Sequential versus Static Screening: An equivalence result

Daniel Krähmer*and Roland Strausz†

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Abstract

We show that every sequential screening model is equivalent to a standard text book static screening model. We use this result and apply well-established techniques from static screening to obtain solutions for classes of sequential screening models for which standard sequential screening techniques are not applicable. Moreover, we identify the counterparts of well–understood features of the static screening model in the corresponding sequential screening model such as the single-crossing condition and conditions that imply the optimality of deterministic schedules.

Keywords: sequential screening, static screening, stochastic mechanisms

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*Universität Bonn, Institut for Mikroökonomik, Adenauer Allee 24-42, D-53113 Bonn (Germany), kraehmer@hcm.uni-bonn.de.

†Humboldt-Universität zu Berlin, Institute for Economic Theory 1, Spandauer Str. 1, D-10178 Berlin (Germany), strauszr@wiwi.hu-berlin.de. We are grateful to Holger Herbst, Johannes Hörner, Matthias Lang, and Urs Schweizer for useful comments. We also thank seminar participants at Paris, Dauphine ("Topics in Information Economics"), in Berlin ("Theoretischer Ausschuss") and Bonn. Both authors gratefully acknowledge financial support by the DFG (German Science Foundation) under SFB/TR-15.

1 Introduction

Recent years have witnessed an increased interest in dynamic adverse selection models in which agents receive novel private information over time (see Pavan et al. (2014) for a general treatment). One of the most fundamental dynamic adverse selection models, the so-called sequential screening model, has been introduced in by Courty and Li (2000). In this model, a seller offers a single good for sale, but in contrast to a static environment, the buyer initially has private information only about the distribution of his valuation, and he fully learns his valuation only after contracting has taken place. Due to its analytical tractability, the sequential screening model has become the workhorse model for analyzing various applied dynamic contracting problems such as ticket pricing, dynamic procurement, or the sale and disclosure of information.[1]

Dynamic adverse selection models are not only practically relevant, they also raise interesting conceptual questions about their relation to well-understood static adverse selection models. In this paper, we demonstrate that for the sequential screening model this relation is actually surprisingly tight. More specifically, we show that any sequential screening model can be equivalently represented as a canonical textbook static screening model (as, eg., described in Fudenberg and Tirole (1991)) so that any contract which is feasible (resp. optimal) in one problem is also feasible (resp. optimal) in the other. Reversely, we identify a class of static screening models that each correspond to an appropriate sequential screening model.

Sequential screening problems are best understood in so-called regular environments which require that the (sequential) virtual trade surplus satisfies a certain monotonicity condition. In contrast, little is known for non-regular environments. In the second step of our analysis, we show that our equivalence result can be used to obtain solutions for classes of non-regular sequential screening problems. More precisely, we identify conditions for which the sequential screening problem is not regular, but the corresponding static screening problem can be solved with well–known techniques from static screening.[2]

Moreover, our equivalence result clarifies the role of the ordering of the agent's private in-

[1]See Dai et al. (2006), Esö and Szentes (2007a, b), Hoffmann and Inderst (2011), Nocke et al. (2011), Krähmer and Strausz (2011, 2015a, b), Inderst and Peitz (2012), Bergemann and Wambach (2014), Deb and Said (2014), Herbst (2015), Liu and Lu (2015), Li and Shi (2015). For a textbook treatment of the sequential screening model, see Krähmer and Strausz (2015c).

[2]While the solutions we obtain may involve bunching, we show how an approach developed in Nöldecke and Samuelson (2007) can be applied in our setting, which does not require optimal control techniques to identify optimal bunches.

formation in sequential screening models. The most often used ordering in sequential screening is that a buyer type is higher if he is more optimistic to obtain favorable subsequent information in the sense of first order stochastic dominance. As it turns out, the equivalent condition in the corresponding static screening model is precisely that the agent's utility function satisfies the single–crossing condition. Reversely, a sequential screening problem in which types are not ordered according to first order stochastic dominance corresponds to a static screening problem without single–crossing.

Key in establishing the connection between the sequential and the static model is to explicitly allow for the use of stochastic contracts in the static model. Intuitively, stochastic contracts enter the picture, because in the sequential model the terms of trade depend on the buyer's valuation that realizes ex post and are, from an ex ante perspective, therefore stochastic. Our insight is that the induced distribution of terms of trade can be replicated by a stochastic contract in the static model so that a party's expected utility in the sequential model, where the expectation is taken with respect to the buyer's future valuation, coincides with its expected utility in the static model, where the expectation is taken with respect to the uncertainty generated by the stochastic contract. By allowing for stochastic contracts, our equivalence result also sheds light on their optimality in sequential screening environments.

It is worth mentioning that our result is *not* implied by a general principle, such as, for example, Pontryagin's maximum principle which states that a dynamic optimization problem can be reduced to *some* static problem subject to *some* constraints. Rather, the insight of our paper is more specific and, therefore, more surprising: the sequential screening model can be represented as a very specific static model, namely exactly as the familiar standard principal agent adverse selection model.

The paper is organized as follows. The next section introduces the two models and derives our two main results. Section 3 discusses our result, and Section 4 concludes.

2 Sequential versus Static Screening

2.1 The sequential screening problem

This subsection considers the sequential screening model of Courty and Li (2000). There is a buyer (the agent, he) and a seller (the principal, she), who has a single unit of a good for sale. The buyer's valuation of the good is $x \in [0, 1]$ and the seller's opportunity costs are $c \geq 0$. The terms of trade specify the probability $q \in [0,1]$ with which the good is exchanged and an expected payment $t \in \mathbb{R}$ from the buyer to the seller. The parties are risk neutral and have quasi-linear utility functions. That is, the seller's profit equals payments minus her expected opportunity costs, $t - cq$, and the buyer's utility equals his expected valuation minus payments, $xq - t$. Each party's reservation utility is normalized to 0.

There are three periods. At the contracting stage in period 1, no party knows the buyer's true valuation, but the buyer privately knows that his valuation x is distributed according to the distribution function $G(x|\theta)$ on the support $[0,1]$ with density $g(x|\theta)$. While the buyer's *ex ante type* θ is his private information, it is commonly known that θ is drawn from the distribution $F(\theta)$ with support $[0,1]$ and density $f(\theta)$. In period 2, after the buyer has accepted the contract, the buyer privately observes his true valuation x. We refer to x as the buyer's *ex post type*. Finally, in period 3, the contract is implemented. We allow the seller's opportunity costs $c = c(\theta, x)$ to depend on the buyer's types.[3]

The seller's problem is to design a contract that maximizes her expected profits. By the revelation principle for sequential games (e.g., Myerson 1986), the optimal contract can be found in the class of direct and incentive compatible contracts which, on the equilibrium path, induce the buyer to report his type truthfully. Formally, a *direct contract*

$$\gamma^d \equiv \{(q^d(\hat{\theta}, \hat{x}), t^d(\hat{\theta}, \hat{x}))|(\hat{\theta}, \hat{x}) \in [0,1]^2\} \tag{1}$$

requires the buyer to report an ex ante type θ in period 1, and an ex post type x in period 2. A contract commits the seller to a *selling schedule* $q^d(\hat{\theta}, \hat{x})$ and a *transfer schedule* $t^d(\hat{\theta}, \hat{x})$.

If the buyer's true ex post type is x and his period 1 report was $\hat{\theta}$, then his utility from reporting $\hat{x}$ in period 2 is

$$\tilde{u}(\hat{x}|\hat{\theta}, x) \equiv x q^d(\hat{\theta}, \hat{x}) - t^d(\hat{\theta}, \hat{x}). \tag{2}$$

We denote the buyer's period 2 utility from truth–telling by

$$u(\theta, x) \equiv \tilde{u}(x|\theta, x). \tag{3}$$

[3]Our equivalence result in Propositions 1 and 2 goes through for discrete (ex ante and/or ex post) type spaces as well. For tractability reasons, the application in Section 3, where we illustrate the usefulness of our equivalence result, is developed for continuous type spaces.

The contract is *incentive compatible in period 2* if it gives the buyer an incentive to announce his ex post type truthfully:

$$u(\theta,x) \geq \tilde{u}(\hat{x}|\theta,x) \quad \forall \hat{x},\theta,x. \tag{4}$$

If the contract is incentive compatible in period 2, the buyer announces his ex post type truthfully no matter what his report in the first period.[4] Hence, if the buyer's true ex ante type is θ, then his period 1 utility from reporting $\hat{\theta}$ is

$$\tilde{U}^d(\hat{\theta}|\theta) \equiv \int_0^1 u(\hat{\theta},x)dG(x|\theta). \tag{5}$$

We denote the buyer's period 1 utility from truth–telling by

$$U^d(\theta) \equiv \tilde{U}^d(\theta|\theta). \tag{6}$$

The contract is *incentive compatible in period 1* if it induces the buyer to announce his ex ante type truthfully:

$$U^d(\theta) \geq \tilde{U}^d(\hat{\theta}|\theta) \quad \forall \hat{\theta},\theta. \tag{7}$$

Finally, an incentive compatible contract is *ex ante individually rational* if it yields the buyer at least his outside option of zero:

$$U^d(\theta) \geq 0 \quad \forall \theta. \tag{8}$$

We say a contract is *feasible* if it is incentive compatible in both periods and ex ante individually rational.

The following lemma is a standard result in monopolistic screening, and we therefore omit the proof.

Lemma 1 *A contract γ^d satisfies the period 2 incentive compatibility constraints (4) if and only if i) $u(\theta,x)$ is absolutely continuous in x; ii) $q^d(\theta,x)$ is increasing in x; and iii) $u_x(\theta,x) = q^d(\theta,x)$ for almost all x.*[5]

[4]Because the buyer's period 2 utility is independent of his ex ante type, a contract which is incentive compatible in period 2 automatically induces truth–telling in period 2 also off the equilibrium path, that is, if the buyer has misreported his ex ante type in period 1.

[5]In what follows, subindices denote partial derivatives.

Since u is absolutely continuous in x, we may use integration by parts to rewrite the agent's period 1 utility as

$$\tilde{U}^d(\hat{\theta}|\theta) = \int_0^1 u(\hat{\theta},x)dG(x|\theta) = \int_0^1 q^d(\hat{\theta},x)[1-G(x|\theta)]dx + u(\hat{\theta},0). \tag{9}$$

The seller's payoff from a feasible contract is the difference between aggregate surplus and the buyer's utility. That is, if the buyer's ex ante type is θ, the seller's conditional expected payoff, conditional on θ, is

$$W^d(\theta) \quad = \quad \int_0^1 [(x-c(\theta,x))q^d(\theta,x) - u(\theta,x)]dG(x|\theta). \tag{10}$$

Using (9), we can rewrite the seller's payoff as

$$W^d(\theta) \quad = \quad \int_0^1 \left[x - c(\theta,x) - \frac{1-G(x|\theta)}{g(x|\theta)}\right] q^d(\theta,x)dG(x|\theta) - u(\theta,0). \tag{11}$$

To present our equivalence result with maximum clarity, we will, without loss of generality, impose the conditions $q^d(\theta,0) = 0$ and $q^d(\theta,1) = 1$.[6] The seller's problem is therefore to find a selling schedule q^d and utility levels $u(\cdot,0)$ for the buyer's lowest ex post type that solves the following maximization problem:

$$\mathscr{P}^d: \qquad \max_{q^d(\cdot,\cdot),u(\cdot,0)} \int_0^1 W^d(\theta)dF(\theta) \quad \text{s.t.}$$

$$\begin{aligned}
&q^d(\theta,x) \text{ increasing in } x,\ q^d(\theta,0) = 0,\ q^d(\theta,1) = 1, \qquad (12)\\
&U^d(\theta) \geq \tilde{U}^d(\hat{\theta}|\theta),\\
&U^d(\theta) \geq 0.
\end{aligned}$$

2.2 A general static screening problem

We now specify a general static screening problem that is based on the formulation in Fudenberg and Tirole (1991), but explicitly allows for stochastic contracts. In particular, we consider a principal and a privately informed agent who can trade some quantity $x \in [0,1]$. An allocation specifies

[6] These restrictions are without loss of generality, because i) if some schedule q^d is feasible then also when we adapt it to satisfy these two endpoint restrictions and ii) the adapted schedule yields identical payoffs since changing q^d at a single point does not affect integrals over $[0,1]$. Hence, if the original schedule q^d is optimal, so is the adapted schedule.

a, possibly stochastic, quantity x to be traded and a transfer $t \in \mathbb{R}$ from the agent to the principal. Before the principal offers a contract, the agent privately learns his *type* $\theta \in [0,1]$, which is drawn from a distribution $F(\theta)$ with support $[0,1]$. Given a type θ and an allocation (x,t), the principal's utility is $S(\theta,x)+t$, and the agent's utility is $V(\theta,x)-t$. Hence, as in Fudenberg and Tirole (1991), our specification allows for arbitrary quasi-linear utility functions, including the interdependent value case where the principal's utility depends directly on the agent's type.

Applying the revelation principle, the principal offers the agent a direct contract

$$\gamma^s = \{(q^s(\hat{\theta},x), t^s(\hat{\theta}))|\hat{\theta} \in [0,1]\}. \tag{13}$$

We explicitly allow the principal to propose a contract with a stochastic quantity schedule. Hence, $q^s(\theta,x)$ represents a cumulative distribution function (cdf) with the interpretation that, if the agent reports θ, then the probability that the quantity traded is at most x is $q^s(\theta,x)$. Consequently, $q^s(\theta,x)$ is positive and increasing in x. Moreover, without loss of generality, we impose the restrictions $q^s(\theta,0)=0$ and $q^s(\theta,1)=1$.[7] The expected utility from a contract γ^s for agent type θ who reports $\hat{\theta}$ therefore corresponds to the (Riemann–Stieltjes) integral with respect to the function $q^s(\hat{\theta},\cdot)$:

$$\tilde{U}^s(\hat{\theta}|\theta) \equiv \int_0^1 V(\theta,x)dq^s(\hat{\theta},x) - t^s(\hat{\theta}). \tag{14}$$

We denote agent type θ's expected utility from truth–telling by

$$U^s(\theta) \equiv \tilde{U}^s(\theta|\theta). \tag{15}$$

A contract is feasible if it is incentive compatible, that is,

$$U^s(\theta) \geq \tilde{U}^s(\hat{\theta}|\theta) \quad \forall \hat{\theta}, \theta \tag{16}$$

and individually rational, that is,

$$U^s(\theta) \geq 0 \quad \forall \theta. \tag{17}$$

[7]Note that in case q^s has a mass point at $x=0$, this departs from the convention that a cdf is right–continuous. Indeed, since $x \in [0,1]$ by assumption, right-continuity implies that $q^s(\theta,1)=1$ and $q^s(\theta,0)=0$ if q^s has no mass point in $x=0$. If q^s does have a mass point in $x=0$, then right-continuity implies $q^s(\theta,0)>0$. In this case, however, since changing q^s in one single point does not affect integrals with respect to $q^s(\theta,\cdot)$ over $[0,1]$, we can re-define q^s so that $q^s(\theta,0)=0$. This means that q^s so re–defined is not right–continuous in $x=0$ anymore. (Alternatively, instead of restricting $q^s(\theta,\cdot)$ to the unit interval, we could define the cdf on $(-\infty,\infty)$. In what follows, we would then integrate over $(-\infty,\infty)$ instead of over $[0,1]$.)

The principal's expected utility from a feasible contract is

$$W^s(\theta) \equiv \int_0^1 S(\theta, x) dq^s(\theta, x) + t^s(\theta). \tag{18}$$

Consequently, an optimal contract (q^s, t^s) in the static principal agent problem solves

$$\mathscr{P}^s: \qquad \max_{q^s(\cdot,\cdot), t^s(\cdot)} \int_0^1 W^s(\theta) dF(\theta) \quad \text{s.t.}$$

$$\begin{aligned} & q^s(\theta, x) \text{ increasing in } x,\ q^s(\theta, 0) = 0,\ q^s(\theta, 1) = 1, && (19) \\ & U^s(\theta) \geq \tilde{U}^s(\hat{\theta}|\theta), \\ & U^s(\theta) \geq 0, \end{aligned}$$

where the first constraint expresses the fact that $q^s(\theta, \cdot)$ is a cdf on $[0, 1]$.

2.3 Equivalence result

We now formalize the sense in which both models are equivalent. We first argue that any sequential screening model with primitives G and c corresponds to a static screening problem with appropriately defined primitives V and S. Before stating this result, note that (12) and (19) imply that any selling schedule q^d in the sequential model corresponds to a stochastic trading schedule in the static model. Therefore, for $t^s(\cdot) = -u(\cdot, 0)$, the choice variables in the two problems $\mathscr{P}^d$ and $\mathscr{P}^s$ are the same. As we now show, there are functions V and S so that also the parties' payoffs are the same.

Proposition 1 *Suppose $(q^d, u(\cdot, 0))$ is a solution to $\mathscr{P}^d$. Then $(q^s, t^s(\cdot)) = (q^d, -u(\cdot, 0))$ is a solution to $\mathscr{P}^s$, where*

$$V(\theta, x) = \int_x^1 1 - G(z|\theta) dz, \tag{20}$$

$$S(\theta, x) = \int_x^1 (z - c(\theta, z)) g(z|\theta) - [1 - G(z|\theta)] dz. \tag{21}$$

Proof of Proposition 1: We show that for $t^s(\theta) = -u(\cdot, 0)$, and for V and S defined in (20) and (21):

$$\tilde{U}^s(\hat{\theta}|\theta) = \tilde{U}^d(\hat{\theta}|\theta), \quad \textit{and} \quad W^s(\theta) = W^d(\theta). \tag{22}$$

This implies that $\mathscr{P}^s$ and $\mathscr{P}^d$ are equivalent and thus the solutions coincide.

To see (22), observe that $V(\theta,x)$ is a decreasing function in x with $dV = -(1-G(x|\theta))dx$. Hence, we can write (9) as a Riemann–Stieltjes integral with respect to V:

$$\tilde{U}^d(\hat{\theta}|\theta) \;=\; -\int_0^1 q^d(\hat{\theta},x)dV(\theta,x)+u(\hat{\theta},0). \tag{23}$$

Applying integration by parts for Riemann–Stieltjes integrals, we obtain

$$\begin{aligned}
\tilde{U}^d(\hat{\theta}|\theta) &= -q^d(\hat{\theta},x)V(\theta,x)\Big|_0^1+\int_0^1 V(\theta,x)dq^d(\hat{\theta},x)+u(\hat{\theta},0) &(24)\\
&= \int_0^1 V(\theta,x)dq^d(\hat{\theta},x)+u(\hat{\theta},0) &(25)\\
&= \int_0^1 V(\theta,x)dq^s(\hat{\theta},x)-t^s(\hat{\theta}) &(26)\\
&= \tilde{U}^s(\hat{\theta}|\theta), &(27)
\end{aligned}$$

where in the second line, we have used that $V(\theta,1)=0$ and $q^d(\hat{\theta},0)=0$, and in the third line we have used the definitions of $q^s(\hat{\theta},x)$ and $t^s(\hat{\theta})$ in the statement of the proposition.

The proof that $W^s(\theta)=W^d(\theta)$ is analogous. Q.E.D.

As illustrated in Figure 1, the main argument behind Proposition 1 can be most easily seen when q^d is a step function. In this case, the agent's expected utility (9) in the sequential model for given $(\theta,\hat{\theta})$ rewrites as

$$\begin{aligned}
\tilde{U}^d(\hat{\theta}|\theta) &= 0\cdot\int_0^{x_1}(1-G(x|\theta))dx+\tilde{q}\cdot\int_{x_1}^{x_2}(1-G(x|\theta))dx+ &(28)\\
&\quad +1\cdot\int_{x_2}^1(1-G(x|\theta))\,dx+u(\hat{\theta},0). &(29)
\end{aligned}$$

The three integrals are illustrated in the left panel of Figure 1. Instead of integrating along dx, we can as well integrate along dq, as illustrated in the right panel of Figure 1. Then the previous expression can be written as

$$0+(\tilde{q}-0)\cdot\int_{x_1}^1(1-G(x|\theta))\,dx+(1-\tilde{q})\cdot\int_{x_2}^1(1-G(x|\theta))\,dx+u(\hat{\theta},0). \tag{30}$$

But, with V as defined in (20) and $t^s(\hat{\theta})=-u(\hat{\theta},0)$, this equals the agent's expected utility $\tilde{U}^s(\hat{\theta}|\theta)$ as defined in (14) for the static model. The same argument works to show that $W^d(\theta)=W^s(\theta)$, implying that $\mathscr{P}^d$ and $\mathscr{P}^s$ are identical.

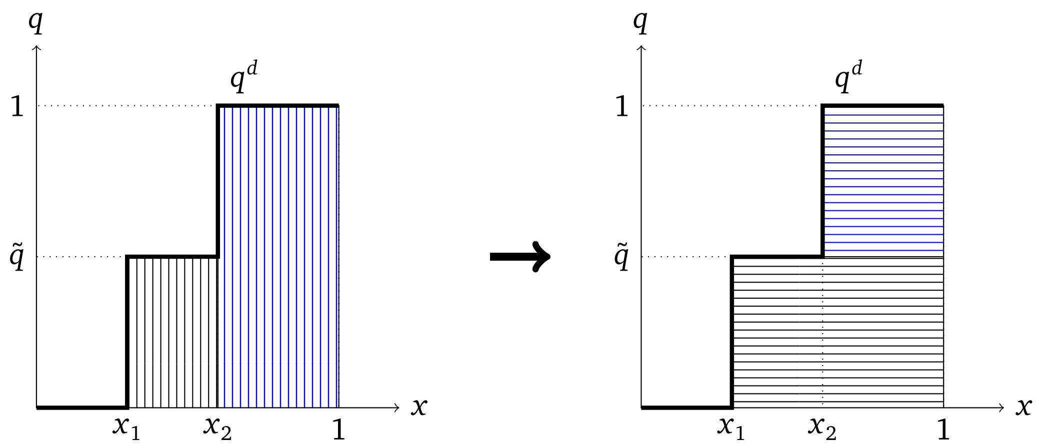

Figure 1: Expected utility in the sequential (left) and static (right) model.

Next, we state a reverse of Proposition 1 which specifies conditions under which a static screening problem with primitives V and S corresponds to a sequential screening problem with primitives G and c. In principle, G and c can be obtained by simply inverting the equations (20) and (21). However, since G needs to be a cdf, additional conditions on V are required to ensure this.[8]

Proposition 2 *Suppose $(q^s, t^s(\cdot))$ is a solution to $\mathscr{P}^s$. If*

$$V_x(\theta,0) = -1; \quad V_x(\theta,1) = 0; \quad V_{xx} \geq 0, \tag{31}$$

then $(q^d, u(\cdot,0)) = (q^s, -t^s(\cdot))$ is a solution to $\mathscr{P}^d$, where

$$\begin{aligned} G(x|\theta) &= V_x(\theta,x) + 1, && (32)\\ c(\theta,x) &= x - \frac{V_x(\theta,x)}{V_{xx}(\theta,x)} + \frac{S_x(\theta,x)}{V_{xx}(\theta,x)}. && (33) \end{aligned}$$

Proof of Proposition 2: Observe first that $G(x|\theta)$ as defined in (32) is a cumulative distribution function by the properties in (31). Next, the same argument as in the proof of Proposition 1 imply that the solutions coincide if the equations (20) and (21) hold, which follow from (32) and (33) by re–arranging terms and integration. Q.E.D.

We point out that while our transformation of the sequential screening model yields a static screening problem that is fully consistent with a stochastic formulation of Fudenberg and Tirole

[8]Without imposing these conditions on V, we still obtain the insight that any static screening problem that allows for stochastic selling schedules q corresponds to a linear optimization problem (linear with respect to q) of the type $\mathscr{P}^d$ with linear constraints, where G is not necessarily a cdf. Such problems are, e.g., considered in Samuelson (1984).

(1991), its economic interpretation is somewhat non-standard. The agent's utility function (20) has the property that

$$V(\theta,0)=\int_0^1 1-G(z|\theta)dz>0, \quad and \quad V_x(\theta,x)=G(x|\theta)-1\leq 0. \tag{34}$$

In other words, the agent derives positive utility when the traded quantity is $x=0$, and his utility is *decreasing* in x. Thus, in contrast to the original sequential screening model, where x represents the agent's valuation as a buyer and his utility was naturally increasing in x, in the static counterpart, the agent is best interpreted as a producer who has costs $V(\theta,0)-V(\theta,x)$ to produce the quantity x and obtains a positive utility $V(\theta,0)$ from being in the relation per se (eg. in the form of "prestige").

Turning to the principal, we first observe from (21) that the principal's utility $S(\theta,x)$ depends directly on the agent's type θ, since it includes the distribution $G(\cdot|\theta)$ explicitly. Hence, even if the original sequential screening problem is one of private values where c is independent of the agent's type θ, its corresponding static version displays interdependent values.

Moreover, for the typical case that the principal's marginal production costs c in the sequential model are type–independent (ie. $c(\theta,x)=c$), the principal's marginal utility in the corresponding static screening model is

$$S_x(\theta,x) \quad = \quad -\{(x-c)g(x|\theta)-[1-G(x|\theta)]\}. \tag{35}$$

If the hazard rate $(1-G(x|\theta))/g(x|\theta)$ is decreasing in x, this means that the principal's utility function $S(\theta,x)$ is hump-shaped: it is increasing in x on $[0,\bar{x}(\theta)]$ and decreasing in x on $(\bar{x}(\theta),1]$, where $\bar{x}(\theta)\in(0,1)$ is the intersection point of $x-c$ with the hazard rate. While somewhat unusual for a static screening problem, this can be interpreted to mean that the principal has horizontally differentiated tastes, and in state θ, his most preferred "variety" is $\bar{x}(\theta)$.

3 Application of the equivalence result

In this section, we apply our equivalence result and show how solution methods from the theory of static screening allow us to solve sequential screening problems to which current approaches in the literature are not directly applicable.

A standard technique in static screening is the so-called local approach which solves a relaxed problem where only "local" incentive constraints are imposed. Under appropriate conditions, the solution to the relaxed problem can then be shown to be a solution of the original problem.

More precisely, the local approach simplifies the static screening problem $\mathscr{P}^s$ by (a) replacing the global incentive constraint $U^s(\theta) \geq \tilde{U}^s(\hat{\theta}|\theta)$ for all $\theta, \hat{\theta}$ by the local incentive constraints $\partial U^s(\theta|\theta)/\partial\hat{\theta} = 0$, which, by an envelope argument pins down the agent's marginal utility as

$$\frac{d}{d\theta}U^s(\theta) = \int_0^1 V_\theta(\theta,x)\,dq^s(\theta,x), \tag{36}$$

and (b) by imposing the participation constraints $U^s(\theta) \geq 0$ for all θ only for the extreme type $\theta = 0$.

After inserting the incentive and the participation constraints in the principal's objective, we obtain the problem

$$\mathscr{R}^s: \qquad \max_{q^s(\cdot,\cdot)\in[0,1]} \int_0^1 \int_0^1 Z(\theta,x)\,dq^s(\theta,x)dF(\theta) \tag{37}$$

$$s.t. \quad q^s(\theta,x) \text{ increasing in } x,\ q^s(\theta,0)=0,\ q^s(\theta,1)=1, \tag{38}$$

where $Z(\theta,x) = S(\theta,x) + V(\theta,x) - (1-F(\theta))/f(\theta) \cdot V_\theta(\theta,x)$ is the (static) virtual surplus. The solution to $\mathscr{R}^s$ is given by the degenerate distribution which places all mass on the point-wise maximizer of Z:

$$x_R^s(\theta) = \arg\max_{x\in[0,1]} Z(\theta,x). \tag{39}$$

We may write the associated distribution function as

$$q_R^s(\theta,x) = \mathbf{1}_{[x_R^s(\theta),1]}(x), \tag{40}$$

where $\mathbf{1}_A(x)$ expresses the indicator function, which equals 1 if $x \in A$ and 0 otherwise.

Following this approach and explicitly allowing for stochastic contracts, Strausz (2006) identifies sufficient conditions for $q_R^s(\theta,x)$ to be a solution to the original problem $\mathscr{P}^s$.[9]

Lemma 2 *(Strausz 2006) The schedule $q_R^s(\theta,x)$ is a solution to $\mathscr{P}^s$ if the following conditions are jointly met:*
(i) $V_{\theta x}(\theta,x) \geq 0$ for all (θ,x);
(ii) $V_\theta(\theta,x) \geq 0$ for all (θ,x);
(iii) $x_R^s(\theta)$ is decreasing in θ.

[9] Strausz (2006) proves Lemma 2 in a model with discrete types , but the extension to continuous types is straightforward.

Condition (i) is the familiar single-crossing condition.[10] It is well-known that if $x_R^s(\theta)$ satisfies (iii), the single-crossing condition ensures that the local incentive compatibility constraints imply the global incentive compatibility constraints that were neglected in the relaxed problem. Condition (ii) ensures that the agent's utility U^s is increasing in his type θ so that the participation constraint for the "least efficient" type $\theta = 0$ implies the participation constraints for all other types that were neglected in the relaxed problem.

The next lemma shows that for the static screening problem to satisfy conditions (i) and (ii), the underlying sequential screening problem *must* have the property that the family of conditional distributions $\{G(x \mid \theta) \mid \theta \in [0,1]\}$ is ranked in the sense of first order stochastic dominance (FOSD).

Lemma 3 *Let V be given by (20).*
(i) $V_{\theta x}(\theta, x) \geq 0$ for all (θ, x) if and only if $G(x \mid \theta)$ is decreasing in θ for all x.
(ii) $V_\theta(\theta, x) \geq 0$ for all (θ, x) if $G(x \mid \theta)$ is decreasing in θ for all x.

The proof of the lemma follows from a straightforward calculation.[11,12]

To relate condition (iii) in Lemma 2 to conditions in the underlying sequential screening model, observe that by (20) and (21), the static virtual surplus Z induced by the sequential screening problem is

$$Z(\theta, x) = \int_x^1 (z - c(\theta, z))g(z|\theta)dz + \frac{1 - F(\theta)}{f(\theta)} \int_x^1 G_\theta(z|\theta)dz. \tag{41}$$

Therefore,

$$Z_x(\theta, x) = -\phi(\theta, x)g(x|\theta), \tag{42}$$

where

$$\phi(\theta, x) = x - c(\theta, x) + \frac{1 - F(\theta)}{f(\theta)} \frac{G_\theta(x|\theta)}{g(x|\theta)}. \tag{43}$$

The function ϕ features prominently in the sequential screening literature because it corresponds to the *sequential* virtual surplus from trade. Courty and Li (2000) refer to ϕ as regular if it is increasing in both arguments and they show that together with the condition that $\{G(x \mid \theta)|\theta \in$

[10] This condition is also referred to as "sorting", "constant sign", or "Spence-Mirrlees" condition.

[11] Note that $V_\theta(\theta, x) = -\int_x^1 G_\theta(z|\theta)dz$, and $V_{\theta x}(\theta, x) = G_\theta(x|\theta)$.

[12] We discuss an economic interpretation of the Lemma in more detail at the end of this section.

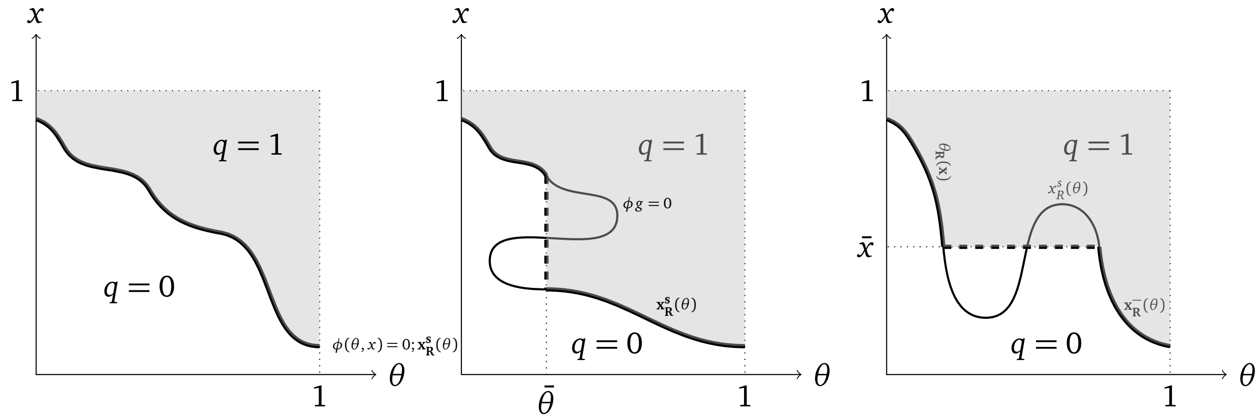

Figure 2: Regular case (left), ϕg increasing in θ (center), resp. in x (left)

$[0,1]\}$ is FOSD-ranked, regularity of ϕ ensures that the trading schedule q^s_R is a solution to the sequential screening problem $\mathscr{P}^d$.

Our equivalence result reveals the significance of the regularity of ϕ from the perspective of static screening: Because ϕ is increasing in x, (42) implies that $x^s_R(\theta)$ is the unique solution to the first–order condition $\phi(\theta, x^s_R(\theta)) = 0$, or is a corner solution. Because ϕ is increasing in θ, $x^s_R(\theta)$ is decreasing in θ. By Lemma 2, $q^s_R(\theta)$ is therefore a solution to $\mathscr{P}^s$, and therefore also to $\mathscr{P}^d$ by Proposition 1. We illustrate the regular case in the left panel of Figure 2.

Little is known about optimal sequential screening contracts if ϕ is not regular. Our next two propositions provide new results by treating cases where ϕ is not necessarily regular. Rather than monotonicity of ϕ in both arguments, we require monotonicity of ϕg in one argument.[13,14]

Proposition 3 *Let $\{G(x \mid \theta) | \theta \in [0,1]\}$ be FOSD-ranked, and let $\phi(\theta, x) g(x \mid \theta)$ be increasing in θ for all x. Then $x^s_R(\theta)$ is a solution to $\mathscr{P}^s$.*

The result is an immediate consequence of Theorem 4 in Milgrom and Shannon (1994) which says that if $Z_{\theta x} \leq 0$, then $x^s_R(\theta)$ is decreasing so that Lemma 2 applies. But by (42), the condition

[13]Because $\phi(\theta, x^s_R(\theta)) = 0$ if and only if $\phi(\theta, x^s_R(\theta)) g(x^s_R(\theta) | \theta) = 0$, the argument in the previous paragraph goes through unchanged if the condition that ϕ is increasing in both arguments is replaced by the condition that ϕg is increasing in *both* arguments. Note however that ϕ being monotone in both arguments does not imply that ϕg is monotone in one argument.

[14]It is easy to see that our subsequent arguments apply to an even larger class of models–those for which ϕ crosses zero at most once from below both in direction x and in direction θ. In particular, Proposition 3 (resp. Proposition 4) also hold for the weaker requirement that ϕg crosses zero at most once from below in direction θ (resp. x). This weaker sufficient condition is however more cumbersome to check.

that $Z_{\theta x} \leq 0$ is the same as the condition that ϕg be increasing in θ. A "typical" shape of ϕg is illustrated in the center panel of Figure 2, where the 0–level set of ϕg (resp. of ϕ) is inverted S-shaped, and x_R displays a downward jump.

Our next proposition considers the case that ϕg is increasing in x. As illustrated in the right panel of Figure 2, $x_R^s(\theta)$ might then not be decreasing in θ, and hence Lemma 2 is not applicable. If $x_R^s(\theta)$ is not decreasing in θ, little is known about how a solution to $\mathscr{P}^s$ in the class of all *stochastic* trading schedules $q^s(\theta, x)$ can be obtained. Instead, the literature on static screening restricts attention to finding deterministic solutions to $\mathscr{P}^s$. In particular, Nöldecke and Samuelson (2007) provide a tractable procedure to find deterministic solutions to static screening problems where the solution to the relaxed problems violates monotonicity. By following their approach, we now show how our equivalence result can be used to identify optimal sequential screening contracts in the class of deterministic sequential screening contracts.

A schedule $q^s(\theta, x)$ is deterministic if it corresponds to a degenerate distribution function which places mass 1 on a distinct quantity $x^s(\theta)$. Thus, we can identify a deterministic schedule $q^s(\theta, x)$ with the schedule $x^s(\theta)$ of quantities it delivers with probability 1. Within the class of deterministic contracts, an optimal static schedule solves the problem

$$\tilde{\mathscr{P}}^s: \qquad \max_{x^s(\cdot)} \int_0^1 Z(\theta, x^s(\theta))\, dF(\theta) \quad s.t. \quad x^s(\theta) \text{ is decreasing in } \theta. \tag{44}$$

In the sequential screening problem, the restriction to deterministic trading schedules corresponds to a restriction to so-called option contracts. To see this, note that under the sequential contract that corresponds to the static contract with selling schedule x^s, trade takes place if and only if ex ante type θ's ex post type x exceeds the threshold $x^s(\theta)$. Such a contract can be implemented by a menu $\{(y(\theta), x^s(\theta)) \mid \theta \in [0,1]\}$ of option contracts where agent type θ initially pays a type-dependent up-front fee $y(\theta)$, and then, after having learned x, has the option to buy the good at a pre-determined exercise price which equals $x^s(\theta)$. Hence, by identifying a deterministic solution to the static problem, we identify an optimal option contract in the sequential problem.

Following Nöldecke and Samuelson (07), let

$$Z^-(x, \theta) = \int_0^\theta Z_x(\tau, x) f(\tau)\, d\tau, \quad and \quad \theta_R(x) = \arg\max_\theta Z^-(x, \theta). \tag{45}$$

For a function $\theta(x)$ which is (not necessarily strictly) decreasing in x, recall the definition of the generalized inverse:

$$x^{-}(\theta) = \inf\{x \in [0,1] \mid \theta(x) \geq \theta\}. \tag{46}$$

With this definition, we can express the following result.

Proposition 4 *Let $\{G(x \mid \theta) | \theta \in [0,1]\}$ be FOSD-ranked, and let $\phi(\theta, x) g(x \mid \theta)$ be strictly increasing in x for all θ. Then $\theta_R(x)$ is decreasing in x, and its generalized inverse denoted by $x_R^{-}(\theta)$ is a solution to $\tilde{\mathscr{P}}^s$.*

To see the intuition, note that by (42), ϕg being strictly increasing in x means that Z is strictly concave in x for all θ. This implies that the point-wise maximizer x_R of Z is the unique solution to the first–order condition $Z_x(\theta, x_R(\theta)) = \phi(\theta, x_R(\theta)) = 0$. However, since x_R does not need to be decreasing in θ, it is generally no solution to $\tilde{\mathscr{P}}^s$. In this case, a solution to $\tilde{\mathscr{P}}^s$ displays "bunching", ie. there are intervals of types θ within which each type trades the same quantity.

For the case that Z is strictly concave in x for all θ, Nöldecke and Samuelson (2007) present a procedure by which the solution can be obtained as the result of an unconstrained optimization problem.[15] The idea is to invert the problem and instead of looking for an optimal quantity for each type, to look for an optimal type for each quantity. More formally, the inverted problem is

$$\tilde{\mathscr{P}}^{s^-}: \quad \max_{\theta(\cdot)} \int_0^1 Z^{-}(x, \theta(x))\, dx \quad s.t. \quad \theta(x) \text{ is decreasing in } x, \tag{47}$$

and $\theta(\cdot)$ is a solution to $\tilde{\mathscr{P}}^{s^-}$ if and only if its generalized inverse is a solution to $\tilde{\mathscr{P}}^s$.[16] As pointed out by Nöldecke and Samuelson (2007), the point-wise maximizer $\theta_R(x)$ of the "inverse virtual surplus" Z^{-} is decreasing in x if Z is strictly concave in x.[17] Because Z is strictly concave in x if and only if ϕg is strictly increasing in x by (42), it follows that $\theta_R(x)$ is a solution to $\tilde{\mathscr{P}}^{s^-}$, and its

[15] The strict concavity of Z corresponds to Assumption 2 in Nöldecke and Samuelson (2007). Their other Assumption 1—that the V is quasi–convex in θ for all x—is trivially satisfied in our setting since $V_\theta \geq 0$.

[16] To see this, note that for a decreasing function $x(\theta)$ with generalized inverse $\theta^{-}(x)$, Fubini's theorem implies:

$$\begin{aligned}
\int_0^1 Z(\theta, x(\theta)) f(\theta)\, d\theta &= \int_0^1 [\int_0^{x(\theta)} Z_x(\theta, x) dx + Z(\theta, 0)] f(\theta)\, d\theta \\
&= \int_0^1 \int_0^{\theta^{-}(x)} Z_x(\theta, x) f(\theta)\, d\theta dx + C = \int_0^1 Z^{-}(x, \theta^{-}(x)) dx + C,
\end{aligned}$$

where $C = \int_0^1 Z(\theta, 0) f(\theta)\, d\theta$. Hence, $x(\theta)$ is a solution to $\tilde{\mathscr{P}}^s$ if and only if $\theta^{-}(x)$ is a solution to $\tilde{\mathscr{P}}^{s^-}$.

[17] Indeed, by Theorem 4 in Milgrom and Shannon (1994), θ_R is decreasing if $Z^{-}_{x\theta} \leq 0$. But observe that $Z^{-}_{x\theta} = Z_{xx} f$.

generalized inverse of $x_R^-(\theta)$ is a solution to $\tilde{\mathscr{P}}^s$. For an illustration, see the right panel in Figure 2.

We conclude this section with remarks that provide additional insights of our analysis.

Remark 1 Under the conditions of Proposition 3 as well as in the regular case, an optimal contract is deterministic and can thus be implemented as a menu of option contracts. This is a helpful insight because in general, it is difficult to find an optimal sequential screening contract in the class of stochastic contracts (as, e.g., in the case of Proposition 4). The difficulty stems from the fact that the set of incentive compatible sequential screening contracts cannot be easily characterized in terms of monotonicity constraints on the schedule $q^d(\theta, x)$. More precisely, the restriction to deterministic sequential screening contracts effectively corresponds to the restriction that the schedule $q^d(\theta, x)$ is increasing in both θ and x. While this suffices to find transfers so that the corresponding contract is incentive compatible, there are incentive compatible contracts where $q^d(\theta, x)$ is not increasing in θ. Given our equivalence result, it is not surprising that identical complications arise in static screening models, where, similarly, monotonicity of the stochastic schedule $q^s(\theta, x)$ in θ is not necessary for incentive compatibility. Therefore, the potential benefit of using stochastic contracts, both in sequential and in static screening, lies precisely in the leeway they provide to relax monotonicity with respect to θ.

Remark 2 Lemma 3 clarifies that the FOSD-ranking of the agent's beliefs in sequential screening simply corresponds to the single-crossing condition from static screening. It therefore guarantees that local incentive compatibility implies global incentive compatibility. Reversely, a sequential screening problem without FOSD-ranking corresponds to a static screening problem without global single-crossing, causing potential problems with global incentive compatibility. One intuitive approach to address these problems is to consider subdomains of the space of types and allocations on which the single-crossing condition holds. (See Araujo and Moreira (2010), Schottmüller (2015) for treatments of adverse selection problems without single-crossing.) Indeed, applying steps in this spirit, Courty and Li (2000) characterize optimal solutions also for sequential screening problems in which the conditional distributions are not FOSD-ranked but are "rotation-ordered" (see Johnson and Myatt, 2006). The rotation-order ensures that the allocations of an optimal solution fall in a subdomain where $V_{\theta x}$ is of constant sign and, hence, the single-crossing condition holds locally.

Remark 3 It is instructive to highlight some economic efficiency features of an optimal contract. These features are best illustrated for the case of type–independent costs: $c(\theta, x) = c$. Under

Proposition 3, the point-wise maximizer of the virtual surplus $x_R^s(\theta)$ is a solution to $\mathscr{P}^s$. It follows from (41) that at the "most efficient" type $\theta = 1$, we have that $x_R^s(1) = c$ and, moreover, that $x_R^s(\theta) > c$ for all $\theta < 1$. Likewise, under Proposition 4, the generalized inverse $x_R^-(\theta)$ of the point-wise maximizer, $\theta_R(x)$, of the inverted virtual surplus is a solution to $\tilde{\mathscr{P}}^s$. It follows from (45) that $\theta_R(c) = 1$, so that we have $x_R^s(1) = c$ and, moreover, $\theta_R(x) < 1$ for all $x > c$. Therefore, the definition of the generalized inverse implies that $x_R^-(1) = c$ and, moreover, $x_R^-(\theta) > c$ for all $\theta < 1$. These observations have two implications. First, under an optimal contract, trade occurs at the most efficient ex ante type if and only if the ex post valuations x is larger than costs c, that is, there is no distortion at the top. Second, since the exercise price is larger than c for all $\theta < 1$, except at the most efficient type $\theta = 1$, trade does not take place for some ex post valuations x larger than costs c, that is, an optimal contract displays downward distortions. Hence, also under our alternative conditions, the key efficiency properties of optimal sequential screening contracts are qualitatively the same as in the regular case.

4 Conclusion

We establish a correspondence between sequential screening and static screening models, and show how our equivalence result allows us to solve classes of sequential screening models for which standard sequential screening techniques are not applicable. We also illuminate a number of salient features of the sequential model in the light of their well–understood counterparts in the static model.

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