup (x, y(x)) & \text{if } y(x) \in [a, b] \\ \mathcal{F}_t & \text{else,} \end{cases}$$

time progresses to $t + 1$ and a new researcher is born observing only $\mathcal{F}_{t+1}$. The initial knowledge $\mathcal{F}_1$ is given. Moreover, we assume researchers are symmetric in the following way.

Assumption 1. Each researcher has the same cost type η. Moreover, researchers condition their strategy only on existing knowledge, $\mathcal{F}_t$, and not on calendar time, t. We focus on symmetric pure strategies: each researcher selects the same pure strategy given the same knowledge.

[20] A computer program to numerically calculate all choices, given $\mathcal{F}_k$, is available on our websites.

In this setting, the evolution of knowledge is a simple corollary to Proposition 3.

Corollary 4. *(i) If a researcher fails to obtain a discovery at time t, all future researchers fail. Knowledge ceases to evolve.*

(ii) Knowledge ceases to evolve within finite time.

(iii) If a researcher finds it optimal to expand knowledge at time t, all future researchers expand knowledge, too.

(iv) As long as knowledge has not ceased to evolve, researchers will eventually find it optimal to expand knowledge.

Corollary 4 highlights two endogenous features of this benchmark. First, the evolution of knowledge will eventually stop. Second, starting with an arbitrary initial knowledge $\mathcal{F}_1$, researchers may begin to close the gaps in existing knowledge. However, ultimately, they will venture into expanding knowledge and remain there until knowledge ceases to evolve.

The first feature follows from the symmetry and homogeneity of researchers imposed in Assumption 1. Two researchers with access to the same knowledge will address the same question with the same approach. Thus, if the first researcher fails, so will the following researchers because we assume that failures cannot be communicated to future generations. We consider this assumption reasonable in the following way: researchers are short lived and—after failing to find an answer—cannot present any credible evidence that they searched at all.

We view this feature as a strength of our model. Discovery fails if the truth takes an unexpected turn. Researchers will continuously fail, and progress stops. It can only resume through exogenous forces, e.g., paradigm shifts (Kuhn, 1962), importing new researcher types (Moser, Voena, and Waldinger, 2014), genius researchers (Benzell and Brynjolfsson, 2019), etc. Consider the alternative assumption about knowledge generation that research outcomes correspond to independent draws from a binary outcome variable with potential realizations *discovery* (finding the answer to the question) and *no discovery* (not finding the answer to the question). The researcher's effort determines the probability of drawing the outcome discovery. This assumption would make the results on moonshots and funding that we show below stronger. Yet, it would introduce an additional source of randomness, which is orthogonal to that created by the random realization of the truth. In particular, we consider it reasonable that researchers with access to the same knowledge draw correlated rather than independent conclusions. Some exogenous alteration of knowledge, information or skill is necessary to restart scientific progress.

The second feature—eventually, researchers will expand knowledge step by step—is a consequence of short-lived researchers. A short-lived researcher does not take the dynamic consequences of her potential finding into account. If available, the researcher takes advantage of "attractive gaps" in knowledge $\mathcal{F}_t$. Gaps are attractive only if they are large enough so that bridging them provides sufficient benefits. If such an attractive gap exists, bridging it is worthwhile because, given effort, discoveries are likely. However, once all gaps are sufficiently small, bridging any of them provides too little benefits. Researchers prefer to take the risk of expanding knowledge. To mitigate their risk at reasonable cost, researchers push the frontier only marginally. Thus, the new areas they create are short and future researchers will not deepen knowledge in them. Instead, they again move beyond the frontier and expand knowledge.

We view this feature as another strength of the model. Absent exogenous frictions, the evolution of knowledge eventually resembles a model in the spirit of the endogenous growth literature, e.g., Romer (1990) and Grossman and Helpman (1991). For example, starting at one initially known point $(x, y(x))$, knowledge evolves according to a step-by-step expansion with stepsize d^∞ until knowledge ceases to evolve. Moreover, if initial knowledge has significant gaps, these are a temporary phenomenon only. Eventually, knowledge becomes dense. From then onward, it follows the step-by-step expansion path.

Neither of the two features is crucial for what follows—we can easily introduce additional frictions, asymmetries or heterogeneity in the baseline. They come, however, at the cost of clarity to the trade-off we highlight next.

5.2 Moonshots

To evaluate the evolution of knowledge from a long-run perspective, we introduce an infinitely-lived decision maker to the model. The decision maker discounts future payoffs with discount factor $\delta \in [0, 1)$. Her within-period payoff is the value of knowledge $v(\mathcal{F}_t)$.

For simplicity, we focus on a setting in which $\mathcal{F}_1$ is such that the researcher's individual choice in period 1 would be to expand knowledge. By Corollary 4, such a situation will eventually arise. As will become clear, the benefits of moonshots are most significant in this instance. Without further loss, we assume $\mathcal{F}_1 = (0, y(0))$ and that knowledge expansion is always to the right.

We define a moonshot as follows.

Definition 4 (Moonshot). Given $\mathcal{F}_k$, a question $x \in \mathbb{R}$ is a moonshot if

1. $x \notin [x_1, x_k]$ and
2. $d(x) > 3q$.

That is, a moonshot is a question that is more novel than the decision maker's myopically optimal question, which by Corollary 2 has distance $3q$.

Our first exercise determines whether a long-lived decision maker prefers a successful moonshot in the first period over the myopically optimal distance. We begin with a negative benchmark result.

Corollary 5. *If research is costless, $\eta = 0$, incentivizing a moonshot is not beneficial. In that case, the static optimum is also dynamically optimal for any δ.*

If the cost of research is positive, $\eta > 0$, a moonshot may be beneficial. The preferences of the decision maker and the researchers do not align for two reasons. First, the cost of research enters the researcher's payoff function directly. However, it enters the payoff function of the decision maker only through the researchers' decisions.[21] Second, a discovery in period $t = 1$ influences the benefits and cost of future generations of researchers in periods $t > 1$ through the implied conjectures. While researchers do not take this effect into account, the long-lived decision maker internalizes this dynamic externality.

Analysis. We now demonstrate why incentivizing moonshots can be beneficial to the decision maker. As an illustration, we sketch the evolution of knowledge for two different initial discoveries in Figure 9.

For now, assume that the decision maker can choose the initial question $\hat{x}$ and can further guarantee the discovery of $y(\hat{x})$. We relax this assumption in Section 5.3. In this case, the decision maker's ex ante payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} \mathbb{E}\left[v(\mathcal{F}_{t+1})\right].$$

[21] If the decision maker were to internalize the researchers' cost as well, it would be more beneficial to incentivize moonshots.

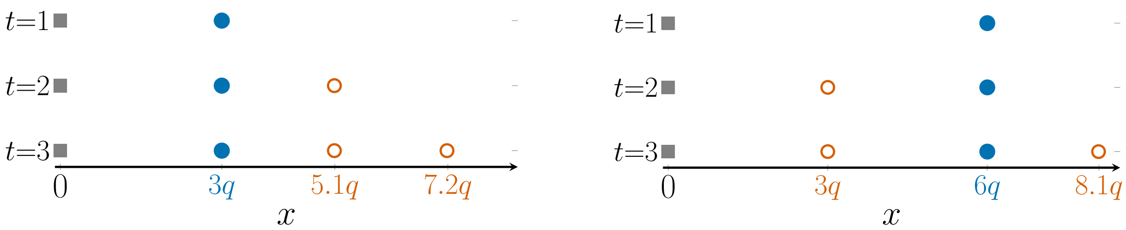


Figure 9: *Evolution of knowledge from $t = 1$ to $t = 3$ for different initial $\hat{x}$'s.* We assume $\mathcal{F}_1 = \{(x_0, y(x_0))\}$ (■). The dots show what questions have a known answer at each point in time t, assuming that discovery has been successful in all periods through t. Apart from the initial discovery (●), all question choices (○) are optimal choices for identical researchers with cost parameter $\eta = 1$. The left panel depicts an initial choice of $\hat{x} = 3q$; the right panel, a choice of $\hat{x} = 6q$.

The myopic benchmark for $\delta = 0$ is given by Corollary 2 and implies $\hat{x} = 3q$. However, if $\delta > 0$, selecting $\hat{x} > 3q$ has two effects on the expected payoff.

One effect comes from the cost function of future researchers. If $\hat{x} > \widehat{X}$, the next researcher aims to deepen knowledge between $\hat{x}$ and 0. Suppose $\hat{x}$ is such that in t=2 the researcher prefers to research on a question $x \in [0, \hat{x}]$ with distance d_2 to $\mathcal{F}_2$. That researcher's cost is determined by the variance $\sigma(d_2; \hat{x})$, which also depends on the choice of $\hat{x}$. A properly chosen moonshot $\hat{x} > 3q$ reduces the cost to future generations of researchers of connecting the initial body of knowledge with the moonshot compared to the alternative of expanding knowledge. Hence, a well-designed moonshot induces higher productivity in future periods.

A second effect comes from persistently shaping knowledge. After τ periods of making discoveries, knowledge is sufficiently dense. All researchers from $t = \tau + 1$ onward act to expand knowledge by d^{∞}. However, the knowledge created during these initial τ periods is different with and without the initial moonshot $\hat{x}$. Figure 9 compares a moonshot with a no-moonshot scenario. We see that $\hat{x} > 3q$ induces a more valuable landscape of knowledge for the decision maker. The first two areas created to the right of 0 provide a higher value of knowledge in the moonshot scenario (right panel) than the first two areas in the no moonshot scenario (left panel).

Overall, an optimal moonshot can increase future output and novelty. The following proposition shows that these benefits may outweigh the short-run cost of a too distant discovery for a sufficiently patient decision maker, provided that the cost of research is in an intermediate range.

Proposition 4. *There is a non-empty range of cost parameters $(\underline{\eta}, \overline{\eta})$ such that the decision maker strictly prefers a moonshot in $t = 1$ for any $\eta \in (\underline{\eta}, \overline{\eta})$ provided δ is larger than the critical discount factor $\underline{\delta}(\eta) < 1$.*

Proposition 4 states that moonshots are optimal if the decision maker is patient enough and the researchers' cost is not extreme. Figure 10 provides an illustration. If the cost is low, the reasoning of the no-cost benchmark above applies. A moonshot has little benefits because the cost does not distort the researchers' decisions much. Yet the loss in benefits of a suboptimal choice in the first period remains. If the cost is high instead, it is optimal for future generations to limit the search to small intervals. The probability of any discovery is low, and it is unlikely that future generations will eventually succeed in closing the gap after a moonshot. In both cases, the decision maker's optimal choice is the myopically optimal $\hat{x} = 3q$.

For intermediate cost levels η, moonshots are beneficial. The positive externality imposed on future researchers is significant, and research is generally productive. Future

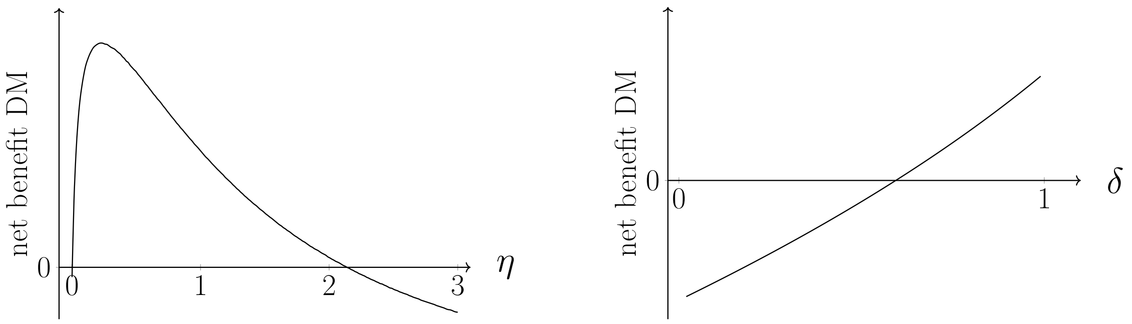


Figure 10: *Moonshot of $d = 6q$ versus the myopic optimum $d = 3q$ for different parameters.* The *left panel* plots the difference between the first-period net present value of a $6q$ moonshot and that of the myopic optimum $d = 3q$ for different η's. The discount factor is $\delta = 0.9$. The moonshot is strictly preferred for the interval $[\underline{\eta}, \overline{\eta}] \approx (0.01, 2.13)$.
The *right panel* plots the difference between the first-period net present value of a $6q$ moonshot and that of the myopic optimum $d = 3q$ for different δ's. The cost parameter is $\eta = 1$. The moonshot is strictly preferred for $\delta > \underline{\delta} \approx 0.6$.

generations benefit from the cost advantage and produce more valuable knowledge at a higher success rate.

The length of the *optimal moonshot* depends on the cost parameter as well. If the cost is low and the decision maker is patient, it might take several generations to close the gap created by the optimal moonshot. The less patient the decision maker and the more costly the research, the shorter the time required to close the gap. The effect of the discount factor is as expected. The reason for the cost effect is that if the cost is high, the chance that a researcher fails to obtain an answer increases. As a result, the *effective* discount factor decreases. The decision maker prefers moonshots that entail less future risk. Figure 11 shows the ex ante value of different moonshots to the decision maker for two cost scenarios.

We want to stress that the value of a moonshot to the decision maker depends crucially on existing knowledge. In particular, it depends on the density of existing knowledge. If there are large areas available, researchers can close an already existing gap instead of

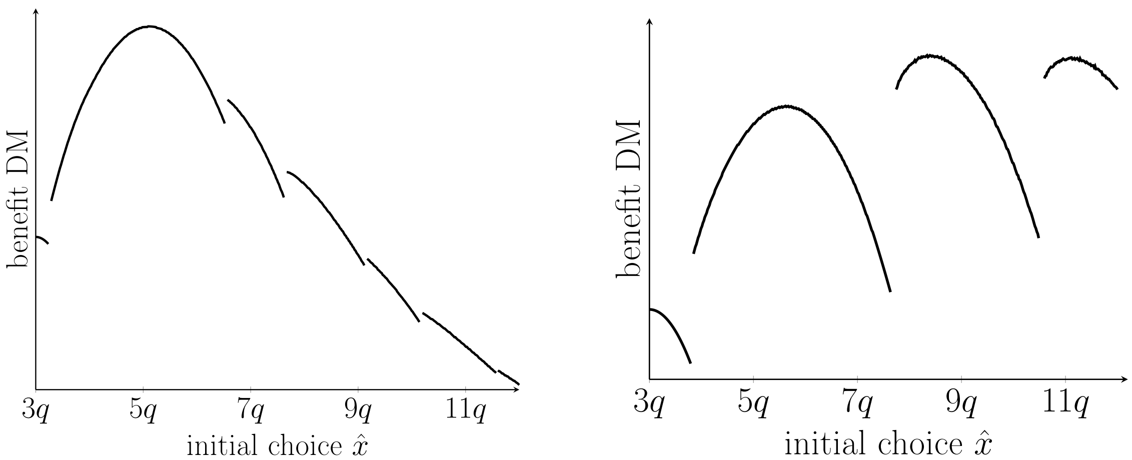


Figure 11: *Optimal moonshots.* The figure plots first-period net present value against the initial choice of moonshot. In the *left panel*, $\eta = 1$ and the optimal moonshot $\hat{x}^*$ is between $5q$ and $6q$. After the initial moonshot, the researcher in $t = 2$ bridges the gap: She deepens knowledge by selecting $x = \hat{x}^*/2$. In the *right panel*, $\eta = 0.1$ and the optimal moonshot is between $8q$ and $9q$. After the initial moonshot, it takes two researchers to bridge that gap. In both figures, $\delta = 0.9$.

conducting a moonshot or expanding knowledge. As a result, the value of the moonshot diminishes. The reason is straightforward. The point of a moonshot is to make a myopically suboptimal discovery to guide future researchers, who will successively close the large area opened by the moonshot. However, if there are already large areas given the existing knowledge, it is attractive to close these gaps first and benefit immediately form closing them. Creating another large area delays potential benefits and is thus less attractive. For the same reasons, the decision maker may not push for moonshots in subsequent periods even if she could. Instead, after one moonshot, she may delay the next moonshot until the gap created by the first moonshot is closed.

5.3 Research Funding

Our previous discussion abstracted from any additional incentives provided to the researchers. Instead, we focused on the effects that an exogenous moonshot has on the evolution of knowledge. We now turn to the incentive provision problem. As our analysis is—in part—motivated by the emphasis on scientific freedom, we assume that a funding institution respects scientific freedom.[22]

We begin with a discussion of myopically optimal funding. After that, we turn to forward-looking funding. Throughout, we consider a funding system with two instruments: *ex ante cost reductions* (for example, through grants) and *ex post rewards* (for example, through prizes). Cost reductions are reductions of the agent's cost parameter η. In particular, the agent's initial cost parameter is η^0 and a cost reduction of h leads to a new cost parameter $\eta = \eta^0 - h$. Rewards are an ex post utility transfer of ζ toward the agent. We assume that rewards are provided for seminal contributions. The more difficult and novel the problem, the larger the chance of receiving a reward. We proxy the ex post relation by the function $f(\sigma_{\mathcal{F}_k}) : \mathbb{R} \to [0, 1]$. It determines the probability of receiving a reward. To keep the funding scheme as simple as possible, we assume a piecewise linear relationship:

$$f(\sigma) = \begin{cases} \frac{\sigma^2}{s} & \text{if } \sigma^2 < s \\ 1 & \text{otherwise,} \end{cases}$$

for some $s \geq 4q$. That is, we assume that the marginal probability of obtaining the reward is constant and positive in difficulty (in terms of variance) up to some level s. Beyond s the marginal probability drops to 0.[23] The parameter restriction on s is to simplify the proofs only.

Further, we assume that the funder is budget constrained and cannot invest more than K in the funding scheme. The relative price of cost reductions is κ such that the funder's budget constraint is

$$K = \zeta + \kappa h.$$

We assume that $\kappa > K/\eta^0$ implying that the funder cannot eliminate the cost of research entirely with her budget.

[22]The NIH, for example, awards most grants via investigator-initiated competitions without a specific research topic suggested. For an empirical investigation of the effects of the alternative, "Request for Applications" grants, on researchers' choices, see Myers (2020). Azoulay, Graff Zivin, and Manso (2011) show that long-term grants guaranteeing freedom of research impact researchers' choices.

[23]In Appendix F, we discuss an alternative, non-linear reward technology, $f(\sigma^2) = 1 - e^{-s\sigma^2}$. Under that assumption, the marginal probability is continuously decreasing. Our results do not change qualitatively, yet we do not obtain a closed-form characterization of the feasible outcome set (see below).

The Feasible Set. Based on this budget constraint, we determine the set of novelty and output combinations the funder can implement with some funding scheme given the parameters (K, κ, s, η^0). The construction is based on the researcher's optimal choice given the funding mix (h, ζ), which is based on the solution to the researcher's problem

$$\max_{d,\rho} \rho\Big(V(d;\infty) + \frac{\sigma^2(d;\infty)}{s}\zeta\Big) - \eta\tilde{c}(\rho)\sigma^2(d;\infty).$$

Because the funder is budget-constrained, any funding choice implies an output $\rho < 1$ bounded away from a guaranteed success. The funder therefore chooses her preferred implementable combination (ρ, d) in the feasible set determined by the parameters (K, κ, s).

Definition 5. The *research-possibility frontier* $d(\rho; K)$ describes the largest distance a funder with budget K can implement for a given level of ρ.

Proposition 5. *For any budget $K < \kappa\eta^0$, there is an $s(K) < \infty$ such that whenever $s > s(K)$, all funding schemes imply novelty $d < s$. Moreover,*

if $\mathbf{s} > \mathbf{s(K)}$. *The set of implementable (d, ρ) combinations for a given cost ratio κ and budget K is described by the research-possibility frontier $d(\rho; K)$ defined over $[\underline{\rho}, \overline{\rho}]$, where $\underline{\rho}$ and $\overline{\rho}$ are the endogenous upper and lower bounds of ρ. These bounds are determined by the extreme funding schemes $(\zeta = 0, \eta = \eta^0 - K/\kappa)$ and $(\zeta = K, \eta = \eta^0)$. The research-possibility frontier is*[24]

$$d(\rho; K) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho}. \tag{1}$$

if $\mathbf{s} < \mathbf{s(K)}$. *Whenever the researcher's choice given (ζ, η) is such that $d \neq s$, then* (1) *describes the relation between $d(\rho)$. Moreover, there is a $\xi > 0$ such that $d \neq s$ for $\zeta < \xi$.*[25]

Myopic Funding. We now turn to the problem of myopic funding in period $t = 1$. We assume that the funder wants to maximize $\rho V(x; \mathcal{F}_1)$. The first-best benchmark follows directly from Corollary 2.

Corollary 6. *A myopic and unconstrained funder optimally sets $\eta = 0$ and induces $d = 3q$ and $\rho = 1$.*

As our funder is budget constrained she cannot eliminate the cost entirely and the optimal myopic funding mix is non-trivial. We now describe the optimal funding scheme of a funder that aims to maximize $\rho V(x; \mathcal{F}_1)$. We begin with a corollary to Proposition 5.

Corollary 7. *A budget-constrained funder cannot implement her first best. Moreover, she cannot implement $d \geq 3q$ with $\zeta = 0$.*

The corollary provides structure on the funder's problem. The funds are insufficient to eliminate the cost friction. Thus, the funder either implements novelty *below* the optimal level or uses research prizes as an instrument. Which option the funder prefers depends on the model parameters.

Inspection of equation (1) reveals that $d(\rho)$ can be an increasing function, a decreasing function or a non-monotone function. Thus, inducing more novelty may imply more or

[24] We define $\tilde{c}_\rho(\rho) := \partial\tilde{c}/\partial\rho(\rho)$.

[25] If $d{=}s$, then ρ is the unique solution to $\frac{V(s;\infty)+\zeta}{\eta s} = c_\rho(\rho)$.

less output depending on parameters and the current level of novelty. The reason is straightforward and follows the discussion on the researcher's perspective in Section 4. If the researcher's benefits increase sufficiently in novelty, she is willing to increase her research efforts significantly despite the cost—output increases. If these benefits do not increase sufficiently, more novelty implies greater risk—output decreases.

By offering the researcher a ceteris paribus higher reward, the funder increases the marginal benefits of research, thereby inducing greater complementarity between output and novelty and thus inducing an increase in effort. However, there is a countervailing force. The cost of increasing rewards is a higher cost parameter η, reducing incentives to exert effort.

The optimal funding mix depends on parametric specifications. We conclude our discussion on optimal myopic funding with a possibility result implied by the discussion above.

Corollary 8. *Myopic optimal funding can involve a combination of the two funding instruments or a focus on either one. It can induce moderately excessive novelty* $d \in (3q, s)$.

Corollary 8 follows from Proposition 7 which we state and prove in Appendix E. The left panel of Figure 12 provides an example in which the optimal myopic funding scheme involves a combination of the two instruments and induces moderately excessive novelty. On the research possibility frontier of Figure 12 (solid line) output and novelty are complements for $d \leq 3.2q$. The funder needs to *increase* distance d if she wants to increases ρ. Dashed lines depict indifference curves, and the myopic payoff for a given ρ is maximized at $d = 3q$ by Corollary 2. However, around $3q$, the funder's problem is the following. Should she increase novelty beyond the optimum to increase output? As we can see in the left panel of Figure 12, moderate excessive knowledge is optimal in that specification even under myopic preferences. The funder provides a funding mix of ex-ante cost reductions and ex-post rewards to obtain it.

Forward-Looking Funding. We conclude this part by considering a forward-looking funder. As we have seen in Section 5.2, incentivizing a moonshot in the first period can be beneficial. However, in that part, we have ignored both the funds needed to incentivize said moonshot and the risks involved.

Invoking Corollary 7, it is immediate that rewards are necessary to implement moonshots. However, a budget-constrained funder cannot guarantee a certain discovery and must trade off the value of a successful moonshot against its (potentially) greater risk.

However, as indicated by Figure 10, the benefits of more novel research today are non-monotone. That implies that—as in the static case—the funder's indifference curves are non-monotone as well. The dashed lines on the right panel of Figure 12 depict the indifference curves of the forward-looking decision maker in the (d, ρ) space. Low levels of novelty of period-1 discovery imply that researchers in period 2 will not choose to deepen knowledge. There is no intertemporal externality. Only if the initial moonshot is sufficiently novel this externality arises. The discontinuity in the funder's indifference curves occurs at the minimum level of knowledge that induces deepening knowledge of the period-2 researcher. To the right of the discontinuity, the funder is willing to accept a lower first-period output in return. The solid line depicts the same research possibility frontier as in the left panel. We conclude with a simple corollary summarizing our discussion.

Corollary 9. *If a moonshot is optimal, the optimal funding mix always includes strictly positive rewards.*

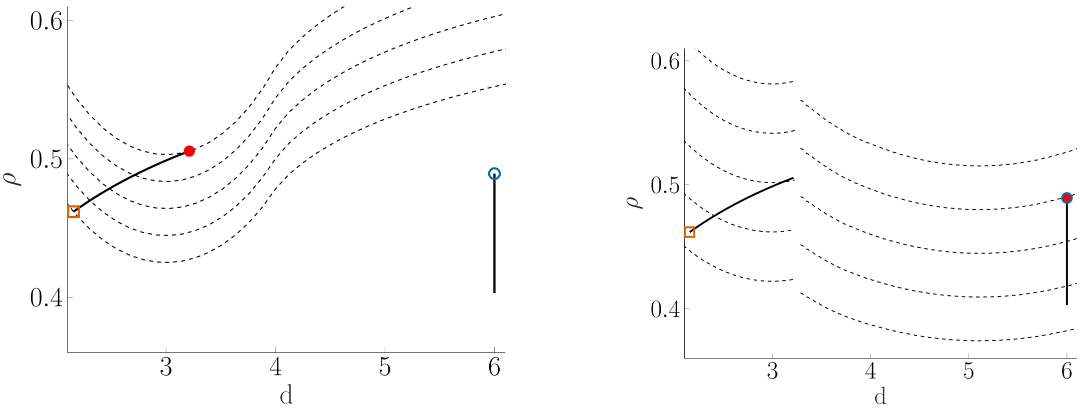


Figure 12: *Static vs Dynamic Optimal Funding.* The solid line is the funder's budget line. Dashed-line depict the decision maker's indifference curves if she is myopic (left panel) and forward looking with discount factor $\delta = 0.9$ (right panel). In both panels, $K = 3, s = 6, q = 1, \kappa = 16$ and $\eta^0 = 1$. The funder's optimal choice (●) consists of a mix of ex-ante cost reductions and ex-post rewards, $(\zeta, h) > 0$ in the left panel and focuses exclusively on rewards in the right panel. The circle (○) depicts the outcome if the funder invest exclusively into rewards, $\zeta = K, h = 0$; the square (□) the outcome if the funder invests exclusively into cost reductions, $\zeta = 0, h = K/\kappa$.

6 Final Remarks

We developed a tractable model of knowledge and research. The starting points of our model are that (i) the pool of available research questions is large, (ii) questions in close proximity to existing knowledge are easier to answer than questions that are far from existing knowledge, and (iii) society applies knowledge when selecting policies. We conceptualized research as the choice of a research question and the subsequent costly search for an answer.

Our model endogenously links novelty and research output and highlights the importance of existing knowledge for research and knowledge accumulation. Novelty and output can be substitutes or complements depending on the research area and the location of the question therein. If research expands the frontier, greater novelty always entails greater risk and thus lower output.

We show that the most valuable and productive research lies in research areas of intermediate length. Discoveries that connect distant—yet not too distant—pieces of knowledge substantially improve society's decisions, and the researcher benefits from relatively precise conjectures in the search for an answer.

We apply our baseline model to a classical question in the economics of science funding: Should we use funds to incentivize more novelty in research? We show that it may be beneficial for society to incentivize moonshots—highly novel discoveries with limited immediate benefits. While suboptimal in the short run, moonshots guide future discoveries. By incentivizing moonshots, the funder allows the evolution of knowledge to benefit from a dynamic externality: the effect of discoveries on future generations of researchers. Properly chosen moonshots increase both the research productivity and the value of knowledge generated in future periods. We derive the funder's feasible set and discuss the role of a funding mix between rewards and cost-reductions for optimal funding.

Our results are in line with recent empirical work—for example, Rzhetsky et al. (2015)—that analyze the impact of scientific findings on future scientific developments. They suggest that scientists choose a dynamically suboptimal strategy when selecting their research questions. Rzhetsky et al. (2015) identify researcher myopia as one of the drivers of that finding. While our model is consistent with this sentiment, we also raise a note of caution. Whether incentivizing moonshots is beneficial depends crucially on the current state of knowledge. Moonshots should be chosen carefully, as too much novelty can hurt the evolution of knowledge.

We began our paper by emphasizing the role of scientific freedom. Preserving that freedom remains a challenging task for science-funding institutions when society designs a funding architecture (see, for example, Bourguignon, 2019). The NSF emphasizes that it aims at funding high-risk/high-reward research to advance the knowledge frontier. Our findings in Section 5 illustrate a particular trade-off that funding institutions face in that context. Several known frictions absent in our model hinder efficient funding in reality. These range from publication bias (Andrews and Kasy, 2019), to the emphasis on priority (Bobtcheff, Bolte, and Mariotti, 2017; Hill and Stein, 2020, 2021), to career concerns (Akerlof and Michaillat, 2018; Heckman and Moktan, 2020). While the question of optimal market design is beyond the scope of this paper, our framework is flexible enough to incorporate these frictions straightforwardly. It may thus be a stepping stone toward developing structural models of science funding that focus on these issues. Such models could be useful for evaluating funding schemes and could provide meaningful counterfactuals to inform decision makers about the optimal provision of research incentives.

Appendix

A Notation and Properties of $\tilde{c}$

Notation: We use argument subscripts to denote the partial derivatives with respect to the argument. We omit function arguments whenever it is convenient and does not cost clarity. We use the notation $\frac{\mathrm{d}f(x,y)}{\mathrm{dx}}$ to indicate the total derivative $(f_x + f_y y_x)$.

Properties of $\tilde{c}$. Some proofs rely on the properties of the inverse error function or more specifically on the representation $\tilde{c}(\rho) = (erf^{-1}(\rho))^2$. The function $\tilde{c}(\rho)$ is convex and increasing on $[0,1)$ with $\tilde{c}(0) = 0$ and $\lim_{\rho \to 1} \tilde{c}(\rho) = \infty$.[26] The derivative

$$\tilde{c}_\rho(\rho) = \sqrt{\pi} erf^{-1}(\rho) e^{\tilde{c}(\rho)}$$

is increasing and convex with the same limits.

We make use of the fact that, for $\rho \in (0,1)$, $\tilde{c}(\rho)$ has a convex and increasing elasticity bounded below by 2 and unbounded above. Its derivative $\tilde{c}_\rho(\rho)$ has an increasing elasticity bounded below by 1 and unbounded above. We want to emphasize that these properties are not special to our quadratic cost assumption. To the contrary, $erf^{-1}(x)^k$ for any $k \geq 2$ admits similar properties with only the lower bounds changing. Formally, the following

[26]Due to this limit and the researcher's ability to choose $\rho = 1$, we augment the support of the cost function to include $\rho = 1$ with $\tilde{c}(1) = \infty$. However, the optimal ρ is always strictly interior unless the cost parameter η is chosen to be zero in which case we assume that $\eta\tilde{c}(\rho = 1) = 0$.

properties are invoked in the proofs:

$$\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} \in (2,\infty) \text{ and increasing,}$$
$$\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)} \in (1,\infty) \text{ and increasing,}$$
$$\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho) \in (0,\infty) \text{ and increasing,}$$
$$\tilde{c}_\rho^{-1}(x) = erf\left(\sqrt{\frac{W(2x^2/\pi)}{2}}\right).$$

with $W(\cdot)$ the principal branch of the Lambert W function. We prove the properties that do not directly follow from the definition of the inverse of the error function in Appendix H.

B Proofs

At various points we make use of inequality relations the proof of which we relegate to Appendix H. In each of these cases, proving the inequalities is done via straightforward algebra that produces little additional insight.

B.1 Proof of Proposition 1

Proof. The value of knowing $\mathcal{F}_k$ is

$$\int_{-\infty}^{\infty} \max\left\{\frac{q-\sigma_x^2(y|\mathcal{F}_k)}{q}, 0\right\} \mathrm{d}x.$$

No matter which point of knowledge $(x, y(x))$ is added to $\mathcal{F}_k$, the value of knowledge outside the frontier is identical for both $\mathcal{F}_k$ and $\mathcal{F}_k \cup (x, y(x))$. Area lengths $X_1 = X_k = \infty$ do not depend on $\mathcal{F}_k$ and neither does the variance for a question $x < x_1$ or $x > x_k$ with a given distance d to $\mathcal{F}_k$. The conjectures about all questions outside $[x_1, x_k]$ deliver a total value of

$$2\int_0^q \frac{q-x}{q}\mathrm{d}x = q,$$

which is independent of $\mathcal{F}_k$.

Moreover, if the answer to a question $\hat{x}$, deepens knowledge, that is, $\hat{x} \in [x_i, x_{i+1}]$ with $(x_i, y(x_i)), (x_{i+1}, y(x_{i+1})) \in \mathcal{F}_k$, it only affects questions in the area $[x_i, x_{i+1}]$, i.e., $G(x|\mathcal{F}_k) = G(x|\mathcal{F}_k \cup (\hat{x}, y(\hat{x})) \; \forall \; x \notin (x_i, x_{i+1})$.

To simplify notation, let us consider the points in terms of distance to the lower bound of the area with X, $d \equiv x - x_i$.

The value of a given area $[x_i, x_{i+1}]$ is (with abuse of notation)

$$v(X) = \int_0^X \max\left\{\frac{q-\frac{d(X-d)}{X}}{q}, 0\right\} \mathrm{d}d.$$

Note that whenever $X \leq 4q$, $\frac{d(X-d)}{X} \leq q$. Hence, we can directly compute the value of any area with length $X \leq 4q$ as

$$v(X) = X - \frac{X^2}{6q}.$$

Whenever $X > 4q$, value is only generated on a subset of points in the area. As the variance is a symmetric quadratic function with $X/2$ as midpoint, there is a symmetric area centered around $X/2$ which has a variance exceeding q. The points with variance equal to q are given by $\overline{d}_{1,2} = \frac{X}{2} \pm \frac{1}{2}\sqrt{X}\sqrt{X-4q}$. On all such points the decision makers losses are limited to 0. Hence, the value of an area with $X > 4q$ is (due to symmetry)

$$\begin{aligned} v(X) &= 2\int_0^{\overline{d}_1} \frac{q - \frac{d(X-d)}{X}}{q} \mathrm{d}d \\ &= X - \frac{X^2}{6q} + \frac{X-4q}{6q}\sqrt{X}\sqrt{X-4q}. \end{aligned}$$

If knowledge expands beyond the frontier, a new area is created and no area is replaced. The value created is thus

$$V(d;\infty) = v(d) = d - \frac{d^2}{6q} + \begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q. \end{cases}$$

If a knowledge point is added inside an area with length X with distance d to the closest existing knowledge, it generates two new areas with length d and $X-d$ that replace the old area with length X. The total value of the two intervals new is

$$\begin{aligned} v(d) + v(X-d) = d - \frac{d^2}{6q} &+ \begin{cases} 0, & \text{if } d \leq 4q \\ \frac{d-4q}{6q}\sqrt{d}\sqrt{d-4q}, & \text{if } d > 4q \end{cases} \\ +X - d - \frac{(X-d)^2}{6q}) &+ \begin{cases} 0, & \text{if } X-d \leq 4q \\ \frac{X-d-4q}{6q}\sqrt{X-d}\sqrt{X-d-4q}, & \text{if } X-d > 4q \end{cases}. \end{aligned}$$

The benefit of discovery is then $V(d;X) = v(d) + v(X-d) - v(X)$. Noticing that $\sigma^2(d;X) = d(X-d)/X$ and replacing accordingly results in the expression from the proposition follow. Taking the limit of $X \to \infty$ corresponds to the value of expanding research beyond the frontier.

□

B.2 Proof of Corollary 2

Proof. The first-order condition for $d \leq 4q$ is

$$\frac{\partial V(d;\infty|d \leq 4q)}{\partial d} = 1 - \frac{d}{3q} = 0.$$

Moreover, the benefit is decreasing in d for $d > 4q$ which can be seen from the derivative with respect to d which is

$$\frac{\partial V(d;\infty|d > 4q)}{\partial d} = -\frac{d}{3q} + 1 + \sqrt{\frac{d-4q}{d}}\frac{d-q}{3q} < 0.$$

The inequality holds by Lemma 23 in Appendix H. □

B.3 Proof of Corollary 3

We prove Corollary 3 via a series of lemmata.

- Lemmata 1 and 2 shows that the distance that maximizes deepening knowledge is $d^0(X) = X/2$ for small X and $d^0(X) < X/2$ for large X.
- Lemma 3 shows that $d^0(X) < X/2$ implies decreasing benefits in X.
- Lemma 4 shows that once $d^0(X) < X/2$ for some X it is true for all $X' > X$ and thus establishes $\widetilde{X}$.
- Lemma 5 shows our convergence and $d^0(X > 6q) > 3q$.
- Lemmata 6 and 7 establishes single peakedness and determine $\check{X}^0$ and $\widehat{X}^0$.
- Lemma 8 determines the order of the cutoffs.

Throughout, we refer to the distance d that maximizes $V(d;X)$ as $d^0(X)$. Recall that $d \leq X/2$.

Proof.

Lemma 1. *$d^0(X) = X/2$ if $X \leq 6q$.*

Proof.
1. Assume $X \leq 4q$.

The benefits of discovery are

$$V(d;X|X \leq 4q) = \frac{1}{3q}(Xd - d^2)$$

which is increasing in d for $d \in [0, X/2]$ and hence maximized at $d = X/2$. Moreover, $V(X/2;X|X \leq 4q) = X^2/(12q)$ which is increasing in X.

2. Assume $X \in (4q, 6q]$**.**

(i) $d \geq X - 4q$ implies (since $d \leq 3q$)

$$V(d;X|d \geq X - 4q, X \in (4q, 6q])) = \frac{1}{6q}\left(2dX - 2d^2 - \sqrt{X}(X - 4q)^{3/2}\right)$$

which is the same as in the first case up to constant $-\sqrt{X}(X - 4q)^{3/2}$. Thus, the optimal d conditional on $d \geq X - 4q$ is $d = X/2$.

(ii) For $d \leq X - 4q$ the benefit becomes

$$V(d;X|d \leq X - 4q, X \in (4q, 6q])) = \\ \frac{1}{6q}\left(2dX - 2d^2 + \sqrt{X-d}(X - d - 4q)^{3/2} - \sqrt{X}(X - 4q)^{3/2}\right),$$

with derivative

$$V_d = \frac{1}{3q}\left(X - 2d - (X - d - q)\sqrt{\frac{X - d - 4q}{X - d}}\right)$$

which is positive for $d \leq X - 4q, X \in (4q, 6q]$ by Lemma 24 from Appendix H. Hence, $V_d(d;X|d \leq X - 4q, X \in [4q, 6q]) > 0$ for all d and X in the considered domain. Thus, $d = X - 4q$ maximizes $V(d;X|d \leq X - 4q, X \in (4q, 6q]))$ and hence $d = X/2$ maximizes $V(d;X|X \in (4q, 6q])$. □

Lemma 2. *If $X > 8q$ then $d^0(X) \neq X/2$. If $d^0(X) \neq X/2$, then $d^0(X) \leq 4q$.*

Proof. Take $\overline{d} = 4q < X/2$. That implies

$$V(\overline{d}; X|X > 8q) = \frac{1}{6q}\left(8Xq - 32q^2 - \sqrt{X}(X-4q)^{3/2} + \sqrt{(X-4q)}(X-8q)^{3/2}\right).$$

By comparison

$$V(X/2; X|X > 8q) = \frac{1}{6q}\left(\frac{X^2}{2} - \sqrt{X}(X-4q)^{3/2} + \frac{1}{2}\sqrt{X}(X-8q)^{3/2}\right)$$

The difference of the two is thus

$$\begin{aligned} V(\overline{d}; X|\cdot) &- V(X/2; X|\cdot) \\ &= \frac{1}{6q}\Big(\sqrt{X-4q}(X-8q)^{3/2} - \frac{\sqrt{X}}{2}(X-8q)^{3/2} - \frac{(X-8q)^2}{2}\Big) \\ &= \frac{1}{6q}\frac{(X-8q)^{3/2}}{2}\Big(2\sqrt{X-4q} - \sqrt{X} - \sqrt{(X-8q)}\Big), \end{aligned}$$

which is positive if

$$4(X-4q) > 2X - 8q \Leftrightarrow X > 4q$$

and holds by assumption.

To establish the second part of the lemma, note that $d > 4q$ can only occur for $X > 8q$. We will show that $V_d(d; X) < 0$ for all $d > 4q$ when $X > 8q$. Towards this, observe that

$$V_{ddX}(d; X) = -\frac{24q^3}{(X-d)^{\frac{5}{2}}(X-d-4q)^{\frac{3}{2}}} < 0.$$

Thus, the $V_{dd}(d; X)$ is lowest for $X \to \infty$ which is

$$V_{dd}(d; X)|_{\lim_{X\to\infty}} = 2\frac{d^2 - d^{\frac{3}{2}}\sqrt{d-4q} - 2q(d+q)}{d^{\frac{3}{2}}\sqrt{d-4q}} > 0.$$

Thus, $V_d(d; X)$ is highest for $d = \frac{X}{2}$ which is

$$V_d(d; X)|_{d=X/2} = 0.$$

Hence, $V(d; X)$ is decreasing in d for $X > 8q$ and $d > 4q$. The optimal distance cannot be larger than $4q$. □

Lemma 3. $d^0(X) < X/2 \Rightarrow \frac{dV(d^0(X);X)}{dX} < 0$.

Proof. By the envelope theorem,

$$\frac{dV(d^0(X); X)}{dX} = V_X(d^0(X); X).$$

This derivative is negative for $X \geq 4q$ and for all $d \in [0, X-4q]$ by Lemma 25 in Appendix H. If $X \geq 8q$, that claim is sufficient. By Lemma 1 we know that $X \geq 6q$ whenever $d^0(X) \neq X/2$. In 2.(i) in the proof of Lemma 25, page S.14, we show that $V_d > 0$ for $d \in [X-4q, X/2)$ if $X \leq 8q$. Hence, if $d^0(X) \neq X/2$, then $d^0(X) \leq X-4q$ and the inequality proved in Lemma 25 proves Lemma 3. □

Lemma 4. *$d^0(X) < X/2$ for some $X \in [6q, 8q) \Rightarrow d^0(X) < X/2$ for all $X' > X$.*

Proof. It suffices to consider $X' < 8q$ by Lemma 2. We prove the claim by showing that $V(d_c^0(X); X)$ for any interior critical point $d_c^0(X) < X/2$ cuts $V(X/2; X)$ from below at *any* potential intersection. Thus, there is at most one switch from $d^0(X) = X/2$ to $d^0(X) < X/2$ and no switch back. Continuity then implies the statement.

$V(d; X)$ is a continuously differentiable function in X and d. Thus, any interior (local) optimum $d_c^0(X)$ is continuous as well and so are $V(d_c^0(X); X)$ and $V(X/2; X)$. We now show that if $V(d_c^0(X); X) = V(X/2; X)$ for some local optimum $d_c^0(X) < X/2$ and $X \in [6q, 8q]$, then $\mathrm{d}V(d_c^0(X); X)/\mathrm{d}X > \mathrm{d}V(X/2; X)/\mathrm{d}X$. Note that $\mathrm{d}V(d_c^0(X), X)/\mathrm{d}X < 0$ by Lemma 3. The first intersection therefore can occur only in a region when $V(X/2, X)$ is decreasing and must be such that $\mathrm{d}V(X/2, X)/\mathrm{d}X < \mathrm{d}V(d_c^0(X), X)/\mathrm{d}X$. We prove that this is the only potential intersection in Lemma 26 in Appendix H where we show that $\mathrm{d}^2V(X/2, X)/(\mathrm{d}X)^2 < 0$ and $\mathrm{d}^2V(d_c^0(X), X)/(\mathrm{d}X)^2 > 0$. □

Lemma 5. *$V(d^0(X); X)$ is continuous in X. As $X \to \infty$, it converges uniformly to $V(d; X)$ and $d^0(X) \to d^0(\infty)$. For any $X > 6q$ we have $d^0(X) > 3q$ and $V(d^0(X), X) > V(3q, \infty)$.*

Proof. Continuity follows because $V(d^0(X); X) = \max_d V(d; X)$ with $V(d; X)$ continuous in both $d \in [0, X/2]$ and X. Now take any sequence of increasing X_n with $\lim_{n\to\infty} X_n = \infty$. For any $\delta(d)$, $\exists n$ such that $V_n(d; X_n) - V(d; \infty) < \delta(d)$ as can be seen from the formulation in the proof of Proposition 1. Hence, $V(d; X_n)$ converges uniformly to $V(d; \infty)$. By uniform convergence the maximizer $d^0(X_n)$ of $V(d; X_n)$ converges too. To see convergence from above, observe that $V(3q; X) > V(3q; \infty)$ for any $6q < X < \infty$.

Finally, from Corollary 2 and the proof of Proposition 1 we know that $V(d; \infty)$ describes the value of an area of length d. That value is increasing for $d < 3q$ and decreasing for $d > 3q$. Now suppose $X > 6q$ and $d^0(X) < 3q$. Then by increasing d both areas created become closer to $3q$ and are thus increasing in value. A contradiction to $d^0(X)$ being the maximizer. □

Lemma 6. *$V(d^0(X); X)$ is single peaked with an interior peak at $\check{X}^0 \approx 6.204q$ with $d^0(\check{X}^0) \approx 3.102q$.*

Proof. Lemma 6 follows from continuity of $V(X/2; X)$ (by Lemma 5) and Lemmata 1 to 4.

The peak can be computed. It is the (real) solution to

$$\frac{X}{X - q} = 2\frac{\sqrt{X - 4q}}{\sqrt{X}}. \tag{2}$$

Defining $m := \frac{X}{q}$ and the above reduces to

$$\frac{m}{m-1} = 2\sqrt{\frac{(m-4)}{m}}.$$

For $m > 4$, the LHS decreases and the RHS increases in m. The solution is:

$$m = \frac{2}{3}\left(4 + (19 - 3\sqrt{2})^{(1/3)} + (19 + 3\sqrt{2})^{(1/3)}\right) \approx 6.204. \quad \square$$

Lemma 7. *Expanding knowledge trumps deepening knowledge if and only if $X < \widehat{X}^0 \approx 4.338q$.*

Proof. $V(3q;X) > V(3q;\infty)$ for $X \geq 6q$ by direct comparison at $X = 6q$ and Lemmata 3, 5 and 6. For $X \in [0, 6q]$ we need to consider only $d^0(X) = X/2$ by Lemma 1. We compare

$$V(X/2;X) = \frac{X^2}{12q} - \frac{\sqrt{X}(X-4q)^{3/2}}{6q}$$

with $V(3q;\infty) = \frac{3q}{2}$. Defining $\ell := \frac{X}{q}$, the two intersect at

$$\left(\frac{\ell^2}{12} - \frac{\sqrt{\ell}}{6}(\ell-4)^{(3/2)} - 3/2\right) = 0$$

which has one solution such that $\ell \leq 6$ at $\ell \approx 4.338$. □

Lemma 8. $4q < \widehat{X}^0 < 6q < \check{X}^0 < \widetilde{X}^0 < 8q$.

Proof. The first two inequalities follow from Lemma 7, the third from Lemma 6. Existence of $\widetilde{X}^0$ and the fourth inequality follow from Lemma 4. Lemma 2 implies the last inequality. □

□

B.4 Proof of Proposition 2

The proof contains several steps and we break it into parts. Part 0 provides preliminary observations used in the following steps. Part 1 proves the results for expanding knowledge. Part 2 proves the results for deepening knowledge.

Throughout, we make use of the first-order necessary conditions for an interior solution which we show are sufficient to characterize the researcher's optimal choice when we use them:

$$\eta\tilde{c}_\rho(\rho) = \frac{V(d;X)}{\sigma^2(d;X)}, \qquad \text{(FOC}^\rho\text{)}$$

$$\rho V_d(d;X) = \eta\tilde{c}(\rho)\sigma_d^2(d;X). \qquad \text{(FOC}^d\text{)}$$

Proof.

Part 0: Preliminaries. We begin by showing that an interior choice of (d,ρ) is optimal and that the relation of d and ρ depends on the ratio of the benefit of discoveries and the variance of the conjecture.

Lemma 9. *There is a non-trivial optimal choice with $\infty > d > 0, 1 > \rho > 0$ on any interval with positive length, $X \in \mathbb{R}^+ \cup \infty$. Solving the first-order condition is a necessary for optimality of $\rho(X)$.*

Proof. The researcher can always guarantee a non-negative payoff by choosing either $d = 0$ or $\rho = 0$. Hence, her value is bounded from below, $U_R(X) \equiv \max_{d,\rho} u_R(d,\rho;X) \geq 0$. Next, note that $u_R(\rho = 0, d > \varepsilon; X) = 0$ for some small $\varepsilon > 0$ and that $\frac{\partial u_R(\rho=0,d>\varepsilon;X)}{\partial\rho} = V(\varepsilon, X) > 0$ by Proposition 1. Therefore, on any interval X there is a maximum with $d > 0, \rho > 0$.

Moreover, by Corollary 3 the value of knowledge is bounded $V(d,X) \leq M < \infty$ and $\lim_{\rho\to 1}\tilde{c}(\rho) = \infty$. Therefore, the optimal $\rho < 1$. Finally, $V(d,\infty)$ is decreasing in d for d

large enough while the cost $\eta\tilde{c}(\rho)\sigma^2(d,\infty)$ is increasing in d. Hence, the optimal distance is bounded $d \leq D < \infty$.

It follows from Lemma 9 and continuous differentiability of the objective that a necessary condition for the optimal $\rho(X)$ is that it solves (FOC$^\rho$). Note that for the distance, $d(X)$, this result is not immediate as for deepening intervals the distance has an exogenous upper bound at $X/2$. □

To determine whether d and ρ are substitutes, we make use of the previous lemma; in particular, of the first-order necessary condition for an optimal $\rho(X)$. The left-hand side of the first order condition (FOC$^\rho$) is increasing in ρ for any ρ and independent of d. Thus, we only need to derive under which conditions the right-hand side of this equation is increasing or decreasing. When the right-hand side is increasing (decresing) in d, ρ and d are complements (substitutes).

Part 1. Expanding knowledge.

Step 1. Proof of Item 1 i.).

When expanding knowledge, the ratio of benefit of discovery and variance of the conjecture is

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{1}{6q}\left(6q - d + \mathbf{1}_{d>4q}\frac{(d-4q)^{3/2}}{\sqrt{d}}\right)$$

which has derivative

$$\frac{1}{6q}\left(-1 + \mathbf{1}_{d>4q}\frac{(d+2q)\sqrt{d-4q}}{d^{3/2}}\right) < 0.$$

Thus, output and novelty are substitutes in the expanding area.

Step 2. Proof of Item 1 ii.).

Next, we characterize the optimal choice when expanding knowledge.

Lemma 10. *When expanding knowledge, the optimal choice is characterized by the first-order conditions (FOCs). The FOCs are sufficient and the optimal $d^\infty \in (2q, 3q)$. The researcher's value is strictly positive $U_R(X = \infty) > 0$.*

Proof. We proceed in three steps. First, we show that the distance is at most $3q$. Second, we show that the first-order conditions are sufficient when expanding knowledge. Third, we characterize the optimal choice of the researcher.

Step 2.1. $d \leq 3q$. Fix any $\rho \geq 0$. Since $\sigma^2(d;\infty)$ is increasing in d, it is immediate that the researcher's utility is non-increasing in d if $V(d;\infty)$ is decreasing in d. Combining this observation with Corollary 2, it is sufficient to restrict attention to $d \leq 3q$.

Step 2.2. FOCs sufficient. By Lemma 9, the researcher's optimal choice is interior and, hence, characterized by the first-order conditions. To see the sufficiency of the first-order conditions, note that the first principal minor of Hessian is $\rho V_{dd} - \eta c\sigma^2_{dd} = -\rho\frac{1}{3q} < 0$ as $\sigma^2_{dd} = 0$ and that the second principal minor is given by the determinant of the Hessian at the critical point:

$$-\rho V_{dd}(d;\infty)\eta\tilde{c}_{\rho\rho}(\rho)\sigma^2(d;\infty) - (V_d - \eta\tilde{c}_\rho(\rho)\sigma^2_d(d;\infty))^2$$

$$=\rho\frac{1}{3q}\eta\tilde{c}_{\rho\rho}(\rho)d-\left(-\frac{d}{3q}+1-\eta\tilde{c}_{\rho}(\rho)\right)^2 \tag{3}$$

$$=\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)}\frac{V(d;\infty)}{3q}-\left(-\frac{d}{3q}+1-\frac{V(d;\infty)}{\sigma^2(d;\infty)}\right)^2.$$

The first equality follows from $V_{dd}=-\frac{1}{3q}$ and $\sigma^2(d;\infty)=d$. The second equality follows from combining $\sigma^2(d;\infty)=d$ with the first-order condition (FOC^ρ) from above via $\eta\sigma^2(d;\infty)=\frac{V(d;\infty)}{\tilde{c}_\rho(\rho)}$ and replacing accordingly.

Substituting for $V(d;\infty)=d-d^2/(6q)$ (as we restrict attention $d\leq 3q$ because of Step 2.1.) yields the following condition for a positive second principal minor:

$$0<\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)}\frac{V(d;\infty)}{3q}-\left(-\frac{d}{3q}+1-\frac{V(d;\infty)}{\sigma^2(d;\infty)}\right)^2$$

$$\Leftrightarrow \rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)}>\left(-\frac{d}{3q}+1-\frac{\frac{-d^2}{6q}+d}{d}\right)^2\frac{3q}{-\frac{d^2}{6q}+d}$$

$$\Leftrightarrow \rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_{\rho}(\rho)}>\frac{d}{2(6q-d)}.$$

The inequality in the last line holds because the properties of $\tilde{c}(\rho)$ imply $LHS\geq 1$ while $RHS\leq\frac{1}{2}$ for $d\leq 3q$.

Step 2.3. Characterization. Substituting the expressions for $V(d;\infty)$ and $\sigma^2(d;\infty)$ for expanding knowledge into the first-order condition (FOC^d) yields

$$\rho\left(1-\frac{d}{3q}\right)=\eta\tilde{c}(\rho). \tag{4}$$

Replacing η via equation (FOC^ρ) and solving for d we obtain

$$d^\infty=3q\left(1-\frac{\tilde{c}(\rho)}{2\tilde{c}_\rho(\rho)\rho-\tilde{c}(\rho)}\right)\in(2q,3q)$$

where the bounds follow from the properties of $\tilde{c}$. □

Part 2. Proof for deepening knowledge.

Step 1. Proof of Item 2 i.).[27] We prove the result for the different regions separately. In particular, we distinguish between different area lengths X and different distances d.

Step 1.1. $X<4q$. Consider deepening knowledge when $X<4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)}=\frac{2X}{6q}.$$

Thus, output and novelty are independent in short research areas.

Step 1.2. $X\in(4q,8q)$.

Step 1.2.(i). $d<4q$, and $X-d>4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)}=\frac{1}{6q}\left(2X-\frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^2(d;X)}+\frac{\sqrt{X-d}(X-d-4q)^{3/2}}{\sigma^2(d;X)}\right)$$

[27] A Mathematica file verifying the computations is available from the authors.

with derivative

$$\frac{1}{6q}\frac{\sigma_d^2(d;X)}{\sigma^4(d;X)}\left(\sqrt{X}(X-4q)^{3/2}-\sqrt{X-d}(X-d-4q)^{3/2}-2\sigma^2(d;X)\frac{(X-d-q)\sqrt{X-d-4q}}{\sqrt{X-d}}\right).$$

Note that evaluated at the limit $d \to 0$ and $d = X - 4q$, this derivative is

$$\frac{\partial}{\partial d}\left(\frac{V(d;X)}{\sigma^2(d;X)}\right)\Big|_{\lim_{d\to 0}} = -\frac{X^2-8qX+10q^2}{6qX^{3/2}\sqrt{X-4q}}$$
$$\frac{\partial}{\partial d}\left(\frac{V(d;X)}{\sigma^2(d;X)}\right)\Big|_{d=X-4q} = X^{3/2}\frac{X-8q}{96q^3\sqrt{X-4q}} > 0.$$

Thus, for $d \to X - 4q$, output and novelty are always complements. However, for $\lim_{d\to 0}$, the derivative is positive (negative) if $X < (>)(4+\sqrt{6})q$.

It remains to show that the derivative has one root for $X \in [(4+\sqrt{6})q, 8q]$ and no root for $X < (4+\sqrt{6})q$ for the result to follow.

To see that this is indeed the case, solve for the root of the derivative of the ratio to obtain

$$\hat{d}_{\pm} = \frac{2}{X-6q}\left((X^2-6qX+6q^2) \pm (X-2q)\sqrt{\frac{18q^2-8qX+X^2}{2}}\right). \quad (5)$$

Note first that $\hat{d}_+ > \frac{X}{2}$. Thus, the only feasible root is $\hat{d}_-$. However, $\hat{d}_- \in [0, X-4q]$ only for $X > (4+\sqrt{6})q$.

Step 1.2.(ii). $d < 4q$ and $X - d < 4q$. In this case,

$$\frac{V(d;X)}{\sigma^2(d;X)} = \frac{1}{6q}\left(2X - \frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^2(d;X)}\right)$$

with derivative

$$\sigma_d^2(d;X)\frac{\sqrt{X}(X-4q)^{3/2}}{\sigma^4(d;X)} > 0.$$

Thus, output and novelty are complements.

Summary Step 1.2. For $X \in (4q, 8q)$ output and novelty are substitutes for small d when $X > (4+\sqrt{6})q$ and complements for large d. They are complements throughout when $X < (4+\sqrt{6})q$.

Step 1.3. $X > 8q$.

Step 1.3.(i). $d < 4q$. This case is analogous to the case of Step 1.2.(i) because for all $X > 8q$ and for all $d < X/2$, $X - d > 4q$.[28] However, the root from Equation (5) $\hat{d}_- > 4q$. Thus, output and novelty are substitutes.

Step 1.3.(ii). $d > 4q$. Note that in this case, $V_d(d;X) < 0$. Thus, the ratio is decreasing for all $d > 4q$ when $X > 8q$ as the variance is always increasing in d.

Step 2. Proof of Item 2 ii.).

The following lemma which we also prove in steps implies the result.

[28] Note that whenever $X - d < 4q$, $d > 4q$ and a simple relabeling of d and $X - d$ brings us back to the discussed case.

Lemma 11. *The researcher's optimal choice of distance is on the midpoint of the area, $d = \frac{X}{2}$, for $X \leq \widetilde{X}$ and interior, $d < \frac{X}{2}$. At $\widetilde{X}$, payoff $U_R(X)$ is decreasing. Further, $\lim_{X\to\infty} d(X) = d^\infty$ from above. Any optimal distance satisfies $d \leq 4q$.*

Proof. Define $d^b := X/2$ which we refer to as the boundary solution, and d^i as the solution d to (FOC^d) assuming $d < X/2$ (if that exists) which we refer to as the interior solution.

Step 2.1. d^b always a candidate solution. Note first that the choice d^b always constitutes a local maximum as the marginal cost of distance is zero at this point, $\frac{\partial\sigma^2(d,X)}{\partial d} = 1 - \frac{2d}{X}$. Moreover, we see in the proof of Corollary 3 that also the marginal benefit is zero at $d = X/2$. Finally, for any choice of d, there is a unique ρ that solves (FOC$^\rho$) because (FOC$^\rho$) given d, has a continuous, strictly increasing, left-hand side that starts at $\tilde{c}_\rho(0) = 0$, has limit $\lim_{\rho\to 1}\tilde{c}_\rho(\rho) = \infty$ and has a constant right-hand side. Hence, the boundary solution with d^b is always a candidate solution.

Step 2.2. $d(X) = X/2$ if $X \leq 4q$. Recall the first-order conditions (FOC$^\rho$) and (FOC^d). Assuming an interior solution d^i, replacing η via (FOC$^\rho$) in (FOC^d) we obtain for (FOC^d)

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{\frac{\tilde{c}(\rho)}{\rho}}{\tilde{c}_\rho(\rho)}.$$

It follows from the properties of $\tilde{c}(\rho)$ that the $RHS \in [0, 1/2]$ and decreasing. Thus, if the $LHS > 1/2$ for all ρ, it is beneficial to increase d if possible and the boundary choice d^b is optimal. For $X \leq 4q$

$$\frac{\frac{V_d}{\sigma_d^2}}{\frac{V}{\sigma^2}} = \frac{\frac{2(X-2d)}{\frac{X-2d}{X}}}{\frac{2(dX-d^2)}{\frac{d(X-d)}{X}}} = 1.$$

Hence, for small areas, the boundary choice is indeed optimal.

Step 2.3. $d(X) < X/2$ if $X > 8q$. Note first that the variance of the question on the boundary is always larger than for any interior question as $\sigma^2 = \frac{d(X-d)}{X}$ is increasing in d. Hence, if the benefit of research V is larger for an interior question than for the boundary question, the researcher can obtain a higher payoff by choosing an interior question with the same ρ as for the boundary question: the cost are lower, the success probability is the same, and the benefit upon success are higher. The benefit of finding an answer on the boundary of an area with $X > 8q$ is always smaller than for some interior distance by Lemma 2 from the proof of Corollary 3. Hence, an interior choice is optimal for $X > 8q$.

Step 2.4. We prove the following (sub-)lemma.

Lemma 12. *If d^i is optimal it must be that $d^i < 4q$ and that $X - d^i > 4q$.*

Proof. For $X \in (4q, 8q)$ and $X - d < 4q$,

$$\frac{\frac{V_d(d,X)}{\sigma_d^2(d,X)}}{\frac{V(d,X)}{\sigma^2(d,X)}} = \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}}$$

which is decreasing in d with limit

$$\lim_{d\to X/2} \frac{2d(X-d)}{-2d^2 + 2dX - \sqrt{X}(X-4q)^{3/2}} = \frac{X^2/2}{X^2/2 - \sqrt{X}(X-4q)^{3/2}}$$

which, in turn, is increasing in X and 1 for $X = 4q$. Hence, any interior solution must be such that $X - d > 4q$ by the same logic as in Step 1.2. For $X - d < 4q$, the first-order condition with respect to d is always positive. For any area with $X > 8q$, $X - d^i > 4q$. That $d^i < 4q$ follows from the benefit of a discovery being decreasing in d whenever $d > 4q$ (see Corollary 3). □

Summary Step 2.1-2.4. We know that (i) in areas with $X < 4q$, the researcher's distance choice on the deepening area will be d^b, (ii) in areas with $X > 8q$ the researcher's distance choice will be d^i, (iii) in areas with $X \in [4q, 8q]$ the researcher's distance choice may d^i or d^b, but (iv) if the solution is d^i, it has to satisfy $X - d > 4q$ and $d < 4q$.[29]

Step 2.5. Single crossing of the payoffs. Next, we show that the payoffs, $U_R(d^b; X)$ and $(U_R(d^i; X)$ cross only once assuming $\rho(d, X)$ is chosen optimally. We use three observations to show this.

1. First, at area length X for which $U_R(d^b; X) = U_R(d^i; X)$, the payoff at the boundary solution must be decreasing faster than the payoff at the interior solution.
2. Second, on the interval $[4q, 8q]$ the payoff of the boundary solution, $U_R(b^d; X)$ has a strictly lower second derivative with respect to X for all X than that of the interior solution $U_R(d^i; X)$. Hence, the two values can cross at most once on this interval.
3. Third, $U_R(d^b; X) \leq U_R(d^i; X)$ if $X \geq 8q$.

The first observation follows because the first switch is from the boundary solution to the interior solution by continuous differentiability of all terms and the observation from above that $d(X) = X/2$ for $X < 4q$. The third observation is shown in Step 2.3 above.

The second observation follows from totally differentiating U_R for the two types of local maxima. Using envelope conditions, we obtain that the payoff is concave in the boundary solution and convex in the interior solution which implies the second observation. Define $\varphi(X) := \max_\rho u(d = X/2, \rho, X)$ for the boundary; we show in Lemma 27 in Appendix H that $\varphi(X)$ is concave. In Lemma 28 in Appendix H we show that $U_R(X) = \max_{\rho,d} u(d, \rho, X)$ is convex in X provided that the maximizer $d(X) < X/2$. The result follows.

Step 2.6. Asymptotics. It remains to show the asymptotics. As $X \to \infty$, $V(d, X)$ converges to $V(d, \infty)$ and $\sigma^2(d, X)$ to $\sigma^2(d, \infty)$ and the researcher's optimization on the deepening interval converges to the optimization on the expanding interval which has a unique and interior maximum at (d^∞, ρ^∞). In particular, if such an interior optimum exists, the envelope condition implies that

$$\frac{\mathrm{d}U_R(d^i(X); X)}{\mathrm{d}X} = \rho V_X(d^i, X) - \eta\tilde{c}(\rho)\sigma_X^2(d^i, X) < 0$$

as $V_X(d, X) < 0$ according to Corollary 3 for $X > 4q$ and $X - d > 4q$ and $\sigma_X^2(d, X) > 0$. Hence, the payoff of any optimal interior choice is decreasing in X. □

□

B.5 Proof of Proposition 3

We prove the statements in Proposition 3 in reverse order. Some parts rely on the lemmata from the proof of Propositions 1 and 2.

Proof.

[29]From Lemmata 2, 4 and 5 any interior choice that maximizes V (ignoring cost) satisfies $X - d > 4q$ and $d < 4q$.

Step 1: Proof of Item 3. We use a series of lemmata to show that a local maximum, $\check{X}$, exists (Lemmata 13 and 14) and that it is global (Lemma 15).

Lemma 13. *Fix $d = X/2$ and assume that an interior optimum exists. Then $U_R(X|d = X/2)$ is maximal only if the total differential $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$.*

Proof. Under the assumption that $d = X/2$, $U_R(X)$ is defined and continuously differentiable for all $X \in [0, \infty)$ despite the indicator functions.[30] Because $X = 0$ implies $U_R(X = 0) = 0$, because $U_R(X)$ declines for X large enough and because Lemma 9 holds, there is an interior X at which $U_R(X)$ is maximized.

Then, because $U_R(X)$ is maximal for some interior X and differentiable, it needs to satisfy

$$\frac{\partial U_R}{\partial X} = 0.$$

By assumption, we have $d(\check{X}) = X/2$ and the first order condition with respect to ρ holds. Thus,

$$\rho \frac{\mathrm{d}V(d = X/2; X)}{\mathrm{d}X} = \frac{\eta}{4}\tilde{c}(\rho).$$

The right-hand side is non-negative, which implies the desired result.[31] □

Lemma 14. *The value of the deepening boundary solution $U_R(d \equiv \frac{X}{2}; X)$ peaks in X at $\check{X} \in (4q, \check{X}^0]$.*

Proof. Note that $U_R'(d = X/2; X) > 0$ for $X \in [0, 4q]$. This follows because in this case $U_R(d = X/2, X) = \rho\frac{X^2}{12q} - \eta\tilde{c}(\rho)\frac{X}{4}$ and, hence, $U_R'(d \equiv X/2, X) = \rho\frac{X}{6q} - \eta\tilde{c}(\rho)\frac{1}{4}$. Using optimality of ρ via the (FOC$^\rho$),

$$\frac{X}{3q} = \eta\tilde{c}_\rho(\rho) \Rightarrow \frac{X}{6q} = \frac{\eta\tilde{c}_\rho(\rho)}{2}$$

which yields

$$\begin{aligned} U_R'(X) =& \rho\frac{\eta\tilde{c}_\rho(\rho)}{2} - \eta\tilde{c}(\rho)\frac{1}{4} \\ =& \frac{\tilde{c}_\rho(\rho)}{4}\rho\eta\left(2 - \frac{\tilde{c}(\rho)}{\rho\tilde{c}_\rho(\rho)}\right) > 0 \end{aligned}$$

where the inequality follows again from the properties of $\tilde{c}(\rho)$.

Moreover, $U_R(X)$ is strictly concave on $[4q, 8q]$ as $V(d = X/2, X)$ is concave on this interval (see the proof of Corollary 3) and $\sigma^2_{XX}(d = X/2, X) = 0$ implying[32]

$$U_R''(X) = \rho\frac{\mathrm{d}\mathrm{d}V(d = X/2; X)}{\mathrm{d}X\mathrm{d}X} < 0.$$

[30] Note that the terms appearing in the indicator functions are of the form $\sqrt{a}(a - 4q)^{3/2}$. Taking the limit of their derivative from above to $4q$ yields zero such that the left and right derivative coincide at the point at which the indicator functions become active.

[31] The RHS is only 0 if $\eta = 0$, $\rho(X) = 1$ and $U_R(X) = V(X)$.

[32] Note that we totally differentiate the value twice and all $\rho'(X)$ and $\rho''(X)$ terms drop out by optimality of ρ by applying the first-order condition directly and total differentiation of the first-order condition.

For $X > \check{X}^0$, $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} < 0$ by the definition of $\check{X}^0$ implying that for $X > \check{X}^0$ the researcher's value is decreasing. By Lemma 13, it follows that the value-maximizing area length $\check{X} \in (4q, \check{X}^0]$. □

Lemma 15. *The researcher's payoff $U_R(X)$ is single-peaked in X with the maximum attained at $\check{X}$.*

Proof. The result follows from 3 observations: First, $\tilde{X} > \check{X} > 4q$ by Lemmata 11 and 14. Second, $V_X(d;X) < 0$ if $X > 4q$ and $d < X/2$ by Lemma 3. Third, by the envelope theorem, if $d(X) < X/2$ it holds that $\partial U_R(X)/\partial X = \rho(X)V_X(d(X);X) - \eta\tilde{c}(\rho(X))\sigma_X(d(X);X) < \rho(X)V_X(d(X);X)$. Thus, the payoff of the interior solution cuts the payoff of the boundary solution from below at an area where both payoffs are decreasing. The single peak is at $\check{X}$. □

Step 2. Proof of Item 2.

Step 2.1 Maximum of $d(X)$ at $\tilde{X}$. By Lemma 11, $d(X)$ is increasing for $X < \tilde{X}$. By Lemma 12, we know that any interior solution d^i is such that $d^i < 4q < X - d^i$ and thus strictly smaller than $X/2$. Thus, $d(X)$ decreases when it switches from the boundary to interior solution. Thus, $d(\tilde{X})$ is a maximum.

Step 2.2 Maximum of $\rho(X)$ at $\dot{X}$. We guess (and verify in step 4. below) that a maximum of $\rho(X)$ exists in the range $[\widehat{X}, \tilde{X}]$, that is the region in which it is optimal to deepen knowledge and to select the mid point $d = X/2$.

Lemma 16. *Suppose $d = X/2$ is optimal for a range $[\underline{X}, \overline{X}]$ such that $d(X) = X/2$. Then the optimal $\rho(X)$ is single-peaked in that range. It is highest at $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}}$*

Proof. By Lemma 13, we know that $\frac{\mathrm{d}V(d=X/2;X)}{\mathrm{d}X} \geq 0$ and by Lemma 14 $\overline{X} > \widehat{X}^0$. Moreover, recall $\sigma^2(d = X/2; X) = X/4$. The first-order condition with respect to ρ becomes

$$\frac{V(X/2;X)}{X} = \frac{\eta}{4}\tilde{c}_\rho(\rho),$$

With

$$\frac{V(X/2;X)}{X} = \frac{X}{12q} - \mathbf{1}_{X>4q}\frac{(X-4q)^{3/2}}{\sqrt{X}6q}.$$

The latter is continuous and concave. Since $\tilde{c}(\rho)$ is an increasing, twice continuously differentiable and convex function, ρ increases in X if and only if $V(X/2;X)/X$ increases in X. By concavity of $V(X/2;X)/X$ that implies single peakedness.

Thus, $\dot{X}$ is independent of η and given by $\dot{X} = \frac{8\cos\left(\frac{\pi}{18}\right)}{\sqrt{3}} \approx 4.548q$. □

Step 3. Proof of Item 1. The following lemma proves the item.

Lemma 17. *$\widehat{X}$ exists, $\lim_{X\searrow\widehat{X}}\rho(X) > \rho^\infty$, and $\widehat{X}$ decreases in η.*

Proof. As $X \to 0$, $d(X) \to 0$ and thus $U_R(X) \to 0$. By Lemma 10, $U_R(\infty) > 0$. Thus, by continuity of $U_R(X)$, $\exists\widehat{X} > 0$ such that expanding research dominates deepening research for all $X < \widehat{X}$. Cost are increasing in X and by Corollary 3, $V(d; X \in (\widehat{X}^0, \infty)) > V(d;\infty)$ which implies $U_R(X \in (\widehat{X}^0, \infty)) > U_R(\infty)$. By Lemma 15 and again continuity of $U_R(X)$, that payoff is maximal at $\check{X}$. Thus, we obtain that $\widehat{X}$ exists and that $\widehat{X} < \check{X}$.

We now show that $\lim_{X \searrow \widehat{X}} \rho(X) > \rho^\infty$ holds if $\widehat{X} < 6q$, then we show $\widehat{X}$ decreases in η which together with the observation that $\widehat{X}^0 < 6q$ is sufficient to prove the lemma.

At $\widehat{X}$ we have

$$\begin{aligned} U_R(\widehat{X}) &= U_R(\infty) \\ \rho(\widehat{X})V(\widehat{X}/2;\widehat{X}) - \eta\tilde{c}(\rho(\widehat{X}))\frac{\widehat{X}}{4} &= \rho^\infty V(d^\infty;\infty) - \eta\tilde{c}(\rho^\infty)d^\infty. \end{aligned} \tag{6}$$

where the fact that $d(\widehat{X}) = \widehat{X}/2$ follows from Lemmata 11, 14 and 15. Moreover, the following has to hold by optimality

$$V(d^\infty;\infty) = \eta\tilde{c}_\rho(\rho^\infty)d^\infty \tag{FOC ρ^∞}$$

$$V(\widehat{X}/2;\widehat{X}/2) = \eta\tilde{c}_\rho(\rho(\widehat{X}))\frac{\widehat{X}}{4} \tag{FOC $\rho^{\widehat{X}}$}$$

Claim 1: $\rho^\infty < \rho(\widehat{X})$ if $\widehat{X} < 6q$. Using (FOC ρ^∞) and (FOC $\rho^{\widehat{X}}$) we obtain that by the properties of the error function $\rho(\widehat{X}) > \rho^\infty$ if and only if

$$4\frac{V(\widehat{X}/2;\widehat{X}/2)}{X} > \frac{V(d^\infty;\infty)}{d^\infty}.$$

Case 1: $\widehat{X} > 4q$. Substituting for the $V(\cdot)$'s the above becomes[33]

$$\begin{aligned} \frac{\widehat{X}}{3q} - \frac{2}{3q}\frac{(\widehat{X}-4q)^{3/2}}{\sqrt{\widehat{X}}} &> 1 - \frac{d^\infty}{6q} \\ \Leftrightarrow \qquad d^\infty + 2\widehat{X} - 4\underbrace{\frac{(\widehat{X}-4q)^{3/2}}{\sqrt{\widehat{X}}}}_{<(\widehat{X}-4q)} &> 6q \end{aligned}$$

A sufficient condition for the above to hold is thus that

$$d^\infty - 2\widehat{X} + 10q > 0$$

Using that $d^\infty > 2q$ by Lemma 10 we obtain that a sufficient condition for $\rho(\widehat{X}) > \rho^\infty$ is that $\widehat{X} < 6q$.

Case 2: $\widehat{X} \in (2q, 4q]$. Performing the same steps only assuming that $\widehat{X} \in [2q, 4q]$ we

$$\begin{aligned} \frac{\widehat{X}}{3q} &> 1 - \frac{d^\infty}{6q} \\ \Leftrightarrow 2\widehat{X} &> 6q - d^\infty > 4q \end{aligned}$$

which implies the desired result.

Case 3: $\widehat{X} < 2q$ We show that case 3 never occurs, that is $\widehat{X} > 2q$. To do so we compare $U_R(d = 2q; \infty)$ with $U_R(d = 1q; X = 2q)$ and show that the former is always larger. Hence, $X = 2q < \widehat{X}$ for any η.

[33]Since $\widehat{X} \leq \check{X} \leq 8q$ that case is irrelevant.

For $X = d = 2q$ we have that

$$\frac{X}{3q} = 1 - \frac{d}{6q},$$

and thus $\rho(X = 2q) = \rho(d; \infty) = \rho$ (cf. case 2). Moreover, we have that

$$V(1q; 2q) = q/3 \qquad\qquad V(2q; \infty) = 4/3q,$$

and (FOC ρ^X) implies

$$4V(1q; 2q)/2q = 2/3 = \eta\tilde{c}_\rho(\rho)$$

Since $\tilde{c}_\rho(\rho) > \tilde{c}(\rho)/\rho$ for any $\rho > 0$ that implies $\eta\tilde{c}(\rho)/\rho < 2/3$.

Now take

$$\begin{aligned} U_R(d = 2q; \infty) - U_R(X = 2q) \\ \rho\frac{4q}{3} - \eta\tilde{c}(\rho) - \rho\frac{q}{3} + \eta\tilde{c}(\rho)\frac{q}{2} \\ q\left(\rho - \frac{3}{2}\eta\tilde{c}(\rho)\right), \end{aligned}$$

which is positive whenever $\eta\tilde{c}(\rho)/\rho < 2/3$ which we know has to hold. Thus, $U_R(d = 2q; \infty) > U_R(X = 2q)$ and therefore $\widehat{X} < 2q$.

Claim 2: If $\rho^\infty < \rho(\widehat{X})$ then $\widehat{X}$ decreases in η.

Using (FOC ρ^∞) and (FOC $\rho^{\widehat{X}}$) to replace the $V(\cdot)$'s in equation (6) and dividing by η we obtain

$$d^\infty\left(\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)\right) = \widetilde{X}/4\left(\rho(\widehat{X})\tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X}))\right)$$

from which we get

$$\widehat{X}/4 = d^\infty\frac{(\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty))}{\left(\rho(\widehat{X})\tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X}))\right)}.$$

Now we use the envelope theorem to calculate

$$\frac{\partial U_R(\widehat{X}) - U_R(\infty))}{\partial\eta} = \tilde{c}(\rho(\widehat{X}))\frac{\widehat{X}}{4} - \tilde{c}(\rho^\infty)d^\infty.$$

Replacing for $\widehat{X}$ implies that the RHS is positive if and only if

$$(\tilde{c}(\rho^\infty)) - \tilde{c}(\rho(\widehat{X}))\frac{\rho^\infty\tilde{c}_\rho(\rho^\infty) - \tilde{c}(\rho^\infty)}{\rho(\widehat{X})\tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X}))} > 0.$$

Using that $\rho\tilde{c}_\rho(\rho) > \tilde{c}(\rho)$ by the properties of the inverse error function and factoring out the denominator of the first term, the above holds if and only if

$$\begin{aligned} \tilde{c}(\rho^\infty)\rho^\infty\tilde{c}_\rho(\rho(\widehat{X})) - \tilde{c}(\rho(\widehat{X})\rho^\infty\tilde{c}_\rho(\rho^\infty) > 0 \\ \frac{\rho(\widehat{X})\tilde{c}_\rho(\rho(\widehat{X}))}{\tilde{c}(\rho(\widehat{X}))} > \frac{\rho^\infty\tilde{c}_\rho(\rho^\infty)}{\tilde{c}(\rho^\infty)} \end{aligned}$$

which holds if and only if $\rho(\widehat{X}) > \rho^\infty$ by the properties of the error function. Thus, $\widehat{X}$ decreases if $\rho(\widehat{X}) > \rho^\infty$.

Conclusion: Since $\widehat{X}^0 \in [2q, 6q]$, $\rho^\infty < \rho(\widehat{X})$ implying that $\widehat{X}$ is decreasing in η. $\square$

Step 4. $\widehat{X} < \dot{X} < \check{X} < \widetilde{X}$**.**
Step 4.1: $\check{X} > \dot{X}$. By the envelope theorem we need for $X = \check{X}$

$$\frac{\partial U_R(\check{X})}{\partial X} = \rho \frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X} - \frac{\eta}{4}\tilde{c}(\rho) = 0. \tag{7}$$

The FOC for ρ implies

$$\frac{V}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho)$$

Now assume for a contradiction that $\rho(\check{X})$ is increasing, then $V(\cdot)/\check{X}$ must be increasing which holds if and only if

$$\frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X}\check{X} > V(d = \check{X}/2; \check{X}).$$

But then we obtain the following contradiction to $U_R(\check{X})$ being maximal

$$\frac{\mathrm{d}V(d = \check{X}/2; \check{X})}{\mathrm{d}X} > \frac{V(d = \check{X}/2; \check{X})}{\check{X}} = \frac{\eta}{4}\tilde{c}_\rho(\rho) > \frac{\eta}{4}\frac{\tilde{c}(\rho)}{\rho}.$$

The first inequality follows because $V(d = \check{X}/2; \check{X})/\check{X}$ must be increasing, the equality follows by equation (7). The last inequality is a consequence of the properties of the inverse error function. By Lemma 16, $\rho(X)$ is single peaked in the relevant range which proves the claim.
Step 4.2: Ordering. By Lemma 17 we know that $\widehat{X} < \widehat{X}^0$. Thus, because $\widehat{X}^0 < \dot{X} \Rightarrow \widehat{X} < \dot{X}$. Moreover, $\widetilde{X} > \check{X}$ by Lemma 11 which concludes the proof. $\square$

B.6 Proof of Proposition 4

Proof. To prove the claim, we show that selecting a moonshot of length $6q$ is preferred to selecting the myopically optimal interval $3q$ for some $(\underline{\eta}, \overline{\eta})$ and $\delta(\eta) < 1$.

We first list the respective data. We restrict attention to η-levels such that $d(6q) = 3q$. These levels exist by continuity of the cost term and the fact that $\widetilde{X}^0 > 6q$ by Lemma 8.

Moonshot:

- *Value created in the first period:* $V(6q; \infty) = \frac{2}{\sqrt{3}}q$
- *Value created in the second period (if successful):* $V(3q; 6q) = \left(3 - \frac{2}{\sqrt{3}}\right)q$
- *Probability of discovery in the second period:* Solution to researcher's first-order condition

$$\frac{4V(3q; 6q)}{6q\eta} = \tilde{c}_\rho(\rho(6q))$$

which implies

$$\rho(6q) = erf\left(\frac{\sqrt{W\left(2\frac{\left(\frac{4V(3q;6q)}{6q\eta}\right)^2}{\pi}\right)}}{\sqrt{2}}\right) = erf\left(\sqrt{\frac{W\left(\frac{8\left(3-2/\sqrt{3}\right)}{9\eta^2\pi}\right)}{2}}\right)$$

where $W(\cdot)$ is the Lambert W function.

- *Continuation payoff:* Conditional on discovery in $t = 2$, the per-period continuation payoff from $t = 3$ onwards is $\rho^\infty V(d^\infty; \infty)$. The node (discovery in $t = 2$) is reached with probability $\rho(6q)$. The $t = 1$ net present value of the decision maker's continuation payoff at $t = 3$ is thus

$$\frac{\delta^2 \rho(6q)\rho^\infty}{1 - \delta\rho^\infty} V(d^\infty; \infty).$$

Myopic Optimum:

- *Value created in the first period:* $V(3q; \infty) = \frac{3}{2}q$
- *Value created in the second period (if successful):* $V(d^\infty; \infty) = d^\infty - (d^\infty)^2/6q$
- *Probability of discovery in the second period:* Given d^∞ it is the solution to

$$\frac{V(d^\infty; \infty)}{\eta d^\infty} = \tilde{c}_\rho(\rho^\infty)$$

which implies

$$\rho^\infty = erf\left(\sqrt{\frac{W\left(\frac{2\left(\frac{6q - d^\infty}{6q\eta}\right)^2}{\pi}\right)}{2}}\right)$$

where $W(\cdot)$ is the Lambert-W (or product log) function.

- *Distance chosen by the researcher in period 2:* Solution to

$$d^\infty = 3q - \eta\frac{\tilde{c}(\rho^\infty)}{\rho^\infty}$$

- *Continuation payoff:* Conditional on discovery in $t = 2$, the continuation payoff from $t = 3$ onwards is $\rho^\infty V(d^\infty; \infty)$. The node (discovery in $t = 2$) is reached with probability ρ^∞. The $t = 1$ net present value of the decision maker's continuation payoff at $t = 3$ is thus

$$\frac{(\delta\rho^\infty)^2}{1 - \delta\rho^\infty} V(d^\infty; \infty).$$

The values follow directly from Proposition 1, the first-order conditions are discussed in the proof of Proposition 2.

The moonshot has two benefits: $\rho(6q) > \rho^\infty$, that is, discovery is more likely in period $t = 2$ (by construction and Proposition 3) and $V(3q; 6q) > V(d^\infty; \infty)$, that is, conditional on a discovery that discovery is more valuable (by Proposition 1). It comes at the cost in the first period as $V(6q; \infty) < V(3q; \infty)$, that is, the first period discovery is suboptimal (by Corollary 2).

We prove a stronger version of moonshot optimality by ignoring the persistent effect of $\rho(6q) > \rho^\infty$. We treat the $t = 1$ net present value of continuation payoffs at $t = 3$ as identical, thereby underestimating the value of the moonshot because:

$$\frac{\delta^2\rho(6q)\rho^\infty}{1 - \delta\rho^\infty} V(d^\infty; \infty) > \frac{\delta^2(\rho^\infty)^2}{1 - \delta\rho^\infty} V(d^\infty; \infty).$$

The losses of a moonshot in $t = 1$ are

$$\left(3/2 - \frac{2}{\sqrt{3}}\right) q. \tag{8}$$

To compute the gains, consider the beginning of period $t = 2$ in both scenarios. The expected payoff from that period is

- for MOONSHOT: $\rho(6q)V(3q; 6q)$,
- for MYOPIC OPTIMUM: $\rho^\infty V(3q; \infty)$.

Total gains in $t = 1$ net present value are thus

$$\delta\left(\rho(6q)V(3q;6q) - \rho^\infty V(3q;\infty)\right) = \delta\left(\rho(6q)\left(3 - \frac{2}{\sqrt{3}}\right)q - \rho^\infty d^\infty\left(1 - \frac{(d^\infty)}{6q}\right)\right). \quad (9)$$

By continuity in η and δ it suffices to show that for $\delta = 1$ and some $\eta > 0$ we have that (8)<(9). (Numerically) solving for $d^\infty, \rho^\infty, \rho(6q)$ using, e.g., $\eta = 1$ verifies that this is the case.[34] In Lemma 29 in Appendix H, we show that d^∞ is linear in q which implies that ρ^∞ is constant in q. Linearity of distance and invariance of probability in the moonshot case can directly be observed. Thus restricting attention to, e.g., $q = 1$ is without loss. □

B.7 Proof of Proposition 5

Proof.

Part 1. Existence of $s(K)$ and ξ.

Step 1. $d \leq s$, interior $d \leq 4q$, and continuity. First notice that $\zeta f(\sigma^2)$ is constant if $d \geq s$. Because $s > 3q$ by assumption it follows from Lemma 10 that the optimal novelty $d^* \leq s$. Moreover, by Corollary 7 $d \leq 3q$ if $\zeta = 0$. The researcher's problem is

$$U_R(d,\rho) = \rho\left(V(3q;\infty) + \frac{\sigma^2(3q;\infty)}{s}\right) - \eta c(\rho(3q;s)\sigma^2(3q;\infty))$$

We can re-write the researcher's problem substituting from the budget constraint as

$$\max_{d,\rho} \rho\Big(V(d;\infty) + \frac{\sigma^2(d;\infty)}{s}\zeta\Big) - \left(\eta - \frac{K-\zeta}{\kappa}\right)\tilde{c}(\rho)\sigma^2(d;\infty)$$

which is continuous in ζ for any (d,ρ). Thus its maximum is continuous too.

Note that by Lemma 24, $V_d < 0, V_{dd} > 0$ for $d > 4q$. Thus, if an interior solution exists, it must be such that $d \leq 4q$. Suppose otherwise and that an interior solution with $d \in (4q, s)$ exists with corresponding ρ. Then, the researcher can increase her payoff by marginally increasing d and keeping ρ constant. By the first-order condition with respect to d, $V_d + \zeta/s - \eta\tilde{c}(\rho)/\rho = 0$. Therefore, a marginal increase of d increases the payoff as $V_{dd} > 0$ implies that $V_d(d+\varepsilon) > V_d(d)$ while all other terms remain the same. Thus, any interior $d \leq 4q$.

Step 2. Existence of ξ. Again because $s \geq 3q$ and Lemma 10, for $\zeta = 0$ it holds that

$$U_R(d^\infty, \rho(d^\infty)) > U_R(s, \rho(s)),$$

with d^∞ the arg max in d. Because both terms are continuous in ζ, the inequality has to hold in a positive neighborhood of $\zeta = 0$.

Define ξ to be $\min_{\zeta \leq K}\{\zeta : U_R(d^\infty, \rho(d^\infty)) = U_R(s, \rho(s))\}$ if it exists or $\xi := K$ otherwise. Then, by definition of ξ and continuity, for $\zeta \in [0, \xi)$, $d < s$.

[34] In this case, $\rho(6q) = 0.453226, \rho^\infty = 0.31075, d^\infty = 2.74272$. This yields as benefit of the moonshot: 0.0283413.

Step 3. Existence of $s(K)$. As $s \to \infty$ the researcher's payoff assuming $d = s$ goes to q. To see this fix ζ, η and consider the researcher's problem assuming $d = s$ and let $s \to \infty$. By construction $f(\sigma^2) = 1$ and by Proposition 1 $\lim_{s\to\infty} V(s;\infty) \to q$. Because $\sigma^2 \to \infty$ the optimal $\rho(s) \to 0$ and so does U_R. By Lemma 10 the researcher's payoff assuming $f(\cdot) \equiv 0$ is positive for any $\eta < \infty$ with some $d^\infty \leq 3q$. Thus, for $\zeta > 0$ and $f(\cdot) > 0$, the payoff for the distance defined in Lemma 10 is strictly larger than for $d = s$. Continuity implies that there is an $\overline{s} > 0$ such that for any $s > \overline{s}$, $d^\infty \leq 4q$ is optimal by step 1.

Note that the cutoff s depends on K. Fist observe that for any K, $s(K) \leq \overline{s}$. However, observe that for $K = 0$, $d < 3q$ by Lemma 10 and by continuity $s(K) = 0$ in a positive neighborhood of $K = 0$.

Part 2. Proof of relationship (1). We make use of the Marginal Rate of Substitution (MRS) between ζ and η for the probability ρ and the distance d. The MRS describes the slope of the iso-ρ curve and the iso-d curve, respectively, in the (η, ζ)-space.

Step 0. Defining the MRS. The MRS for ρ is

$$MRS^{\rho}_{\zeta\eta} := -\frac{\frac{\partial \rho}{\partial \eta}}{\frac{\partial \rho}{\partial \zeta}},$$

and $MRS^{d}_{\zeta\eta}$ analogously.

Lemma 30 in Appendix H derives

$$MRS^{\rho}_{\zeta\eta} = s\left(2\tilde{c}_\rho(\rho) - \tilde{c}(\rho)/\rho\right), \tag{10}$$

and

$$MRS^{d}_{\zeta\eta} = \tilde{c}_\rho \frac{\tilde{c}/\rho - \tilde{c}_\rho + \frac{\tilde{c}}{\tilde{c}_\rho}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}}. \tag{11}$$

Step 1. Deriving the Research Possibility Frontier. Note that because rewards do not increase beyond s and $s > 3q$, the researcher is never selecting a distance $d > s$.

Assuming $d < s$ can use the two first order conditions of the researcher and solve for ζ and η. We obtain

$$\begin{aligned} \eta &= \frac{d}{6q}\frac{\rho}{\rho\tilde{c}_\rho - \tilde{c}} \\ \zeta &= \left(\frac{d}{3q} - 1 + \frac{d}{6q}\frac{\tilde{c}}{\rho\tilde{c}_\rho - \tilde{c}}\right)s. \end{aligned} \tag{12}$$

Replacing in $MRS^{\rho}_{\zeta\eta}$ and $MRS^{d}_{\zeta\eta}$ we observe that any (ρ, d) can be implemented through at most one (η, ζ)-combination because each iso-ρ curve crosses each iso-d curve at most once: both slopes (the respective MRS) are positive and $MRS^{\rho}_{\zeta\eta} > MRS^{d}_{\zeta\eta}$ if $s > 0.1$ by the properties of $\tilde{c}(\rho)$.

Plugging $h = \eta^0 - \eta$ as well as conditions (12) into the budget line, $K = \zeta + \kappa h$, and solving for d yields the interior solution

$$d(\rho) = 6q(K + s - \kappa\eta^0)\frac{\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)}{2s\rho\tilde{c}_\rho(\rho) - s\tilde{c}(\rho) - \kappa\rho}. \tag{13}$$

The minimum then constitutes the research possibility frontier.

Step 2. Deriving the bounds $\underline{\rho}$, $\overline{\rho}$.

Step 2.1. Assuming $s > s(K)$.

Recall from the proof of Proposition 2 that

$$d^{\infty} = 3q\left(1 - \frac{\tilde{c}(\rho)}{2\tilde{c}_{\rho}(\rho)\rho - \tilde{c}(\rho)}\right).$$

Replacing d in its first order condition (FOC^d) (equation (4) on page 38) yields

$$\underline{\eta}(\rho) = \frac{\rho}{(2\rho\tilde{c}_{\rho}(\rho) - \tilde{c}(\rho))}$$

which describes the largest cost parameter $\underline{\eta}(\rho)$ (assuming $\zeta = 0$) that implies a probability ρ selected by the researcher. The parameter $\underline{\eta}$ is decreasing in ρ.

Next, recall that $MRS^{\rho}_{\zeta\eta}$ describes the slope of the iso-ρ curve in the (η, ζ)-plane. As $MRS^{\rho}_{\zeta\eta}(\rho)$ is independent of η that slope is constant and each iso-ρ curve is given by

$$\zeta(\eta; \rho) = (\eta - \underline{\eta}(\rho))MRS^{\rho}_{\zeta\eta}(\rho).$$

Because $MRS^{\rho}_{\zeta\eta}(\rho)$ is increasing (and convex) in ρ and $\underline{\eta}(\rho)$ decreases in ρ, iso-ρ curves are ordered in the (η, ζ)-space. If $\rho' > \rho$ the iso-ρ curve of ρ' is steeper and than the iso-ρ curve of ρ.

Now, consider the budget line in the (η, ζ)-plane which is

$$\zeta = K - \kappa(\eta^0 - \eta)$$

which is linearly increasing with slope κ and root at $\check{\eta} = \eta^0 - K/\kappa$, the polar case $(\zeta = 0, h = K/\kappa)$. Let $\check{\rho}$ be the probability of discovery at that root. Then, by construction $\check{\eta} = \underline{\eta}(\check{\rho})$.

If $MRS^{\rho}_{\zeta\eta}(\check{\rho}) > \kappa$, then the iso-$\rho$ curve for $\check{\rho}$ is steeper than the budget line. Because iso-ρ curves are ordered and the budget line is increasing, all iso-ρ curves that cross the budget line must have $\rho < \check{\rho}$ which implies $\overline{\rho} = \check{\rho}$. The minimum implementable $\underline{\rho}$ crosses the budget line at the largest attainable $\eta = \eta^0$ and hence corresponds to the other polar case.

If instead $MRS^{\rho}_{\zeta\eta}(\check{\rho}) < \kappa$ then all iso-ρ curves that cross the budget line must have $\rho > \check{\rho}$ which implies hat $\underline{\rho} = \check{\rho}$ and the largest attainable $\overline{\rho}$ is induced by $\eta = \eta^0$ and $\zeta = K$.

Step 2.2. Assuming s<s(K).

Restricting the domain of ζ to $[0, \xi)$ and applying the arguments from Step 2.1. yields the result.

Part 3. Substitutes or Complements. We focus on the case $s > s(K)$.[35] To show that d and ρ can be both substitutes and complements from the funder's perspective, we need to consider the slope of (13). The first term in brackets is independent of ρ but may be positive or negative depending on parameter.

For the second term, let $num(\rho)$ be the numerator of the last term of (13) and $den(\rho)$ its denominator. Then, that last term is increasing in ρ if and only if

[35] For the case of $s < s(K)$ observe that for a (generic) funding schemes such that $d = s$, d does not vary with local changes in to the funding scheme and only ρ adjusts, the results are thus not particularly interesting.

$$num'(\rho)den(\rho) > num(\rho)den'(\rho)$$

or equivalently using that $num'(\rho) = \rho\tilde{c}_{\rho\rho}(\rho) > 0$, $den'(\rho) = s\left(2\rho\tilde{c}_{\rho\rho} + \tilde{c}_\rho\right) - \kappa$ if and only if

$$\frac{\kappa}{s} < \underbrace{\frac{\tilde{c}_\rho(\rho)\tilde{c}(\rho) + \rho\tilde{c}(\rho)\tilde{c}_{\rho\rho}(\rho) - \rho\left(\tilde{c}_\rho(\rho)\right)^2}{\tilde{c}_{\rho\rho}(\rho)\rho^2 - \rho\tilde{c}_\rho(\rho) + \tilde{c}(\rho)}}_{=MRS^d_{\zeta\eta}(\rho)}.$$

Thus, $d(\rho)$ is increasing if and only if

$$(K + s - \kappa\eta^0)(sMRS^d_{\eta\zeta}(\rho) - \kappa) > 0. \tag{14}$$

Which, depending on parameters, may or may not hold. The figures in the main text provide examples for both cases. □

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Supplementary Material

C Graphical example

Here, we present a short graphical example to highlight our model ingredients and fosters intuition. Suppose the following snapshot of the realization of the Brownian path constitutes the truth on $[-2, 2]$.

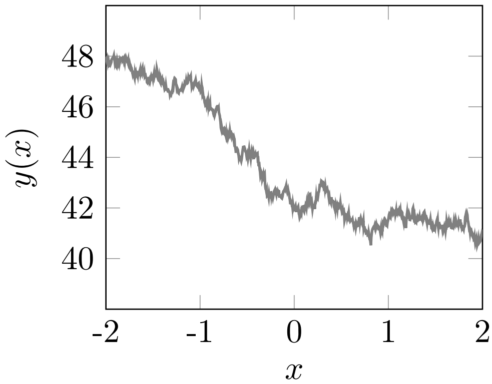

Figure 13: *The color of the truth is gray.*

The next graphs depict knowledge if the answer to a single question is known, $\mathcal{F}_1 = \{(0, 42)\}$, and in if two answers are known, $\mathcal{F}_2 = \{(-1.2, 46.6), (0, 42)\}$.

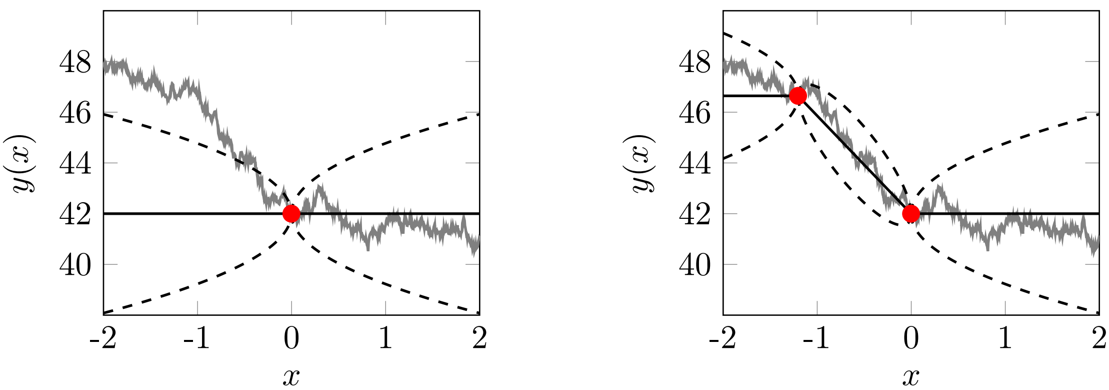

Figure 14: *Conjectures and their precision under $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right).* The red dots represent known question-answer pairs. The solid lines represent the expected answer to each question x given the existing knowledge. The dashed line represents the 95-percent prediction interval—that is, the interval in which the answer to question x lies, with a probability of 95 percent, given $\mathcal{F}_k$.

In the situation represented in the left panel of Figure 14, under $\mathcal{F}_1$, only the answer to question 0, which is 42, is known. We represent that knowledge by a dot (●). Given the martingale property of a Brownian motion, the current conjecture is that the answer to all other questions is normally distributed with mean 42. We represent the mean of the conjecture by the solid lines. However, the farther a question is from 0, the less precise is the conjecture (see Figure 2). We depict the level of precision by the dashed 95-percent prediction interval. For each question x, the truth lies, with a probability of 95 percent, between the two dashed lines given the knowledge $\mathcal{F}_k$.

In the right panel of Figure 14, in addition to $\mathcal{F}_1$, the answer to question $x = -1.2$, which is 46.6, is known. The additional knowledge changes the conjectures for questions in the negative domain compared to the left panel. The conjecture about questions between -1.2 and 0 is represented by a Brownian bridge. The expectation of answers is decreasing from -1.2 to 0 and is 46.6 to the left of -1.2. Moreover, uncertainty decreases for all questions in the negative domain, and the prediction bands become narrower. The positive domain is unchanged because of the martingale property of the Brownian motion.

Now, consider moving to knowledge $\mathcal{F}_3 = \{(-1.2, 46.6), (0, 42), (1.2, 41.8)\}$ (left panel of Figure 15) and then to $\mathcal{F}_4 = \{(-1.6, 46.6), (0, 42), (0.8, 40.8), (1.2, 41.8)\}$ (right panel of Figure 15).

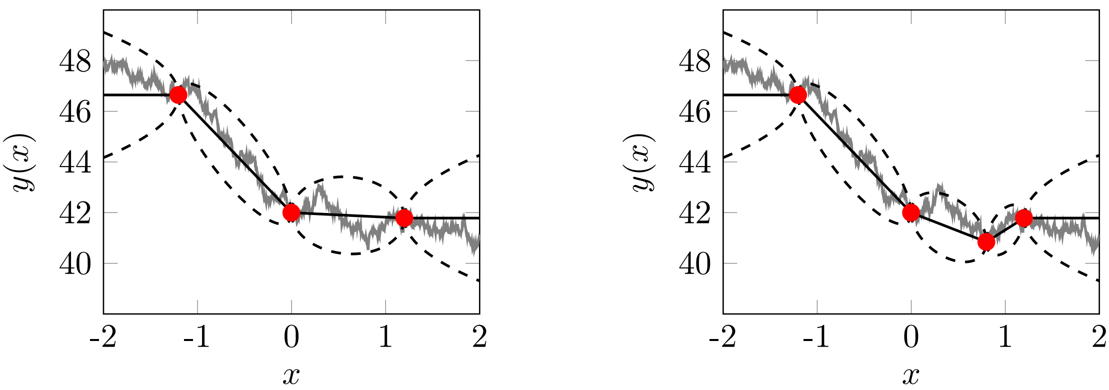

Figure 15: *Conjectures and their precision under $\mathcal{F}_3$ (left) and $\mathcal{F}_4$ (right).*

Moving from $\mathcal{F}_2$ to $\mathcal{F}_3$, the change is similar to that from $\mathcal{F}_1$ to $\mathcal{F}_2$, but this time in the positive domain. All conjectures in the positive domain become more precise, but the negative domain is unaffected. Further, a Brownian bridge between the known points $(0, 42)$ and $(1.2, 41.8)$ arises.

Moving from $\mathcal{F}_3$ to $\mathcal{F}_4$, knowledge of an answer to a question that lies between two already-answered questions is added. Conjectures about answers to questions between 0 and 1.2 become more precise. Further, since $40.8 < 41.8$, answers to all questions between 0 and 1.2 are expected to be lower compared to the conjecture based on knowledge $\mathcal{F}_3$. Moreover, the expected answers are decreasing in x from 0 to 0.8 and increasing from 0.8 and 1.2.

D The Cost of Research

In this section, we provide a microfoundation for the cost function assumed in Section 4. The cost implies an endogenous measure of the productivity of research. Here, we conceptualize research as the search for an answer. That is, we model research as sampling a set of candidate answers to question x with the goal of discovering the actual answer, $y(x)$.

Formally, we assume that the sampling decision consists of selecting an interval $[a, b] \in \mathbb{R}$. If the true answer lies inside the chosen interval, such that $y(x) \in [a, b]$, research succeeds and a discovery is made. If $y(x) \notin [a, b]$, research fails and no discovery is made. Thus, the choice of the research interval entails an ex ante probability of successful research. Restricting the sampling decision to a single interval $[a, b]$ comes without loss for our

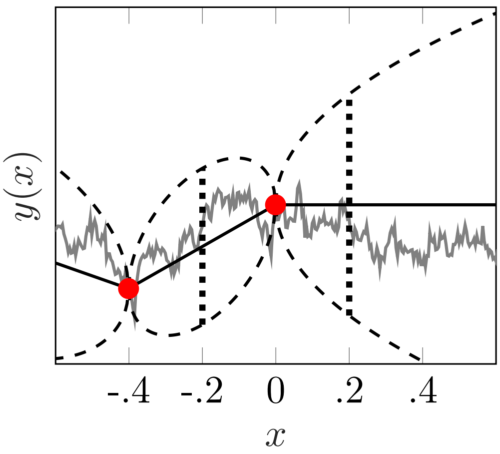

Figure 16: *Cost of research and interference.* The dotted vertical lines represent the 95 percent prediction intervals for the answers to questions $x = -0.2$ and $x' = 0.2$, assuming the answer to questions 0 and -0.4 are known. Both x and x' have distance $d = 0.2$ to existing knowledge. However, the 95 percent prediction interval at question x is shorter because the variance is smaller because researching $x = -0.2$ deepens knowledge. Research on question $x = 0.2$ expands knowledge, which implies a larger variance.

purposes, as conjectures $G_x(y|\mathcal{F}_k)$ follow a normal distribution.

We now characterize the cost of research in terms of three variables of interest: the research area, the novelty of the question, and the expected output. The area length, X, and the novelty, $d(x)$, of a research question are familiar concepts from Section 3. Output describes the expected probability that search leads to discovery which denote that probability by ρ.

We begin by defining a prediction interval.

Definition 6 (Prediction Interval)**.** The prediction interval $\alpha(x, \rho)$ is the shortest interval $[a, b] \subseteq \mathbb{R}$ such that the answer to question x is in the interval $[a, b]$ with probability ρ.

Next, we describe the prediction interval $\alpha(x, \rho)$ based on the conjecture $G_x(y|\mathcal{F}_k)$.

Proposition 6. *Suppose $\alpha(x, \rho)$ is the prediction interval for probability ρ and question x when answer $y(x)$ is normally distributed with mean μ and standard deviation σ. Then, any prediction interval has the following two features:*

1. *The interval is centered around μ.*
2. *The length of the prediction interval is $2^{3/2} erf^{-1}(\rho)\sigma$, where erf^{-1} is the inverse of the Gaussian error function.*

Proof. The normal distribution is symmetric around the mean with a density decreasing in both directions starting from the mean. It follows directly that the smallest interval that contains the realization with a particular likelihood is centered around the mean.

Take an interval $[z_l, z_r]$ of length $Z < \infty$ that is symmetric around the mean μ and let it be such that it contains a total mass of $\rho < 1$ in the interval. Then, a probability mass of $(1 - \rho)/2$ lies to the left of the interval by symmetry of the normal distribution. Moreover, the left bound z_l of the interval has (by symmetry of the interval around the mean μ) a distance $\mu - Z/2$ from the mean. From the properties of the normal distribution,

$$\Phi(z_l) = 1/2\left(1 + erf\left(\frac{z_l - \mu}{\sigma\sqrt{2}}\right)\right) = 1/2\left(1 + erf\left(\frac{-Z/2}{\sigma\sqrt{2}}\right)\right).$$

Solving using symmetry of erf yields

$$1/2\left(1-erf\left(\frac{Z}{\sigma 2^{3/2}}\right)\right)=\frac{1-\rho}{2}$$

or equivalently

$$erf\left(\frac{Z}{\sigma 2^{3/2}}\right)=\rho$$
$$\Leftrightarrow Z=2^{3/2}erf^{-1}(\rho)\sigma.$$

□

The properties of the prediction interval can be seen in the figures depicting the Brownian path. The dashed lines depict the $\rho = 95$ percent–prediction interval (as, for example, in Figure 16). Figure 16 indicates that the prediction interval depends on the location of the question. Two questions with the same distance from existing knowledge (that is, distance from question $x = 0$) have different 95 percent prediction intervals depending on whether research deepens knowledge or expands it. That difference translates into different costs.

Proposition 6 implies that if the cost function is homogeneous of any degree in interval length $(b - a)$, we can represent it with an alternative cost function proportional to $c(\rho, d, X)$ that is multiplicatively separable in (d, X) and ρ without having to keep track of the exact location of the search interval $[a, b]$, which proves to be convenient.

It also implies that, fixing ρ, the changes in the cost with respect to novelty d and area length X vary in their effect on $\sigma(d; X)$ only. Similarly, holding distance and area length constant, changes in ρ translate into cost changes according to a function of $erf^{-1}(\rho)$—a convex increasing function.

Proposition 6 intuitively links the cost of research effort to the probability of a discovery. Because the inverse error function is increasing and convex, the cost of finding an answer with probability ρ is increasing and convex in ρ. Discovering an answer with certainty implies an infinitely large interval; short of certainty, there is always a chance that the answer is outside the sampled interval.

Importantly, Proposition 6 also links output and novelty: for a given level of effort, the probability of success depends on the precision of the conjecture about a question. Research on a more novel question inside the same research area with the same level of effort entails a higher risk.

In the paper we assume, for concreteness, we assume that cost is proportional to $(a-b)^2$. As should be clear from Proposition 6, the quadratic formulation is for convenience only. What matters for our results qualitatively is that the cost is (i) homogeneous, (ii) increasing, and (iii) convex in the sampling interval $(a - b)$. Under the quadratic assumption, the cost function is characterized by a simple corollary to Proposition 6.

Corollary 10. *For knowledge $\mathcal{F}_k$, probability ρ, and question x, the minimal cost of obtaining an answer to question x with probability ρ is proportional to*

$$c(\rho, d; X) = \tilde{c}(\rho)\sigma^2(d; X).$$

Given Corollary 10, cost is increasing in d and X and concave in d; the concavity decreases in X with the limiting case in which cost is linear in d as $X \to \infty$.

E Maximizing the Myopic Return to Funding

Here, we provide a general result on optimal myopic funding. Corollary 8 in the main text follows from it.

Proposition 7. *Suppose the funder aims at maximizing the myopic expected benefit from research, $\rho V(d;\infty)$. The optimal funding scheme can be a combination of the two instruments ($\zeta > 0, h > 0$) or can focus only on one of the two ($\zeta = 0, h > 0$ or $\zeta > 0, h = 0$). Moreover, the following statements are true :*

1. *If output decreases in novelty on the research possibility frontier throughout, optimal funding cannot induce excessive novelty.*
2. *Otherwise, optimal funding may induce excessive novelty. If output increases in novelty for funding schemes that induce $d < s$, moderate excessive novelty $d \in (3q, s)$ can be optimal.*

Proof.

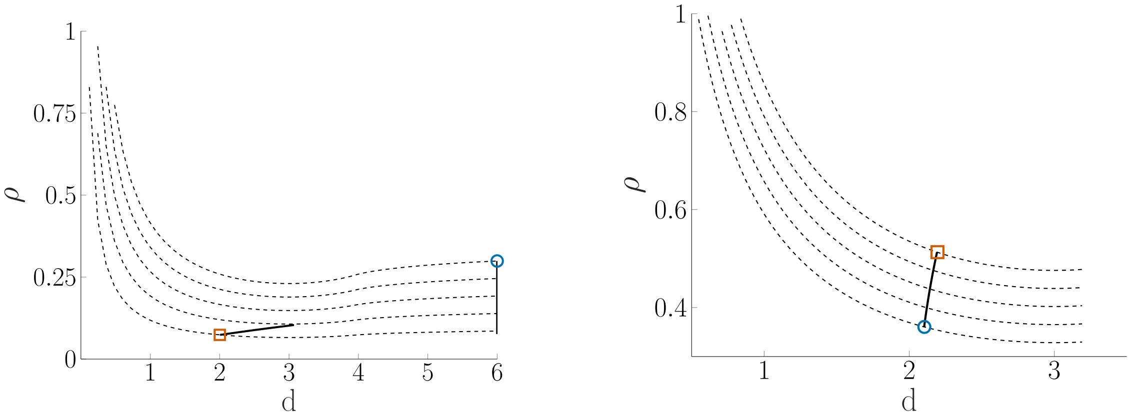

Figure 17: *Funding schemes that maximize immediate benefits.* In both panels we have $\kappa = 7, q = 1$. In the left panel we have in addition $K = 30, \eta^0 = 10, s = 6$, in the right panel we have $K = 3, \eta^0 = 1, s = 600$.

Step 1. Restraint Novelty. Assume that $d(\rho; K)$ is monotone and decreasing. It follows that it is beneficial for the funder to induce a marginally higher ρ whenever $d \in (3q, s)$. This increase in ρ decreases d marginally. Both effects increase $\rho V(d;\infty)$ if $d > 3q$. What remains is to show that inducing $d = s$ is never optimal from the funder's perspective.

Consider the (ζ, h)-combination that induces the largest $\tilde{d}$ such that (13) applies.

Because $d(\rho; K)$ decreases by assumption and $\tilde{d} \leq s$ we have that the associated $\rho(\tilde{d}) \geq \rho(s)$. Thus, for any implementable $d < s$, $\rho(d < s) > \rho(s)$ because $d(\rho; K)$ is decreasing. It suffices to find an implementable $\tilde{d} < s$ such that $V(\tilde{d};\infty) \geq V(s;\infty)$ to prove the claim.

Let $\underline{d}$ be the distance induced by the funding scheme $(\zeta, \eta) = (0, K/\kappa)$. Because $\zeta = 0$, Proposition 2 implies $\underline{d} > 2q$.

Now, recall from Proposition 1 that V is symmetric around $d = 3q$ on the interval $d \in [2q, 4q]$, increasing in d if $d < 3q$ and decreasing if $d > 3q$. Because $s \geq 4q$ we have that $V(s;\infty) < V(4q;\infty) = V(2q;\infty) < V(\underline{d};\infty)$ and hence $\rho(\underline{d})V(\underline{d};\infty) \geq \rho(s)V(s;\infty)$ which proves the statement.

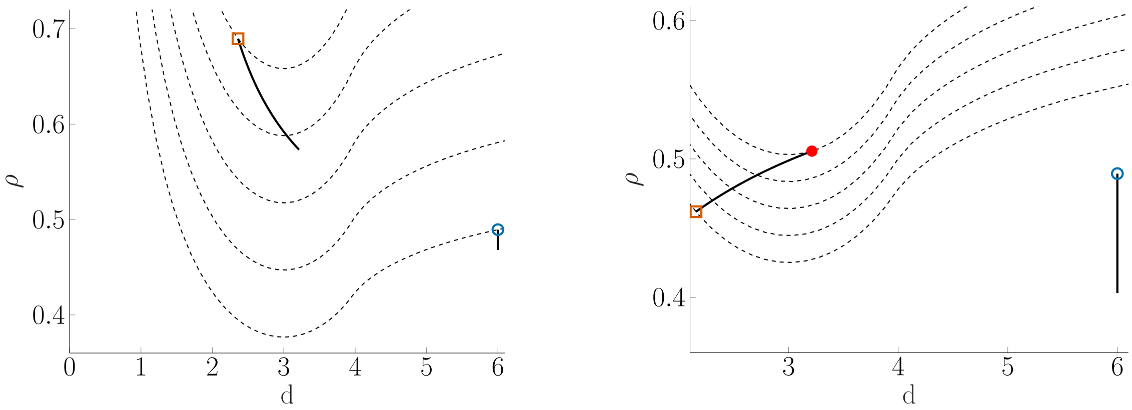

Figure 18: *Funding schemes that maximize immediate benefits.* The dashed elliptical curves depict all points that deliver the same expected value $\rho V(d;\infty)$. The solid line is the funder's budget line. In both panels, $K = 3, s = 6, q = 1$, and $\eta^0 = 1$. In the left panel, the relative price of cost reductions is $\kappa = 7$; on the right, that price is $\kappa = 16$. The funder's optimal choice (●) in both cases consists of a mix of ex-ante cost reductions and ex-post rewards, $(\zeta, h) > 0$. The circle (○) depicts the outcome if the funder invest exclusively into rewards, $\zeta = K, h = 0$; the square (□) the outcome if the funder invests exclusively into cost reductions, $\zeta = 0, h = K/\kappa$.

Step 2. Excessive Novelty. The parameters used to calculate the example leading to Figure 12, right panel provide an example of moderate excessive novelty, $d \in (3q, s)$. Using, e.g., parameters $K = 30, \eta^0 = 10, \kappa = 7, q = 1, s = 6$ provides an example in which it is optimal to incentives $d = s$ and to focus exclusively on rewards. However, even if ρ and d are complements throughout, excessive novelty need not be optimal. An example is $K = 3, \eta^0 = 1, \kappa = 7, q = 1, s = 600$. Here it is optimal to focus entirely on cost reductions. Figure 17 provides the respective graphs.

□

Figure 18 illustrates Proposition 7. It highlights the fundamental difference between the case in output and novelty complement each other in the budget constraint, and when they are not. In the left panel, output and novelty do not complement each other. Thus, the funder trades off novelty and output and settles optimally for a funding mix in the interior of what can be achieved in terms of novelty and output. The optimal funding scheme is a mix of both instruments. In the right panel, there are complementarities. The funder chooses to combine the two instruments. The funder's optimal solution includes excessive novelty: the novelty induced is larger than the value-maximizing level $d = 3q$. The reason for excessive novelty is that it comes with higher output. The researcher's desire to win the award induces her to work harder on finding a solution, meaning output increases. However, the funder does not want to go to the extreme $d = s$ as that would imply a reduction in output.

The optimal funding scheme combines ex ante cost reductions and ex post rewards. If the funder were to concentrate on awards alone, she would induce novelty $d = s$. In response, the researcher takes too much risk in her effort to win the award. Output—and thus the expected benefits—decline.

F Different Rewarding Technology

In this section, we briefly discuss a variant of the model from Appendix E. The model is identical to that in Appendix E apart from the functional form $f(\sigma^2)$. Instead of assuming a linear relationship, we assume

$$f(\sigma^2) = 1 - e^{-s\sigma^2}.$$

Changing the reward technology in this way has two implications. First, rewards are not guaranteed no matter how difficult to answer the question is. Second, the likelihood to receive an ex-post reward is now strictly concave in the variance, which implies a decreasing return to novelty in the reward function.

Using this specification, we lose the closed-form expression of the research possibility frontier from Proposition 5; however, the findings we discuss around Proposition 7 remain largely unchanged as Figure 19 illustrates: d and ρ can be substitutes (left panel) or complements (right panel) from the funder's perspective; if they are complements, it may be optimal to induce excessive novelty to increase output (right panel); if they are substitutes, excessive novelty is never optimal (left panel). A combination of the two funding schemes may be optimal to maximize the expected benefits to society (both panels).

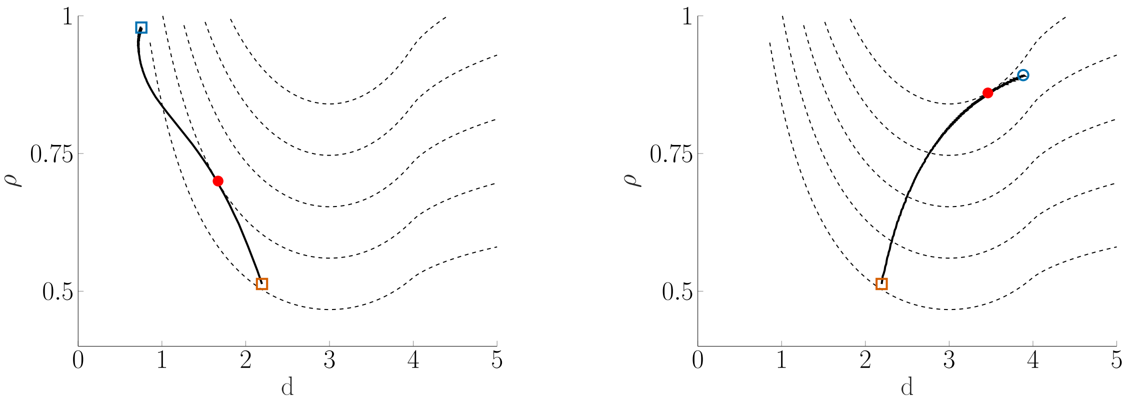

Figure 19: *Funding schemes that maximize immediate benefits.* The dashed elliptical curves depict all points that deliver the same expected value $\rho V(d;\infty)$. The solid line is the funder's budget line. In both panels, $K = 30, \kappa = 70, q = 1$, and $\eta^0 = 1$. In the left panel, the return parameter $s = 6$; in the right panel, that parameter is $s = .6$. The funder's optimal choice (●) in both cases consists of a mix of ex-ante cost reductions and ex-post rewards, $(\zeta, h) > 0$. The circle (○) depicts the outcome if the funder invests exclusively into rewards, $\zeta = K, h = 0$; the square (□) the outcome if the funder invests exclusively into cost reductions, $\zeta = 0, h = K/\kappa$.

G Different Universe of Questions

Our baseline model assumes that the universe of questions can be represented on the real line. That is, we assume an implicit order on questions. In this part, we show that all our results extend to a more general question space.

To begin with, consider our baseline model and fix some knowledge $\mathcal{F}_m$. As described in Section 2, knowledge pins down $\mathcal{X}_k$—a set composed of (half-)open intervals: bounded intervals $[x_i, x_{i+1})$ of length X_i each, and two unbounded intervals $(-\infty, x_1)$ and $[x_k, \infty)$ of length ∞. As we describe in Propositions 1 to 3, we can determine the benefits and cost of every new discovery to both researcher and decision maker by replacing the exact identity of the question x with the tuple (d, X), that is, through the distance of the question x to existing knowledge and length of the research area in which x lies.

Now, consider any set $\widehat{\mathcal{X}}_m = \widehat{\mathcal{X}}_k \cup \widehat{\mathcal{X}}_n$ that contains $k + n$ elements: k convex-valued and bounded intervals on $\mathbb{R}$ with Euclidean distance between its upper and lower bound, $X_{i \in \widehat{\mathcal{X}}_k}$, and n convex-valued but unbounded intervals on $\mathbb{R}$ of infinite length, $X_{i \in \widehat{\mathcal{X}}_n} = \infty$. For any tuple (d, X) with $X \in \widehat{\mathcal{X}}_m$ and $d \in [0, X/2]$ all our definitions and expressions for benefits and cost are well-defined regardless of how $\widehat{\mathcal{X}}_m$ was generated.

For any given set $\widehat{\mathcal{X}_m}$ generated by some existing knowledge $\mathcal{F}_m$, suppose that the truth-generating process Y is such that the answer to question x characterized by (d, X) is normally distributed with a variance of $\sigma^2(d; X)$.[36] Then, all of our results continue to hold.

G.1 Generalization to a Multidimensional Universe of Questions

Here, we show a mapping from a model with an $n-$dimensional independent Brownian motion as the truth-generating process to our baseline model with a one-dimensional Brownian motion as truth-generating process.

Assume that there is a standard n-dimensional Brownian motion $W_z = (W_z^1, ..., W_z^n)$ whose components W_z^i are independent one-dimensional standard Brownian motions.[37] Suppose $\mathcal{F}^i_{j(i)}$ is the finite set of $j(i)$ known realizations of the Brownian path in dimension i and $\mathcal{F}_k = \cup_{i=1}^n \mathcal{F}^i_{j(i)}$ is knowledge. As described in Section 2, each $\mathcal{F}^i_{j(i)}$ determines a partition of the domain of W_z^i denoted by $\mathcal{X}^i_{j(i)}$ with $j(i) + 1$ elements. As in the baseline case, the knowledge in dimension i decomposes the dimension-i process into $j(i) - 1$ independent Brownian bridges each associated with a length X_l^i, $l = \{1, ...j(i)\}$ and two independent Brownian motions. Therefore, the union $\mathcal{F}_k$ determines $k = \sum_{i=1}^n j(i) - 1$ independent Brownian bridges of length X_l^i each and $2n$ Brownian motions. By the martingale property of the Brownian motion and the fact that realizations are not directly payoff relevant, the setting is isomorphic to one in which we have k independent *standard* Brownian bridges of length X_l^i each and $2n$ *standard* Brownian motions. Thus, the set $\widehat{\mathcal{X}}_k = \{X^i_{l(i)}\} \cup \infty$ is a sufficient statistic to calculate any of the results in the text. However, the set $\widehat{\mathcal{X}}_k = \{X^i_{l(i)}\} \cup \infty$ can also be generated with an appropriate realized path of a one-dimensional Brownian motion with a corresponding $\mathcal{F}_k$.

G.2 Seminal Discoveries

We conclude this part by presenting a model with *seminal discoveries*—discoveries that open new fields of research—that builds on the multidimensional universe of questions described above. For example, Friedrich Miescher's isolation of the "nuclein" in 1869 was initially intended to contribute to the study of neutrophils, yet, in addition, it opened up the new and, to a large extent, orthogonal field of DNA biochemistry.

Formally, consider the following model of the evolution of knowledge. Initially, there is a single field of research A and a single known question-answer pair, $(x_0, y(x_0)) = (0, 0)$. The set of all questions in field A is known to be one-dimensional and represented by $\mathbb{R}$. The truth is known to be generated by a standard Brownian path Y passing through $(0, 0)$. However, with an exogenous probability $p \in [0, 1]$ any discovery $(x, y(x))$ is *seminal* and opens a new, independent field of research B_x. A seminal discovery is a question-answer

[36] Note that the dependence of the variance of the conjecture depends only on d and X. Thus, the truth-generating process has to satisfy a Markov property as the Brownian motion on the real line in our main model. Moreover, note that the specification of the expected value of the answer is not relevant for our results as long as it is well-defined given $\mathcal{F}_m$

[37] Thus, each process starts at an initial point $(0, 0)$, has a drift of zero, a variance of one and independent, normal increments.

pair $(x, y(x))$ that is an element of two independent Brownian paths crossing only at $(x, y(x))$. Thus, upon occurrence, a seminal discovery generates knowledge in multiple dimensions. Because it is a priori unknown whether a discovery is seminal, the payoff from generating knowledge in another dimension is constant in expected terms—it does not influence a researcher's (or funder's) choices. After the seminal discovery, the updated model of truth and knowledge is the one described above with the multi-dimensional universe of questions. As we showed above, that model can, in turn, be mapped into our baseline. The special case $p = 0$ is our baseline model.

It should become clear from our discussion that even the case in which the probability of a seminal discovery depends on the question is qualitatively similar to what we discuss in the baseline model. The quantitative differences in such a model come from the fact that questions which are likely to be a seminal discovery are more attractive to address for all parties involved.

H Omitted Proofs

H.1 Properties of $\tilde{c}(\rho)$

To simplify notation, we suppress the argument ρ and denote the inverse error function by $\iota := erf^{-1}(\rho)$.

Lemma 18. *The derivatives of the inverse error function satisfy*

1. $\frac{d}{d\rho}\iota = \frac{1}{2}\sqrt{\pi}e^{\left(\iota^2\right)}$
2. $\frac{d^2}{d\rho^2}\iota = 2\iota\iota'^2$
3. $\frac{d^3}{d\rho^3}\iota = 2\iota'^3\left(1+4\iota^2\right)$.

Proof. See Dominici (2008). □

Lemma 19. 1. $\lim_{\rho\to 0}\rho\frac{\iota'}{\iota} = 1$

2. $\lim_{\rho\to 1}\rho\frac{\iota'}{\iota} = \infty$
3. $\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = 0$
4. $\lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\pi}{3}$

Proof. We will make use of L'Hôpital's rule and the derivative properties from Lemma 18 in the following.

The first item follows from

$$\begin{aligned}\lim_{\rho\downarrow 0}\rho\frac{\iota'}{\iota} &= \lim_{\rho\downarrow 0}\frac{\iota'+\rho\iota''}{\iota'}\\ &= \lim_{\rho\downarrow 0}\frac{\iota'+2\rho\iota\iota'^2}{\iota'}\\ &= \lim_{\rho\downarrow 0}(1+\rho\iota\iota')\\ &= 1.\end{aligned}$$

The second item follows from

$$\begin{aligned}\lim_{\rho\uparrow 1}\rho\frac{\iota'}{\iota} &= \lim_{\rho\uparrow 1}\frac{\iota'+\rho\iota''}{\iota'}\\ &= \lim_{\rho\uparrow 1}\frac{\iota'+2\rho\iota\iota'^2}{\iota'}\\ &= \lim_{\rho\uparrow 1}(1+2\rho\iota\iota')\\ &= \infty.\end{aligned}$$

The third item follows from

$$\begin{aligned}\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) &= \lim_{\rho\to 0}\frac{\iota'}{\iota}\left(1-\rho\frac{\iota'}{\iota}\right)+\lim_{\rho\to 0}\rho\frac{\iota''}{\iota}\\ &= \underbrace{\lim_{\rho\to 0}\iota'}_{=\sqrt{\pi}/2}\lim_{\rho\to 0}\frac{\iota-\rho\iota'}{\iota^2}+\underbrace{\lim_{\rho\to 0}2\rho\iota'^2}_{=0} = -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\frac{\rho\iota''}{2\iota\iota'}\\ &= -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\frac{\rho\iota(\iota')^2}{2\iota\iota'} = -\lim_{\rho\to 0}\frac{\sqrt{\pi}}{2}\rho\iota' = 0.\end{aligned}$$

The fourth item follows from[38]

$$
\begin{aligned}
\lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) &= \lim_{\rho\to 0} 2\frac{\iota''\iota-\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) + \underbrace{\lim_{\rho\to 0} 4\rho \underbrace{\iota'\iota''}_{=2(\iota')^3\iota}}_{=0} \\
&= \lim_{\rho\to 0} 2\frac{\iota''\iota-\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) \\
&= \lim_{\rho\to 0} 2\frac{\iota''\iota}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) - 2\lim_{\rho\to 0}\frac{\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) \\
&= \underbrace{\lim_{\rho\to 0} 4\iota'^2\left(1-\rho\frac{\iota'}{\iota}\right)}_{=0} - 2\lim_{\rho\to 0}\frac{\iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) \\
&= -2\lim_{\rho\to 0}\left(\rho\frac{\iota'}{\iota}\right)^2\frac{\iota-\rho\iota'}{\rho^2\iota} \\
&= 2\lim_{\rho\to 0}\frac{\rho\iota''}{2\rho\iota+\rho^2\iota'} \\
&= 4\lim_{\rho\to 0}\frac{\iota'^2}{2+\rho\frac{\iota'}{\iota}} \\
&= \frac{4}{3}\lim_{\rho\to 0}\iota'^2 = \frac{\pi}{3}.
\end{aligned}
$$

□

Lemma 20. *The following statements hold:*

1. *For all* $\rho \in (0,1)$, $\frac{d}{d\rho}\left(\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)\right) > 0$
2. *For all* $\rho \in (0,1)$, $\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho) > 0$
3. $\lim_{\rho\to 0}\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = 2$
4. $\lim_{\rho\to 1}\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = \infty$

Proof. The first statement holds because

$$\frac{d}{d\rho}\left(\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)\right) = \rho\tilde{c}_{\rho\rho}(\rho) > 0.$$

by convexity of the inverse error function.

The second statement holds because of the first statement and $\left(\rho\tilde{c}_\rho(\rho) - \tilde{c}(\rho)\right)|_{\rho=0} = 0$.

The third statement holds by observing that the elasticity is equal to $2\rho\frac{\iota'}{\iota}$ and the first statement of Lemma 19.

The fourth statement holds by the same observations and the second statement of Lemma 19. □

Lemma 21. *The elasticity of* $\tilde{c}(\rho)$, $\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)}$, *is increasing in* ρ.

Proof. Recall that $\rho\frac{\tilde{c}_\rho(\rho)}{\tilde{c}(\rho)} = 2\rho\frac{\iota'}{\iota}$ and that it is therefore sufficient to prove that the inverse error function has an increasing elasticity.

[38] To arrive at the first line let $\lambda := \iota'/\iota$ and observe that $(\rho\lambda)'' = (\lambda+\rho\lambda')' = 2\lambda' + \rho\lambda''$ and $\lambda' = 2(\iota')^2 - \lambda^2$ which implies $\lambda'' = 4\iota'\iota'' - 2\lambda\lambda'$.

Note that

$$\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\iota'}{\iota} + \rho\frac{\iota''\iota - \iota'^2}{\iota^2}.$$

From Lemma 19 know that

$$\lim_{\rho\to 0}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) = 0$$
$$\lim_{\rho\to 0}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right) = \frac{\pi}{3}.$$

Thus, there exists an $\varepsilon > 0$ such that the elasticity is increasing for $\rho \in (0,\varepsilon)$. To show that it is increasing for all $\rho \in (0,1)$ suppose –toward a contradiction– that the derivative of the elasticity crosses 0. In this case, it has to hold that

$$\frac{\iota''\iota - \iota'^2}{\iota^2} = -\frac{\iota'}{\rho\iota}.$$

Consider the second derivative of the elasticity at such a critical point

$$\begin{aligned}\frac{d^2}{d\rho^2}\left(\rho\frac{\iota'}{\iota}\right)|_{\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right)=0} &= 2\frac{\iota''\iota - \iota'^2}{\iota^2}\left(1-\rho\frac{\iota'}{\iota}\right) + \rho\frac{\iota'''\iota - \iota''\iota'}{\iota^2}\\ &= -2\frac{\iota'}{\iota\rho}\left(1-\rho\frac{\iota'}{\iota}\right) + \rho\frac{\iota'''\iota - \iota''\iota'}{\iota^2}\\ &= 2\frac{\iota'}{\iota\rho}\left(\rho\frac{\iota'}{\iota}-1\right) + 2\rho\frac{\iota'^3}{\iota}4\iota^2\\ &> 0\end{aligned}$$

where the last inequality follows because the elasticity is weakly greater than one and all other terms are positive.

Thus, any critical point must be a minimum. However, the elasticity is continuous and increasing at $\rho \in (0,\varepsilon)$. Thus, there is no interior maximum and the elasticity is increasing throughout. □

Lemma 22. *The elasticity of $\tilde{c}_\rho(\rho)$, $\rho\frac{\tilde{c}_{\rho\rho}(\rho)}{\tilde{c}_\rho(\rho)}$, is increasing in ρ.*

Proof. The derivative of the corresponding inverse error function elasticity (which is one half the one of our cost function) is

$$\begin{aligned}\frac{d}{d\rho}\left(\rho\frac{\iota'}{\iota}\right) &= \frac{\iota''}{\iota'} + \rho\frac{\iota'''\iota' - \iota''^2}{\iota'^2}\\ &= \frac{\iota''}{\iota'} + 2\rho\iota''^2(1+2\iota(2\iota-1)).\end{aligned}$$

Next, we will show that $1+2\iota(2\iota-1) > 0$. Note that this is a convex function of ρ with a minimum at $\iota\iota' = \frac{1}{4}$ which is solved by $\rho = erf\left(\sqrt{\frac{W(\frac{1}{2\pi})}{2}}\right) \approx 0.29$ where W denotes the principal branch of the Lambert-W function. Evaluating $1+2\iota(2\iota-1)$ at this minimum yields

$$1+\left(\sqrt{2W\left(\frac{1}{2\pi}\right)}-1\right)\sqrt{2W\left(\frac{1}{2\pi}\right)} \approx 0.75.$$

□

H.2 Omitted Steps in Proofs

Here, we provide the steps that we have omitted in the proofs because they involve cumbersome algebraic manipulation with little economic or mathematical insight.

Lemma 23. $\frac{\partial V(d;\infty|d>4q)}{\partial d} < 0$.

Proof.

$$\frac{\partial V(d;\infty|d>4q)}{\partial d} = -\frac{d}{3q} + 1 + \sqrt{\frac{d-4q}{d}}\frac{d-q}{3q}$$

Letting $\tau := d/q (> 4$ by assumption) the statement is negative if

$$\frac{3-\tau}{3} + \sqrt{\frac{\tau-4}{\tau}}\frac{\tau-1}{3} < 0$$

The left-hand side is increasing in τ and converges to 0 as $\tau \to \infty$. □

Lemma 24. *$V(d;X) > 0$ if $d \in [0, X-4q]$ and $X \in (4q, 6q]$.*

Proof. We show that the derivative V_d is a convex function which is positive at its minimum on $[0, X-4q]$ and hence throughout on that domain.

The relevant derivatives to consider are

$$V_d = \frac{1}{3q}\left(X - 2d - (X-d-q)\sqrt{\frac{X-d-4q}{X-d}}\right).$$

$$V_{dd} = \frac{1}{3q}\left(-2 + \frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}\left((X-d-4q)(X-d) + (X-d-q)2q\right)\right).$$

$$V_{ddd} = \frac{4q^2}{(X-d)^{5/2}(X-d-4q)^{3/2}} > 0.$$

where $V_{ddd} > 0$ follows immediately from $(X-d) > 0$ and $(X-d-4q) > 0$. It follows that, V_d is strictly convex over the relevant range. The maximal distance in this range, $d = X-4q$, $V_d|_{d=X-4q} = \frac{8q-X}{3q} > 0$.

Hence, the minimum of the first derivative is either at $d = 0$ or at some interior d such that $V_{dd} = 0$. Suppose the minimum is at $d = 0$, then $V_d|_{d=0} = \frac{1}{3q}\left(X - (X-q)\sqrt{\frac{X-4q}{X}}\right) > 0$ because $\frac{X-4q}{X} < 1$.

Hence, the only remaining case is when V_d attains an interior minimum. In this case, $V_{dd} = 0$ must hold at the minimum and hence

$$\sqrt{X-d-4q}(X-d)^{3/2} = \frac{(X-d-4q)(X-d) + (X-d-q)2q}{2}.$$

The first derivative can be rewritten as

$$V_d = \frac{1}{3q}\left(X - 2d - \frac{1}{\sqrt{X-d-4q}(X-d)^{3/2}}(X-d-q)(X-d-4q)(X-d)\right)$$

and plugging in for the minimum condition we obtain

$$V_d|_{V_{dd}=0}$$

$$= \frac{1}{3q}\left(X - 2d - \frac{2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}\right)$$
$$= \frac{1}{3q}\frac{(X-2d)((X-d-4q)(X-d)+(X-d-q)2q)-2(X-d-q)(X-d-4q)(X-d)}{(X-d-4q)(X-d)+(X-d-q)2q}.$$

As the denominator and $\frac{1}{3q}$ are both positive, the sign of V_d at its minimum is determined by the sign of its numerator only. Note that the numerator is increasing in d because its derivative is $2(X-6q)(X-d-q) > 0$. Thus, the numerator of the derivative of V_d evaluated at the interior minimum d such that $V_{dd} = 0$ is greater than

$$-X(X^2 - 8qX + 10q^2) = -X((X-4q)^2 - 6q^2) > 0.$$

□

Lemma 25. *$V_X(d^0(X); X) < 0$ if $X \geq 4q$ and $d \in [0, X-4q]$.*

Proof. Observe that for any $X \geq 4q$ and $d \leq X - 4q$

$$V_{Xd} = \frac{1}{24q}\left(8 - 3\sqrt{\frac{X-d}{X-d-4q}} - (5(X-d)+4q)\frac{\sqrt{X-d-4q}}{(X-d)^{3/2}}\right).$$

Denote $a := X - d$, this is an increasing function in a as

$$\frac{dV_{Xd}}{da} = \frac{4q^2}{a^{5/2}(a-4q)^{3/2}} > 0.$$

Hence, the highest value of V_{Xd} is attained for $a \to \infty$ and

$$\lim_{a\to\infty}\frac{1}{24q}\left(8 - 3\underbrace{\sqrt{\frac{a}{a-4q}}}_{\to 1} - 5\underbrace{\frac{a\sqrt{a-4q}}{a^{3/2}}}_{\to 1} + 4q\underbrace{\frac{\sqrt{a-4q}}{a^{3/2}}}_{\to 0}\right) = 0.$$

It follows that the V_{Xd} converges to zero from below implying that $V_{Xd} < 0$. Thus, $V_X(d^0(X), X) < V_X(d = 0, X)$ and we obtain

$$\begin{aligned}
&V_X(d, X | d \leq 4q, X - d \geq 4q)\\
&\quad = \frac{1}{3q}\left(d + (X-d-q)\sqrt{\frac{X-d-4q}{X-d}} - (X-q)\sqrt{\frac{X-4q}{X}}\right)\\
&\quad < V(d = 0, X | d \leq 4q, X - d \geq 4q)\\
&\quad = \frac{1}{3q}\left((X-q)\sqrt{\frac{X-4q}{X}} - (X-q)\sqrt{\frac{X-4q}{X}}\right) = 0.
\end{aligned}$$

as desired. □

Lemma 26. *If $X \in [6q, 8q)$, $\mathrm{d}^2V(X/2, X)/\mathrm{d}X^2 < 0$ and $\mathrm{d}^2V(d^0(X), X)/(\mathrm{d}X)^2 > 0$.*

Proof. Considering the boundary solution we obtain

$$\frac{d^2V(X/2, X)}{dX^2} = -\frac{X^2 - 2qX - 2q^2}{3qX^{3/2}\sqrt{X-4q}} + \frac{1}{6q}$$

$$\frac{d^3V(X/2,X)}{dX^3} = \frac{4q^2}{X^{5/2}(X-4q)^{3/2}} > 0$$

implying that $\frac{d^2V(X/2,X)}{dX^2} \leq \frac{d^2V(4q,8q)}{dX^2}$ with

$$\frac{d^2V(4q,8q)}{dX^2} = -\frac{64q^2-16q^2-2q^2}{3q8^{3/2}q^{3/2}2q^{1/2}} + \frac{1}{6q} = -\frac{46q^2}{96\sqrt{2}q^3} + \frac{1}{6q} = \frac{8-23/\sqrt{2}}{48q} < 0.$$

Next, consider the value of any interior solution and apply the envelope and implicit function theorem to obtain

$$\begin{aligned}
\frac{dV(d^0(X),X)}{dX} &= V_X + d'(X)\underbrace{V_d}_{=0 \text{ by optimality of } d} = V_X \\
\frac{d^2V(d^0(X),X)}{dX^2} &= V_{XX} + d'(X)V_{dX} + d'(X)\underbrace{(V_{Xd} + V_{dd}d'(X))}_{=0 \text{ by IFT on FOC}} + d''(X)\underbrace{V_d}_{=0 \text{ by optimality}} \\
&= V_{XX}(d^0(X),X) + d'(X)V_{dX} \\
&= V_{XX}(d^0(X),X) \underbrace{-\frac{V_{dX}^2}{V_{dd}}}_{>0 \text{ as } V_{dd}<0}.
\end{aligned}$$

Observing that

$$V_{XXd}(d,X|d \leq 4q, X-d \geq 4q) = \frac{4q^2}{(X-d)^{5/2}(X-d-4q)^3/2} > 0$$

we can compute as lower bound for

$$\begin{aligned}
V_{XX}(d^0(X),X) = \frac{1}{24q}&\left(3\left(\sqrt{\frac{X-d}{X-d-4q}} - \sqrt{\frac{X}{X-4q}}\right) + 6\left(\sqrt{\frac{X-d-4q}{X-d}} - \sqrt{\frac{X-4q}{X}}\right)\right. \\
&\left. + \left(\frac{X-4q}{X}\right)^{3/2} - \left(\frac{X-d-4q}{X-d}\right)^{3/2}\right) \\
&\geq V_{XX}(d=0,X) = 0
\end{aligned}$$

implying that $\mathrm{d}^2V(d^0(X),X)/(\mathrm{d}X^2) \geq 0$. □

Lemma 27. *Assume* $X \in [4q, 8q]$, *then* $\mathrm{d}^2U_R(d = X/2; X)/(\mathrm{d}X)^2 < 0$.

Proof. Take the case of the boundary solution: we are analyzing a one-dimensional optimization problem with respect to ρ. Denote the objective $f(\rho; X)$ and the optimal value by $\varphi(X) = \max_\rho f(\rho; X)$. Then, the optimal ρ solves $f_\rho = 0$. We obtain

$$\begin{aligned}
\varphi'(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + f_X \\
\varphi''(X) &= \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) + \underbrace{(f_{\rho\rho}\rho'(X) + f_{X\rho})}_{=0 \text{ by total differentiation of FOC}} \rho'(X) + f_{XX} + \rho'(X)f_{X\rho} \\
&= f_{XX} - \frac{f_{X\rho}^2}{f_{\rho\rho}}
\end{aligned}$$

$$= \rho(X) V_{XX}(X/2;X) + \frac{(V_X - \frac{V}{X})^2}{V\frac{c''}{c'}}$$

which yields as condition for the value to be concave

$$\rho(X)\frac{c''}{c'} > -\frac{(V_X - \frac{V}{X})^2}{V_{XX}V}$$

where the inequality sign changed direction as $V_X X < 0$.

Note that at the boundary solution the right-hand side simplifies to

$$\frac{X^{3/2} - 2(X+2q)\sqrt{X-4q}}{X^{3/2} - 2(X-4q)\sqrt{X-4q}} \; \frac{16q^2 + 4qX - 2X^2 + X^{3/2}\sqrt{X-4q}}{8q^2 + 8qX - 4X^2 + 2X^{3/2}\sqrt{X-4q}}$$

where both fractions are less than one. Finally, we know that the left-hand side is above two by the properties of the inverse error function. Hence, the optimal value at the boundary solution is strictly concave as $\sigma^2_{XX}(X/2;X) = 0$ and $V_{XX} < 0$ in the region considered by Corollary 3 □

Lemma 28. *Let $d^i < X/2$ be a local maximum of $u_R(\rho, d, X)$. If $d^i(X)$ exists on $X \in [4q, 8q]$, then $\mathrm{d}^2 U_R(d = d^i(X); X)/(\mathrm{d}X)^2 > 0$.*

Proof. The implicit function theorem yields for $d'(X)$ and $\rho'(X)$

$$\begin{pmatrix} d'(X) \\ \rho'(X) \end{pmatrix} = -\frac{1}{f_{dd}f_{\rho\rho} - f^2_{\rho d}} \begin{pmatrix} f_{dX}f_{\rho\rho} - f_{\rho X}f_{d\rho} \\ f_{\rho X}f_{dd} - f_{dX}f_{d\rho} \end{pmatrix}.$$

Note that $-\frac{1}{f_{dd}f_{\rho\rho} - f^2_{\rho d}} < 0$ as this is $-\frac{1}{det(\mathcal{H})}$ and the determinant of the second principal minor being positive is a necessary second order condition for a local maximum given that the first ($f_{\rho\rho}$) is negative.

Denote the objective $f(\rho, d; X)$ and the optimal value by $\varphi(X) = \max_{\rho,d} f(d, \rho; X)$. Then, the optimal (d, ρ) solves $f_\rho = 0$ and $f_d = 0$. Differentiating the value of the researcher twice with respect to X yields

$$\begin{aligned}
\varphi'(X) = & \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho'(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) + f_X \\
\varphi''(X) = & \underbrace{f_\rho}_{=0 \text{ by optimality}} \rho''(X) + \underbrace{f_d}_{=0 \text{ by optimality}} d'(X) \\
& + d'(X) \underbrace{\left(f_{dX} + f_{dd}d'(X) + f_{d\rho}\rho'(X)\right)}_{=0 \text{ by total differentiation of foc wrt } d} \\
& + \rho'(X) \underbrace{\left(f_{\rho X} + f_{\rho d}d'(X) + f_{\rho\rho}\rho'(X)\right)}_{=0 \text{ by total differentiation of foc wrt } \rho} \\
& + f_{dX}d'(X) + f_{\rho X}\rho'(X) + f_{XX} \\
& = f_{dX}d'(X) + f_{\rho X}\rho'(X) + f_{XX}.
\end{aligned}$$

Observe first that $f_{XX} > 0$ as $f_{XX} = \rho V_{XX}(d;X) - \eta\tilde{c}(\rho)\sigma^2_{XX}(d;X)$ and $V_{XX} > 0$ by proof of Corollary 3 (in particular, Lemma 26) and $\sigma^2_{XX}(d;X) = -\frac{2d^2}{X^3}$. Next, we show $f_{dX}d'(X) + f_{\rho X}\rho'(X) > 0$ using the implicit function theorem together with the property of the local maximum that $f_{\rho\rho} f_{dd} > f_{\rho d}^2$.

$$f_{dX} d'(X) + f_{\rho X} \rho'(X) = -f_{dX} \left(\frac{f_{dX} f_{\rho\rho} - f_{\rho X} f_{d\rho}}{f_{dd} f_{\rho\rho} - f_{\rho d}^2} \right) - f_{\rho X} \left(\frac{f_{\rho X} f_{dd} - f_{dX} f_{d\rho}}{f_{dd} f_{\rho\rho} - f_{\rho d}^2} \right).$$

As we only need the sign of this expression we can ignore the positive denominator to verify

$$\begin{aligned} -f_{dX}(f_{dX} f_{\rho\rho} - f_{\rho X} f_{d\rho}) - f_{\rho X}(f_{\rho X} f_{dd} - f_{dX} f_{d\rho}) &> 0 \\ f_{dX}^2 f_{\rho\rho} + f_{\rho X}^2 f_{dd} - 2 f_{dX} f_{\rho X} f_{d\rho} &< 0 \\ \frac{f_{dX}}{f_{\rho X}} \frac{f_{\rho\rho}}{f_{d\rho}} + \frac{f_{\rho X}}{f_{dX}} \frac{f_{dd}}{f_{\rho d}} &> 2. \end{aligned}$$

where we used the signs of the terms that follow because

$$\begin{aligned} f_{\rho\rho} &= -\eta \tilde{c}_{\rho\rho}(\rho) \sigma^2 < 0 \\ f_{\rho X} &= V_X - \eta \tilde{c}_\rho(\rho) \sigma_X^2 \\ &< V_X - \eta \frac{\tilde{c}(\rho)}{\rho} \sigma_X^2 < 0 \\ f_{d\rho} &= V_d - \eta \tilde{c}_\rho(\rho) \sigma_d^2 \\ &< V_d - \eta \frac{\tilde{c}(\rho)}{\rho} \sigma_d^2 = 0 \\ f_{dX} &= \rho V_{dX} - \eta \tilde{c}(\rho) \sigma_{dX}^2 < 0 \end{aligned}$$

which in turn follow from the first-order conditions and Corollary 3.

Because $f_{\rho\rho} f_{dd} - f_{\rho d}^2 > 0$, we can replace $\frac{f_{\rho\rho}}{f_{d\rho}}$ with $\frac{f_{d\rho}}{f_{dd}}$ as $\frac{f_{\rho\rho}}{f_{d\rho}} > \frac{f_{d\rho}}{f_{dd}}$ yielding

$$2 < \frac{f_{dX}}{f_{\rho X}} \frac{f_{d\rho}}{f_{dd}} + \frac{f_{\rho X}}{f_{dX}} \frac{f_{dd}}{f_{\rho d}}$$

which is true as the right-hand side can be written as $g(a) = a + \frac{1}{a}$ with $a = \frac{f_{dX}}{f_{\rho X}} \frac{f_{d\rho}}{f_{dd}} > 0$. Note that $g(a)$ is a strictly convex function for $a > 0$ and minimized at $a = 1$ with $g(a = 1) = 2$. □

Lemma 29. *d^∞ is linear in q and ρ^∞ is constant in q.*

Proof. The lemma follows because $\sigma^2(mq; \infty) = mq$ and thus (by Proposition 1) the functions $f(m, q) := V(mq; \infty)/\sigma^2(mq; \infty)$ and $g(m, q) := V_d(mq; \infty)$ are homogeneous of degree 0 in q.

It is then immediate from (FOC^d) and (FOC$^\rho$) that d^∞ is homogeneous of degree 1 in q and ρ^∞ is homogeneous of degree 0. Noticing that $d^\infty(q = 0) = 0$ implies the result. □

Lemma 30. *$MRS_{\zeta\eta}^\rho = s(2\tilde{c}_\rho(\rho) - \tilde{c}(\rho)/\rho)$ and $MRS_{\zeta\eta}^d = \tilde{c}_\rho \frac{\tilde{c}/\rho - \tilde{c}_\rho + \frac{\tilde{c}}{\tilde{c}_\rho} \tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_\rho + \rho \tilde{c}_{\rho\rho}}$.*

Proof. For any (η, ζ) the system of first-order conditions for a non-boundary choice is given by

$$\begin{aligned} V_d(d, \infty) + \zeta \sigma_d^2(d, \infty)/s &= \eta \tilde{c}(\rho)/\rho \\ \frac{V(d, \infty) + \zeta \sigma_d^2(d, \infty)/s}{d} &= \eta \tilde{c}_\rho(\rho) \end{aligned}$$

For an interior optimal choice of (d, ρ), we obtain using $\sigma^2(d, X) = d, \sigma_d^2(d, X) = 1$ and $\sigma_{dd}^2(d, X) = 0$

$$\begin{pmatrix} \frac{\mathrm{d}d}{\mathrm{d}\eta} \\ \frac{\mathrm{d}d}{\mathrm{d}\zeta} \\ \frac{\mathrm{d}\rho}{\mathrm{d}\eta} \\ \frac{\mathrm{d}\rho}{\mathrm{d}\zeta} \end{pmatrix} = -\frac{1}{det(\mathcal{H})}$$

$$\begin{pmatrix} d(\tilde{c}_\rho(V_d + \zeta/s - \eta\tilde{c}_\rho) + \eta\tilde{c}\tilde{c}_{\rho\rho}) \\ -d(V_d + \zeta/s - \eta\tilde{c}_\rho + \rho\eta\tilde{c}_{\rho\rho}) \\ -\rho\sigma^2\tilde{c}_\rho V_{dd} + \tilde{c}(V_d + \zeta/s - \eta\tilde{c}_\rho) \\ -\rho/s(V_d + \zeta/s - \eta\tilde{c}_\rho - dV_{dd}) \end{pmatrix}$$

where $det(\mathcal{H})$ is the determinant of the Hessian matrix of the objective function which is given by

$$-\eta\sigma^2\tilde{c}_{\rho\rho}\rho V_{dd} - (V_d + \zeta/s - \eta\tilde{c}_\rho)^2 > 0.$$

Note that the determinant of the Hessian matrix for a local maximum is positive as the Hessian is negative semidefinite and the first principal minor $-\eta\tilde{c}_{\rho\rho}\sigma^2 < 0$ by convexity of the inverse error function.[39]

It follows that the sign of the derivatives are determined only by the negative of the sign of the respective terms in the matrix. Using the first-order conditions to rewrite these equations yields

$$\frac{\mathrm{d}d}{\mathrm{d}\eta} = -\frac{d\eta}{det(\mathcal{H})}\left(\tilde{c}_\rho\left(\frac{\tilde{c}}{\rho} - \tilde{c}_\rho\right) + \tilde{c}\tilde{c}_{\rho\rho}\right) < 0$$
$$\frac{\mathrm{d}d}{\mathrm{d}\zeta} = \frac{d\eta}{det(\mathcal{H})}\left(\frac{\tilde{c}}{\rho} - \tilde{c}_\rho + \rho\tilde{c}_{\rho\rho}\right) > 0$$

where the inequalities hold due to the properties of the inverse error function.

$$\frac{\mathrm{d}\rho}{\mathrm{d}\eta} = -\frac{\rho\eta}{det(\mathcal{H})}$$
$$(2\tilde{c}_\rho - \tilde{c}/\rho)(\tilde{c}_\rho - \tilde{c}/\rho) < 0$$

where we have used that $\sigma^2 V_{dd} = -\frac{d}{3q}$ and from the first-order conditions we know that $\frac{d}{3q} = 2\eta(\tilde{c}_\rho - \tilde{c}/\rho)$. The properties of $\tilde{c}$ imply that $\tilde{c}_\rho > \tilde{c}/\rho$. Finally,

$$\frac{\mathrm{d}\rho}{\mathrm{d}\zeta} = \frac{\rho\eta/s}{det(\mathcal{H})}(\tilde{c}_\rho - \tilde{c}/\rho) > 0$$

where the analogous reasoning as for the previous inequality applies. To conclude, we have:

[39] In our case, one can actually show that this has to hold given that $d < \infty$. Plugging in from the first-order conditions yields

$$\eta^2(\tilde{c}_\rho - \tilde{c}/\rho)2(\rho\tilde{c}_{\rho\rho} - \tilde{c}_\rho + \tilde{c}/\rho) > 0$$

where the inequality follows from the properties of $\tilde{c}$.

$$\begin{array}{ll} \frac{\mathrm{d}d}{\mathrm{d}\eta} < 0 & \frac{\mathrm{d}d}{\mathrm{d}\zeta} > 0 \\ \frac{\mathrm{d}\rho}{\mathrm{d}\eta} < 0 & \frac{\mathrm{d}\rho}{\mathrm{d}\zeta} > 0. \end{array}$$

We obtain for the marginal rate of substitution between ζ and η on the expanding interval

$$-\frac{\frac{\mathrm{d}\rho}{\mathrm{d}\eta}}{\frac{\mathrm{d}\rho}{\mathrm{d}\zeta}} = MRS^{\rho}_{\zeta\eta} = s(2\tilde{c}_{\rho} - \tilde{c}/\rho)$$

where we used the simplifications from above.

Similarly, we obtain

$$-\frac{\frac{\mathrm{d}d}{\mathrm{d}\eta}}{\frac{\mathrm{d}d}{\mathrm{d}\zeta}} = MRS^{d}_{\zeta\eta} = \tilde{c}_{\rho}\frac{\tilde{c}/\rho - \tilde{c}_{\rho} + \frac{\tilde{c}}{\tilde{c}_{\rho}}\tilde{c}_{\rho\rho}}{\tilde{c}/\rho - \tilde{c}_{\rho} + \rho\tilde{c}_{\rho\rho}}.$$

□