

# Free Riding and Duplication in R&D\*

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## Abstract

We study a model of R&D race in the exponential-bandit learning framework (Choi 1991; Keller, Rady, and Cripps 2005), in which two research firms, each endowed with an independent R&D process, choose when to irreversibly exit the R&D race. Each R&D process can be either good or bad. In the absence of a research breakthrough (innovation), a firm becomes more pessimistic about its R&D process over time. We show that strict patent protection may lead to excessive duplication of research efforts, while the lack of patent protection leads to free riding and under-experimentation of research opportunities. The choice of optimal patent system involves a trade-off between duplication in the early stage of R&D when both firms are optimistic and under-experimentation in the later stage when one firm has already exited and the remaining firm is pessimistic. While the optimal patent system is in general inefficient, we propose an asymmetric distribution policy that implements the social optimum.

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# 1 Introduction

The recent advancements in theoretical studies on strategic experimentation (e.g., Keller et al. 2005) have renewed interests of economists to study experimentation in R&D races. Various authors have applied learning models to investigate R&D races and obtained fresh insights into this classic problem. Some authors (e.g., Das 2014) argue that winner-take-all R&D race leads to excessive duplication of research efforts. Since only one success counts, resources put into research by competitors could be wasted. On the other hand, Acemoglu et al. (2011) study a model of experimentation in which firms can copy the innovation of others. In the absence of patent protection, free riding occurs in the form of delayed experimentation. They further show that an appropriately designed patent system can implement the optimal allocation by encouraging experimentation while maintaining efficient transfer of knowledge ex post. Their analysis suggests that patent could be an efficient means of reducing free riding.

In this paper, we employ an exit game to study the phenomena of free riding and duplication in R&D races. In particular, we are interested in how a patent system may create or resolve these inefficiencies. In our model, each firm is endowed with an independent R&D process (project) and each chooses privately when to exit the R&D race irreversibly. Each R&D process can be either good or bad and each has a different prior probability of being good. We refer to the firm whose project is more likely to be good as the *strong firm* and the other firm the *weak firm*. Given that a firm's project is good, a breakthrough (innovation) occurs to her in a Poisson process as long as she remains in the R&D race. Each firm incurs a flow cost to stay in the race. Once a breakthrough occurs, the game ends. A parameter  $\alpha \in [\frac{1}{2}, 1]$  describes how the post-innovation surplus  $v$  is split between the two firms. The innovating firm receives  $\alpha v$  and the non-innovating firm receives  $(1 - \alpha)v$ . When  $\alpha = 1$ , the innovating firm receives all the surplus  $v$ . When  $\alpha = \frac{1}{2}$ , both the innovating firm and the non-innovating firm receive half of the surplus  $\frac{v}{2}$ . Thus,  $\alpha$  can be considered as a measurement of the *patent strength*.

We find that the heterogeneity in the firms' projects has important welfare and policy consequence. Unless the projects are homogeneous, strict patent, i.e.,  $\alpha = 1$ , is suboptimal

and leads to excessive duplication of research efforts.<sup>1</sup> Imposing no patent protection, i.e.,  $\alpha = \frac{1}{2}$ , is likewise suboptimal and leads to excessive free riding behavior and under-experimentation of research opportunities. We further show that under the optimal patent system, the strong firm never over-experiments and the weak firm never under-experiments. As a result, duplication occurs in the early stage of R&D when both firms are optimistic and under-experimentation occurs in the later stage when the weak firm has already exited and the remaining strong firm is pessimistic. Unless the projects are homogeneous, the optimal patent system is inefficient. Nevertheless, we propose an asymmetric distribution policy that implements the social optimum. Under the optimal distribution policy, strict patent protection is granted if the discovery is made by the strong firm and innovation by the weak firm is only partially protected. Indeed, the optimal distribution policy may even grant protection to the strong firm when the discovery is made by the weak firm. By doing so, it discourages the weak firm from wasteful duplication of the strong firm’s effort.

## 1.1 Related Literature

This paper belongs to the literature of strategic experimentation. Many works in this literature take the payoff structure of the game as given and belong to one of two classes: models of “collaboration” in which experimentation has a positive externality and models of “competitive experimentation” in which the success of one agent imposes a negative externality on the others.<sup>2</sup> Papers in the first class include Bolton and Harris (1999), Keller et al. (2005), Keller and Rady (2010), Bonatti and Hörner (2011), Klein and Rady (2011), Klein (2013), and Das (2017). Papers in the second class focus mostly on the case of winner-takes-all contest, these include Choi (1991), Malueg and Tsutsui (1997), Chatterjee and Evans (2004), Mason and Välimäki (2010), Moscarini and Squintani (2010), and Das (2014). Latest applications of models of strategic experimentation to R&D competition

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<sup>1</sup>Our notion of duplication is similar to that of Loury (1979) but different from that of Chatterjee and Evans (2004), Das (2014) and Matros and Smirnov (2016), who consider models with more than one line of research; duplication occurs when the two players (firms) conduct the same line of research. In contrast, we say that duplication occurs in our model when too many firms conduct R&D relative to the efficiency benchmark, even though each firm has her own line of research.

<sup>2</sup>Beyond this simple dichotomy are models with private payoffs and observational learning. These include Rosenberg et al. (2007), Murto and Välimäki (2011), Rosenberg et al. (2013), Heidhues et al. (2015), and Wagner (2017b).

focus on various aspects of information including private discoveries (Akcigit and Liu, 2016; Das, 2015) strategic information disclosure (Song and Zhao, 2016) and transparency of patent systems (Salish, 2016).

More closely related to the present paper are works on contest and contract designs with learning. Acemoglu et al. (2011) study a model of innovation and patent protection similar to ours in spirit. Their model differs from ours in that experimentation is completed instantaneously in their model. As a result, beliefs do not evolve over time and their model does not address the issue of over-experimentation and under-experimentation over time. They also find that the social optimum can be implemented without resorting to asymmetric policy, which is not the case in our setting. Halac et al. (2017) study the optimal contest in a learning model with homogenous agents where the designer chooses how to allocate a prize and what information about contestants' successes to disclose over time to maximize the contestants' efforts. Our model differs from theirs in three respects. First, the patent designer in our model designs a patent to maximize the total welfare of the contestants (firms) instead of the total efforts. Second, we consider contests with heterogeneous agents. Third, we consider in this paper a more restrictive class of prize-sharing schemes and do not allow the designer to vary the disclosure policy. These differences lead to different trade-offs in the design of optimal contest (patent). Moroni (2017) and Wagner (2017a) consider dynamic moral hazard models with multiple agents and learning. In contrast to our model, they assume that fully dynamic, history-contingent contract with transfer can be made.<sup>3</sup>

There is a sizable literature, pioneered by Nordhaus (1969), on patent policy focusing various aspects of innovation and patent races. Much of this literature studies models without learning and focuses on different trade-offs from those considered in this paper. The design of optimal patent policy is studied by, for example, Denicolò (1999) and Judd, Schmedders and Yeltekin (2012); see also the references therein.

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3

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<sup>3</sup>Other works that take a principal-agent approach to experimentation include Gerardi and Maestri (2012), Hörner and Samuelson (2014), Chen and Ishida (2015), Guo (2016), Halac et al (2016), Klein (2016), Hidir (2017) and Wolf (2017).

presents the equilibrium characterizations. Section 4 studies the planner’s problem where the firms cooperate to maximize joint expected payoffs and analyzes the welfare consequences of increasing the projects’ qualities. Section 5 presents the characterization of the optimal patent. Section 6 extends the model to allow general distribution policy and shows that the optimal policy implements the planner’s solution. Section 7 discusses the roles of the modelling assumptions and some possible extensions. Section 8 concludes. Some of the proofs are relegated to the Appendix.

## 2 Model

### 2.1 Setup

Time is continuous and the horizon is infinite. Two research firms, 1 and 2, each endowed with an R&D process (project), decide when to stop experimenting and exit the R&D race (abandon the project). Each firm incurs a flow cost  $c$  to stay in the race. Once a firm exits, prohibitively high sunk costs make re-entry infeasible. This assumption captures the idea that large amounts of intellectual and financial resources are often needed for R&D activities. Moreover, the decision to exit is private and unobservable to the competing firm. The roles of these assumptions are discussed in Section 7.

Each firm’s project may be either good or bad. With prior probability  $p_{i,0}$ , firm  $i$ ’s project is good, independent of firm  $j$ ’s project.<sup>4</sup> At time  $t$ , a breakthrough (innovation) occurs to firm  $i$  with instantaneous probability equal to  $\lambda$  if firm  $i$ ’s project is good and she has not exited the race. On the other hand, a breakthrough never occurs if the project is bad. We refer to the firm whose project has a strictly higher prior probability of being good as the *strong firm* and use the upper case letter  $S$  to indicate her. The other firm is referred to as the *weak firm* and the lower case letter  $w$  is used to indicate her. When the

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<sup>4</sup>Choi (1991) and follow-up works (e.g., Besanko and Wu, 2013; Salish, 2016; Song and Zhao, 2016) assume perfectly positively correlated R&D processes. However, Chatterjee and Evans (2004), Klein and Rady (2011), and Das (2014) argue that negative correlation is also possible in R&D. To simplify the exposition, independence is assumed in this paper. We interpret the two projects as two approaches to develop the same product. The approaches could be different but the end products are assumed to be close enough so that only one firm can patent her product. Note that when the correlation of the two R&D processes is less than 1, R&D investment of the second firm has an additional social value as it can serve as an insurance against the bad outcome of the first firm’s experimentation.

prior probabilities are equal, the strong (weak) firm refers to firm 1 (2).

The game ends either when a firm has achieved a breakthrough or when both firms exit.<sup>5</sup> Suppose firm  $i$  is the first to achieve a breakthrough, then firm  $i$  receives an award  $\alpha v$  while firm  $j$  receives  $(1 - \alpha)v$ , where  $\alpha \in [\frac{1}{2}, 1]$ . Costs and profits are undiscounted. The description of the game is common knowledge among the firms.

The post-innovation payoffs of the game can be interpreted as follows. After the breakthrough occurs, a patent is granted to the innovating firm. The non-innovating firm copies the patented innovation and makes a royalty payment  $(\alpha - \frac{1}{2})v$  to the innovating firm. Each firm then earns a surplus of  $\frac{v}{2}$  in the product market. When  $\alpha$  equals to  $\frac{1}{2}$ , both firms get the same payoff  $\frac{v}{2}$ . When  $\alpha$  equals to 1, the non-innovating firm gets 0. Thus, the parameter  $\alpha$  is referred to in this paper as the *patent strength*. The case of  $\alpha = 1$  is also referred to as the *strict* patent case,  $\alpha \in (\frac{1}{2}, 1)$  the *weak* patent case, and  $\alpha = \frac{1}{2}$  the *no* patent case. The social welfare of this economy is defined to be the sum of the expected payoffs of the two firms.

Notice that we have made the assumption that the total surplus in the product market is always  $v$ , independent of the patent system. From a theoretical perspective, we suggest that this is a useful benchmark and starting point as it allows us to abstract away from the well-studied source of inefficiency due to monopoly power and concentrate on the trade-off between experimentation and duplication, which is the main focus of this paper. Similar approach is, for example, adopted by Loury (1979). From a practical perspective, the assumption that patent protection does not lead to efficiency loss in the product market, while highly restrictive, may not be unrealistic at times. In the 2006 case of *eBay Inc. v. MercExchange, L.L.C.*,<sup>6</sup> the Supreme Court of the United States determined that an injunction should not be automatically issued upon a finding of infringement, instead, sometimes “legal damages may well be sufficient to compensate for the infringement.” Moreover, many standard setting organizations (SSOs) require the owners of *standard-essential patents* to grant licenses on fair, reasonable and nondiscriminatory (FRAND)

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<sup>5</sup>Strictly speaking, one should also specify the outcome of the game if both firms choose to experiment forever. However, our assumption of nonextreme prior makes sure that such a strategy is never optimal. We, therefore, omit this detail.

<sup>6</sup>547 U.S. 388 (2006).

terms. In such cases, it may be reasonable to think of patent strength as the reasonable royalty that the innovator can expect to receive and efficiency loss in the product market can be limited. The implication of adding consumer welfare and product market efficiency considerations in the picture will be discussed in Section 7.

A strategy of a firm in this game is an exit time  $t \in \mathbb{R}_+$ , with the interpretation that the firm will exit at time  $t$  if no breakthrough has occurred prior to  $t$ . The solution concept we use is Nash equilibrium. We focus on the pure strategy equilibrium that is total welfare maximizing. An equilibrium strategy of firm  $i$  is denoted by  $t_i^*$ .

We make the following assumption on the prior  $(p_{1,0}, p_{2,0})$ .

**Assumption (Nonextreme Prior):**  $\forall i = 1, 2, p_{i,0} \in (\frac{c}{\lambda v}, 1)$ .

Under this assumption, degenerate priors are ruled out. Unless a breakthrough occurs, the firms are not sure that if a project is good. Moreover, the assumption that  $p_{1,0}, p_{2,0} > \frac{c}{\lambda v}$  makes sure that both firms must have incentives to experiment for a small period of time under *some* patent arrangement. Otherwise, at least one of the firms will always exit at time 0 and the problem becomes one with a single firm. Since  $p_{i,0} < 1$ , this assumption also entails that  $\lambda v > c$ .

## 2.2 Preliminaries

Since the arrival of a breakthrough is publicly observed and a firm's exit time is known to her opponent in any equilibrium in pure strategies, given the firms' common prior  $(p_{1,0}, p_{2,0})$ , the firms share a common posterior on the equilibrium path at each  $t \geq 0$  denoted by  $(p_{1,t}, p_{2,t})$ . Suppose at time  $t$ , no breakthrough has occurred and firm  $i$  has not exited, then, by Bayes' rule, the posterior belief about firm  $i$ 's project at time  $t$  is given by

$$p_{i,t} = \frac{p_{i,0}e^{-\lambda t}}{p_{i,0}e^{-\lambda t} + 1 - p_{i,0}}. \quad (1)$$

The law of motion of the posterior beliefs can be derived accordingly, suppose that over the time interval  $[t, t + dt)$ , firm  $i$  engages in R&D and has not achieved a breakthrough

by the end of the period, the belief about her project is updated to

$$p_{i,t} + dp_{i,t} = \frac{p_{i,t}(1 - \lambda dt)}{p_{i,t}(1 - \lambda dt) + 1 - p_{i,t}}.$$

Simplifying, we have

$$dp_{i,t} = -p_{i,t}(1 - p_{i,t})\lambda dt. \quad (2)$$

Once there is a breakthrough, the posterior belief jumps to 1. Notice that the expression in (2) is not constant. Thus, when the prior probabilities are different, the posterior beliefs do not move at the same rate even when both firms are experimenting. Denote the belief path taken by two firms that experiment forever by  $\gamma$ . That is,

$$\gamma(p_{1,0}, p_{2,0}) \equiv \left\{ (p_1, p_2) \in [0, 1]^2 : \exists t \in \mathbb{R}_+, \forall i \in \{1, 2\}, p_i = \frac{p_{i,0}e^{-\lambda t}}{p_{i,0}e^{-\lambda t} + 1 - p_{i,0}} \right\}. \quad (3)$$

Figure 1 illustrates some of the possible belief paths.

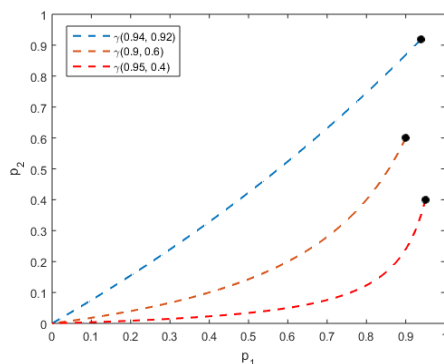


Figure 1: Belief Paths. Parameter Value:  $\lambda = 1$ .

If firm  $i$  exits at some time  $t'$ , the posterior belief will stay at  $p_{i,t'}$  forever.

### 2.3 The Single Firm Decision Problem

In this subsection, we introduce and solve the single firm problem, which is not only a useful benchmark for the noncooperative problem, but also an important intermediate step in solving it.

Suppose at time 0, firm  $j$  exits the R&D race. Knowing that, firm  $i$  chooses a time



$t_i \geq 0$  to exit conditional on no breakthrough. This problem, which we call the *single firm decision problem*, is standard, and the optimal policy is to exit when the posterior belief for firm  $i$ 's project reaches the threshold value  $\frac{c}{\lambda\alpha v}$ , which we also refer to as the *single firm threshold belief*. We have,

**Lemma 1** *Given that firm  $j$  exits at time 0, firm  $i$ 's optimal policy is to exit when the posterior belief falls below the single firm threshold belief  $\frac{c}{\lambda\alpha v}$ . The optimal exit time  $t_i^\dagger(\alpha)$  has the expressions*

$$t_i^\dagger(\alpha) = \begin{cases} -\frac{1}{\lambda} \log\left(\frac{1-p_{i,0}}{p_{i,0}} \frac{c}{\lambda\alpha v - c}\right) & \text{if } p_{i,0} > \frac{c}{\lambda\alpha v}, \\ 0 & \text{if } p_{i,0} \leq \frac{c}{\lambda\alpha v}. \end{cases} \quad (4)$$

Moreover, firm  $i$ 's continuation value at time  $t$  is given by

$$\hat{W}(p_{i,t}; \alpha) = \begin{cases} p_{i,t}\alpha v - \frac{c}{\lambda} + (1-p_{i,t}) \frac{c}{\lambda} \log\left(\frac{1-p_{i,t}}{p_{i,t}} \frac{c}{\lambda\alpha v - c}\right) & \text{if } p_{i,t} > \frac{c}{\lambda\alpha v}, \\ 0 & \text{if } p_{i,t} \leq \frac{c}{\lambda\alpha v}, \end{cases} \quad (5)$$

and firm  $j$ 's continuation value at time  $t$  is given by

$$\check{W}(p_{i,t}; \alpha) = \begin{cases} \left(\frac{p_{i,t}\lambda\alpha v - c}{\lambda\alpha v - c}\right) (1-\alpha)v & \text{if } p_{i,t} > \frac{c}{\lambda\alpha v}, \\ 0 & \text{if } p_{i,t} \leq \frac{c}{\lambda\alpha v}, \end{cases} \quad (6)$$

where  $p_{i,t}$  is given by (1).

To see the optimality of the policy, we first write down firm  $i$ 's objective function. Suppose firm  $i$  experiments at time  $t \geq 0$ . At time  $t$ , the posterior belief that firm  $i$ 's project is good is  $p_{i,t}$ , the instantaneous probability of a breakthrough conditional on a good project is  $\lambda$  and firm  $i$  receives  $\alpha v$  from a breakthrough. Thus, the instantaneous expected payoff at time  $t$  is simply  $p_{i,t}\lambda\alpha v - c$ . To calculate firm  $i$ 's payoff at time 0, the instantaneous expected payoff must be weighted by the probability that a breakthrough has not occurred by time  $t$ , which is  $\exp\{-\int_0^t p_{i,s}\lambda ds\}$ . (Otherwise, the game has ended and the experimentation at time  $t$  becomes irrelevant.) Thus, in the single firm problem,

firm  $i$  chooses  $t_i$  to maximize the expression

$$\int_0^{t_i} (p_{i,t} \lambda \alpha v - c) e^{-\int_0^t p_{i,s} \lambda ds} dt. \quad (7)$$

By (1),  $p_{i,t}$  is strictly decreasing in  $t$  and tends to 0 as  $t$  increases. As a result, the instantaneous expected payoff,  $p_{i,t} \lambda \alpha v - c$ , is strictly decreasing and eventually becomes negative and (7) is maximized at the point  $t_i$  at which  $p_{i,t_i} \lambda \alpha v = c$ . If such a point does not exist,  $p_{i,t} \lambda \alpha v < c$  for all  $t \geq 0$ . In this case, (7) is maximized at  $t_i = 0$ . These two cases lead to the two expressions for  $t_i^\dagger(\alpha)$  in (4). The continuation values (5) and (6) are then computed from the optimal exit time (4) by evaluating the integrals

$$\begin{aligned} \hat{W}(p_{i,t}; \alpha) &\equiv \int_t^{t_i^\dagger(\alpha) \wedge t} (p_{i,s} \lambda \alpha v - c) e^{-\int_t^s p_{i,z} \lambda dz} ds, \\ \check{W}(p_{i,t}; \alpha) &\equiv \int_t^{t_i^\dagger(\alpha) \wedge t} p_{i,s} \lambda (1 - \alpha) v e^{-\int_t^s p_{i,z} \lambda dz} ds. \end{aligned}$$

Notice that even though firm  $j$  has exited R&D at time 0, her continuation value could still be strictly positive. This is because as long as firm  $i$  undertakes an innovation at some point in the future, firm  $j$  receives  $(1 - \alpha)v$  without having to pay the flow cost  $c$ . As we shall see in Section 3, this possibility of being a copycat creates incentives for firm  $j$  to exit earlier than her single firm benchmark to “free ride” on firm  $i$ ’s R&D efforts in the noncooperative problem. Such behavior, however, is not necessarily bad for social welfare as it may help to reduce duplications of research efforts, as we will find in Section 4.

### 3 Equilibrium Characterizations

In this section, we give several characterizations of the equilibria of our game. We first provide a general characterization for *all* equilibria in pure strategies (Proposition 1). Then, we divide equilibria in pure strategies into two distinct classes and study them separately. An equilibrium is *regular* if and only if the strong firm’s equilibrium exit time is weakly later than the weak firm’s. In Lemma 2, we completely characterize nonregular equilibria. After establishing the existence of regular equilibrium, we provide characterizations of regular

equilibrium over different ranges of patent strength.

Building on Lemma 1, Proposition 1 provides the basic characterization of the equilibria of our game, on which much of the subsequent analysis is based.

**Proposition 1** *Consider an equilibrium  $(t_1^*, t_2^*)$ . Suppose  $t_i^* \geq t_j^*$ , then,  $t_i^*$  is the smallest  $t$  that satisfies  $p_{i,t}\lambda\alpha v \leq c$ , which has the expressions*

$$t_i^* = t_i^\dagger(\alpha) = \begin{cases} -\frac{1}{\lambda} \log\left(\frac{1-p_{i,0}}{p_{i,0}} \frac{c}{\lambda\alpha v - c}\right) & \text{if } p_{i,0} > \frac{c}{\lambda\alpha v}. \\ 0 & \text{if } p_{i,0} \leq \frac{c}{\lambda\alpha v}. \end{cases} \quad (8)$$

Moreover,  $t_j^*$  satisfies

$$INB(p_{j,t_j^*}, p_{i,t_j^*}; \alpha) \equiv p_{j,t_j^*} \lambda (\alpha v - \check{W}(p_{i,t_j^*}; \alpha)) - c \leq 0, \quad (9)$$

where the function  $\check{W}(\cdot; \alpha)$  is given by (6). Moreover, if  $t_j^* > 0$ , (9) also holds with equality.

Proposition 1 has two parts. The first part of Proposition 1 states that the equilibrium exit time of the firm that exits later in equilibrium must be identical to the optimal exit time of the single firm decision problem. In other words, the firm whose equilibrium exit time is higher acts *as if the opponent chooses to exit at time 0*.

To understand the characterization, suppose  $t_i^* \geq t_j^*$  and  $t_j^* = 0$ . In this case, the assertion is trivially true. Suppose  $t_j^* > 0$ , consider a time  $t \in [0, t_j^*]$  when both firms are still active. The probability that a breakthrough has not occurred by that time is  $\exp\{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds\}$ . The posterior probability that firm  $i$ 's (firm  $j$ 's) project is good is  $p_{i,t}$  ( $p_{j,t}$ ) and the instantaneous probability of a breakthrough by firm  $i$  (firm  $j$ ) is thus  $p_{i,t}\lambda$  ( $p_{j,t}\lambda$ ). It follows that firm  $i$ 's instantaneous expected payoff at time  $t$  is given by  $(p_{i,t}\lambda\alpha + p_{j,t}\lambda(1-\alpha))v - c$ . Moreover, with probability  $\exp\{-\int_0^{t_j^*} (p_{1,s} + p_{2,s}) \lambda ds\}$ , firm  $j$  exits at time  $t_j^*$  and firm  $i$  faces the single firm decision problem from  $t_j^*$  onwards. It

follows that given an exit time  $t_i \geq t_j^*$ , firm  $i$ 's expected payoff is given by

$$V_i(t_i, t_j^*) = \int_0^{t_j^*} \{(p_{i,t}\alpha + p_{j,t}(1-\alpha))\lambda v - c\} e^{-\int_0^t (p_{1,s}+p_{2,s})\lambda ds} dt + e^{-\int_0^{t_j^*} (p_{1,s}+p_{2,s})\lambda ds} \left( \int_{t_j^*}^{t_i} (p_{i,t}\lambda\alpha v - c) e^{-\int_{t_j^*}^t p_{i,s}\lambda ds} dt \right).$$

Since  $t_i^* \geq t_j^* > 0$ , the first-order condition has an interior solution

$$\frac{\partial V_i(t_i, t_j^*)}{\partial t_i} \Big|_{t_i=t_i^*} = e^{-\int_0^{t_i^*} p_{i,s}\lambda ds - \int_0^{t_j^*} p_{j,s}\lambda ds} \{p_{i,t_i^*}\lambda\alpha v - c\} = 0. \quad (10)$$

By (1),  $p_{i,t}\lambda\alpha v - c$  is strictly decreasing in  $t$ . Thus, (10) is both necessary and sufficient.

Next, consider the characterization of  $t_j^*$  in the second part of Proposition 1. Since  $t_i^*$  is characterized by Lemma 1, given an exit time  $t_j \leq t_i^*$ , we can write firm  $j$ 's expected payoff as

$$V_j(t_j, t_i^*) = \int_0^{t_j} \{(p_{j,t}\alpha + p_{i,t}(1-\alpha))\lambda v - c\} e^{-\int_0^t (p_{1,s}+p_{2,s})\lambda ds} dt + e^{-\int_0^{t_j} (p_{1,s}+p_{2,s})\lambda ds} \check{W}(p_{i,t_j}; \alpha), \quad (11)$$

where  $\check{W}(\cdot; \alpha)$  is given by (6). Differentiating (11) with respect to  $t_j$  and simplifying the expression, we obtain firm  $j$ 's *marginal* net benefit of experimentation,  $MNB$ , at time  $t_j$

$$\frac{\partial V_j(t_j, t_i^*)}{\partial t_j} = e^{-\int_0^{t_j} (p_{1,s}+p_{2,s})\lambda ds} \{p_{j,t_j}\lambda(\alpha v - \check{W}(p_{i,t_j}; \alpha)) - c\} \quad (12)$$

The first order condition implies that firm  $j$  exits only if the  $MNB$  is nonpositive. We refer to the term in the bracket in (12) as firm  $j$ 's *instantaneous* net benefit of experimentation and denote it by

$$INB(p_{j,t}, p_{i,t}, \alpha) \equiv \underbrace{p_{j,t}\lambda(\alpha v - \check{W}(p_{i,t}; \alpha))}_{\text{Instantaneous benefit}} - \underbrace{c}_{\text{Instantaneous cost}}. \quad (13)$$

The *marginal* net benefit of experimentation  $\frac{\partial V_j(t_j, t_i^*)}{\partial t_j}$  differs from the *instantaneous* net

benefit of experimentation (13) in that the *MNB* is weighted by the probability that a breakthrough has not occurred by time  $t$ . The *MNB* can be thought of as the net benefit from an infinitesimal amount of additional experimentation at time  $t$  *calculated at time 0*. The net benefit is therefore “discounted,” as the additional experimentation may not be carried out if a breakthrough has occurred prior to time  $t$ . On the other hand, the *INB* can be interpreted as the net benefit from an infinitesimal amount of additional experimentation at time  $t$  *calculated at time  $t$* . As a result, no discounting is needed. It is easy to see that the *MNB* and the *INB* always has the same sign. Thus, the condition (9) is equivalent to the first order condition of firm  $j$ 's problem. The advantage of working with the *INB* over the *MNB* is that the *INB* refers only to the posterior  $(p_{j,t}, p_{i,t})$ . Neither the prior  $(p_{1,0}, p_{2,0})$  nor the current time  $t$  is involved. This allows us to analyze firm  $j$ 's problem using a 2-dimensional representation of the belief space.

To understand the meaning of (13), suppose, at time  $t$ , firm  $j$  experiments for a short amount of time  $dt$  and exits at  $t + dt$ . Three events may occur during this time interval. Firstly, firm  $i$  may achieve a breakthrough. The probability of this event is approximately  $p_{i,t}\lambda dt$ . If it occurs, firm  $j$  receives  $(1 - \alpha)v$ . Secondly, firm  $j$  may achieve a breakthrough. The probability of this event is approximately  $p_{j,t}\lambda dt$ . If it occurs, firm  $j$  receives  $\alpha v$ . (The probability that both firms achieve a breakthrough between  $t$  and  $t + dt$  is of the order of  $(dt)^2$ , which can be ignored when  $dt$  is small enough.) Thirdly, it may happen that no firm achieves a breakthrough during the time interval. The probability of this event is approximately  $1 - (p_{1,t} + p_{2,t})\lambda dt$ . In this case, firm  $j$  exits and receives the continuation value  $\check{W}(p_{i,t} + dp_{i,t}; \alpha)$  at  $t + dt$ . Thus, firm  $j$ 's continuation payoff at time  $t$  is approximately

$$-cdt + (p_{i,t}\lambda dt)(1 - \alpha)v + (p_{j,t}\lambda dt)\alpha v + (1 - (p_{1,t} + p_{2,t})\lambda dt)\check{W}(p_{i,t} + dp_{i,t}; \alpha). \quad (14)$$

Moreover, for  $dt$  small enough, the following approximation holds:

$$\check{W}(p_{i,t}; \alpha) \approx (p_{i,t}\lambda dt)(1 - \alpha)v + (1 - p_{i,t}\lambda dt)\check{W}(p_{i,t} + dp_{i,t}; \alpha) \quad (15)$$

This is because a breakthrough occurs to firm  $i$  with probability approximately equal to  $p_{i,t}\lambda dt$  in a time interval with duration  $dt$ . Using (15), we can rewrite (14) into

$$\check{W}(p_{i,t}; \alpha) + (p_{j,t}\lambda (\alpha v - \check{W}(p_{i,t} + dp_{i,t}; \alpha)) - c) dt. \quad (16)$$

Further using the approximation

$$\check{W}(p_{i,t} + dp_{i,t}; \alpha) \approx \check{W}(p_{i,t}; \alpha) - p_{i,t}(1 - p_{i,t})\lambda \check{W}'(p_{i,t}; \alpha) dt \quad (17)$$

to replace  $\check{W}(p_{i,t} + dp_{i,t}; \alpha)$  in (16) and ignore the term of the order of  $(dt)^2$ , we finally arrive at the following expression for the continuation payoff at time  $t$ ,

$$\check{W}(p_{i,t}; \alpha) + (p_{j,t}\lambda (\alpha v - \check{W}(p_{i,t}; \alpha)) - c) dt. \quad (18)$$

By exiting at time  $t$ , firm  $j$  receives the continuation value  $\check{W}(p_{i,t}; \alpha)$ . Thus, (18) implies that firm  $j$  should never exit when the instantaneous net benefit of experimentation (13) is strictly positive, or, equivalently, firm  $j$  should exit only if the instantaneous net benefit of experimentation is nonpositive, i.e., (9) holds. Notice that, in general, the objective function (11) is nonconcave and condition (9) is necessary but not sufficient for optimality.

In the general characterization in Proposition 1, we make no assertion about the relationship between the exit times and the firms' probabilities. In particular, the weak firm may exit later than the strong firm, and vice versa. Distinguishing these two classes of equilibria allows us to give finer characterizations for each of them.

**Definition 1** *An equilibrium  $(t_1^*, t_2^*)$  is regular if and only if the strong firm's equilibrium exit time is weakly later than the weak firm's, i.e.,  $t_S^* \geq t_w^*$ . An equilibrium that is not regular is nonregular.*

Intuitively, being more optimistic, the strong firm should exit at a later time than the weak firm. Indeed, Proposition 2 confirms this intuition and shows that such an equilibrium always exists.

**Proposition 2** *There always exists a regular equilibrium.*

While nonregular equilibrium may also exist, Lemma 2 shows that these equilibria exist only in the case of extreme free riding in which the strong firm exits immediately as the game starts and free rides on the weak firm's efforts.

**Lemma 2** *In any nonregular equilibrium, the strong firm must exit at time 0, i.e.,  $t_w^* > t_s^* \Rightarrow t_s^* = 0$ .*

Whenever a nonregular equilibrium exists, the exit times of the weak and strong firms are uniquely pinned down by (8) in Proposition 1 and Lemma 2, respectively. Therefore, nonregular equilibria are completely characterized by them. Next, suppose that a nonregular equilibrium exists so that the strong firm has enough incentives to exit at time 0 and free rides on the weak firm's effort, then, the weak firm, being more pessimistic, must have enough incentives to do the same and free rides on the strong firm's effort. This intuition suggests that whenever a nonregular equilibrium exists, there exists a regular equilibrium in which the weak firm exits at time 0. Our next lemma shows that this is indeed the case. Moreover, the regular equilibrium must be superior in terms of total welfare, as the strong firm is the remaining firm in this equilibrium. ((5) and (6) are both increasing in  $p_{i,t}$ .) Thus, we have

**Lemma 3** *Whenever a nonregular equilibrium exists, there exists a regular equilibrium that dominates it in terms of total welfare.<sup>7</sup>*

As our focus is on the total welfare, from this point on, we will focus on regular equilibria. To start, we define

$$E(\alpha) \equiv \bigcup_{i=1,2} \left\{ (p_1, p_2) \in [0, 1]^2 : p_i \geq p_j, INB(p_j, p_i; \alpha) > 0 \right\}, \quad (19)$$

which is the set of beliefs under which the *weak firm's* instantaneous net benefit of experimentation (13) is strictly positive given the patent strength  $\alpha$ . Proposition 1 implies that

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<sup>7</sup>In fact, Lemma 2, Proposition 7 and Proposition 9 together imply that any nonregular equilibrium is dominated by *any* regular equilibrium.

in a regular equilibrium the weak firm will never exit when the posterior belief  $(p_{1,t}, p_{2,t})$  is inside the set  $E(\alpha)$ . Further define

$$\partial E(\alpha) \equiv \bigcup_{i=1,2} \left\{ (p_1, p_2) \in [0, 1]^2 : p_i \geq p_j, INB(p_j, p_i; \alpha) = 0 \right\}.$$
<sup>8</sup>

We have,

**Lemma 4** *The set  $E(\alpha)$  is convex.*

Our next lemma establishes a lower bound of the patent strength, below which the weak firm will exit at time 0 regardless of the prior  $(p_{1,0}, p_{2,0})$ . Define

$$\underline{\alpha} \equiv \sqrt{\frac{c}{\lambda v}}. \tag{20}$$

Notice that the assumption of nonextreme prior entails that  $\underline{\alpha} < 1$ . We have,

**Lemma 5**  *$E(\alpha)$  is empty if and only if  $\alpha \leq \underline{\alpha}$ .*

If  $E(\alpha)$  is empty, the *weak firm's* instantaneous net benefit of experimentation (13) is nonpositive everywhere on the belief space and thus nonpositive for all time  $t \geq 0$  regardless of the prior  $(p_{1,0}, p_{2,0})$ . Thus, there is no point for the weak firm to experiment at all. We have,

**Proposition 3** *Suppose  $\alpha \leq \underline{\alpha}$ , then in the unique regular equilibrium, the weak firm exits at time 0, and the strong firm exits at  $t_S^*$  given by (8).*

Therefore, Proposition 3 completely characterizes regular equilibria when  $\alpha \leq \underline{\alpha}$ . Before turning into the characterizations of regular equilibria for  $\alpha > \underline{\alpha}$ , we first establish a fundamental result on comparative statics of the exit times (Proposition 4) through the help of Lemma 6.

**Lemma 6** *Suppose  $\alpha'' > \alpha' \geq \underline{\alpha}$ , then, for all  $p_i \geq p_j > 0$ ,  $INB(p_j, p_i; \alpha'') > INB(p_j, p_i; \alpha')$ .*

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<sup>8</sup>Notice that this departs from the more standard use of notations in which  $\partial E$  denotes the boundary of the set  $E$ .



To derive Lemma 6, substitute (6) into (13) and differentiate, we have, for  $p_i < \frac{c}{\lambda\alpha v}$ ,

$$\frac{\partial INB(p_j, p_i; \alpha)}{\partial \alpha} = p_j \lambda v > 0;$$

for  $p_i > \frac{c}{\lambda\alpha v}$ ,

$$\frac{\partial INB(p_j, p_i; \alpha)}{\partial \alpha} = p_j \lambda \left( v + \frac{p_i \lambda \alpha v - c}{\lambda \alpha v - c} v - (1 - p_i) \frac{\lambda v^2 (1 - \alpha) c}{(\lambda \alpha v - c)^2} \right). \quad (21)$$

An increase in the patent strength  $\alpha$  has three effects on the weak firm's  $INB$  when  $p_{S,t} > \frac{c}{\lambda\alpha v}$ . The first term is positive and it represents the direct incentives to the weak firm: since the award is higher, the benefit of experimentation is higher. The second is also positive and it represents the indirect incentives. Under a stricter patent, the benefit from the opponent's breakthrough is lower, so there is less incentive to free ride. The third term is *negative* and it represents the strategic effect. An increase in the patent strength  $\alpha$  increases the strong firm's amount of experiment through (8). This reduces the weak firm's incentives to experiment. In general, the third effect is not dominated by the other two effects. But this is the case when  $\alpha \geq \underline{\alpha}$ .

**Proposition 4** *Suppose that under the patent system  $\alpha$ , firm  $i$  conducts a positive amount of experimentation in a regular equilibrium, i.e.,  $t_i^* > 0$ , then, an increase in the patent strength to  $\alpha' > \alpha$  strictly increases the amount of the experimentation by firm  $i$  in the new regular equilibrium.*

The proof of Proposition 4 is straightforward. By Proposition 1, the strong firm's exit time  $t_S^*$  is given by (8) and is strictly increasing in  $\alpha$  when  $t_S^* > 0$ . By Proposition 3,  $t_w^* > 0$  only if  $\alpha > \underline{\alpha}$ . By Lemma 6, for all time  $t$ , the weak firm's  $INB$  strictly increases with  $\alpha$ . As a result, the weak firm must have incentives to experiment beyond the original equilibrium exit time after an increase in  $\alpha$ .

By Lemma 6, as  $\alpha$  increases,  $INB(p_j, p_i; \alpha)$  increases for all  $p_i \geq p_j$  and  $\alpha \geq \underline{\alpha}$ . Lemma 4 and (19) then imply that  $E(\alpha)$  expands as  $\alpha$  increases. Define  $\bar{\alpha}$  as the patent strength

at which the set  $E(\alpha)$  touches the point  $(1, 1)$ . Substituting (6) into (13) and solving

$$INB(1, 1; \alpha) = \lambda(\alpha v - (1 - \alpha)v) - c = 0,$$

we find that  $\bar{\alpha}$  has the expression

$$\bar{\alpha} \equiv \frac{1}{2} + \frac{c}{2\lambda v}. \quad (22)$$

It is straightforward to check that  $\underline{\alpha} < \bar{\alpha} < 1$ . Moreover,  $\bar{\alpha} > \frac{1}{2}$ , so that  $E(\frac{1}{2})$  does not contain the point  $(1, 1)$ .

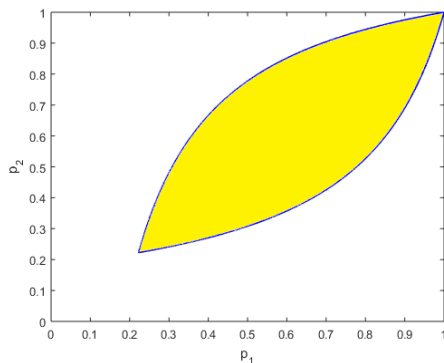


Figure 2: The set  $E(\bar{\alpha})$ . Parameter Values:  $(v, \lambda, c, \bar{\alpha}) = (8, 1, 1, 0.5625)$ .

It turns out that the parameter  $\bar{\alpha}$  is very useful for characterizing regular equilibria. To see that, consider the weak firm's  $MNB$  at some  $t \leq t_S^*$  in a regular equilibrium. Plugging (1) and (6) into (12) and solving the integral in  $\exp\left(-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds\right)$ , we obtain the expression

$$\begin{aligned} & \frac{\partial V_w(t, t_S^*)}{\partial t} \\ = & p_{S,0} p_{w,0} (\lambda v (2\alpha - 1) - c) \left(e^{-\lambda t}\right)^2 \\ & + \left( p_{w,0} (1 - p_{S,0}) \left( \frac{\lambda (\lambda \alpha^2 v + (1 - 2\alpha) c) v}{\lambda \alpha v - c} - c \right) - c p_{S,0} (1 - p_{w,0}) \right) e^{-\lambda t} \\ & - c (1 - p_{S,0}) (1 - p_{w,0}) \end{aligned} \quad (23)$$

for  $t \leq t_S^*$ . Observe that the  $MNB$  is a quadratic function of  $e^{-\lambda t}$ . Moreover, when  $\alpha < \bar{\alpha}$ , the weak firm's  $MNB$  is strictly concave in  $e^{-\lambda t}$ ; when  $\alpha \geq \bar{\alpha}$ , it is convex in  $e^{-\lambda t}$ . This

distinction leads to different equilibrium characterizations (Propositions 5 and 6). To state the result, define the belief path

$$\begin{aligned} & \mu(p_{1,0}, p_{2,0}) \\ \equiv & \left\{ (p_1, p_2) : p_S = \frac{p_{S,0}e^{-\lambda t_S^*}}{p_{S,0}e^{-\lambda t_S^*} + 1 - p_{S,0}}, \exists t \in (t_S^*, \infty), p_w = \frac{p_{w,0}e^{-\lambda t}}{p_{w,0}e^{-\lambda t} + 1 - p_{w,0}} \right\} \\ \cup & \left\{ (p_1, p_2) : \exists t \leq t_S^*, \forall i \in \{1, 2\}, p_i = \frac{p_{i,0}e^{-\lambda t}}{p_{i,0}e^{-\lambda t} + 1 - p_{i,0}} \right\}, \end{aligned} \quad (24)$$

which is the belief path taken by a strong firm that exits at time  $t_S^*$  and a weak firm that experiments forever. The belief path  $\mu(p_{1,0}, p_{2,0})$  is useful for the characterization of the weak firm's strategy because choosing a time  $t$  to exit is equivalent to choosing a point on the set  $\mu(p_{1,0}, p_{2,0})$  to exit. This alternative representation allows us to analyze the weak firm's problem graphically. We have,

**Proposition 5** *Suppose  $\alpha < \bar{\alpha}$ , given a nonextreme prior  $(p_{1,0}, p_{2,0})$ , there are three possible scenarios in a regular equilibrium.*

1.  $\mu(p_{1,0}, p_{2,0}) \cap E(\alpha) = \phi$ : *The weak firm exits at time 0.*
2.  $(p_{1,0}, p_{2,0}) \in E(\alpha)$ : *The weak firm exits when the posterior reaches the boundary of the set  $E(\alpha)$ .*
3.  $(p_{1,0}, p_{2,0}) \notin E(\alpha)$  and  $\mu(p_{1,0}, p_{2,0}) \cap E(\alpha) \neq \phi$ : *There are two possible cases.*
  - (a) *The weak firm exits at time 0.*
  - (b) *The weak firm exits when the posterior reaches the boundary of the set  $E(\alpha)$  for the second time. In this case, the posterior enters  $E(\alpha)$  prior to the exit and leaves  $E(\alpha)$  at the time of the exit.*

The scenarios in Proposition 5 are illustrated in Figures 3-6. The set  $E(\alpha)$  is represented by the yellow (shaded) areas. The single firm threshold belief for firm 1 (2) is indicated by the blue vertical (horizontal) lines. The dotted lines in the left panels represent the set  $\mu(p_{1,0}, p_{2,0})$ . Those in the right panels represent the equilibrium belief paths. In all of the examples, firm 1 is the strong firm.

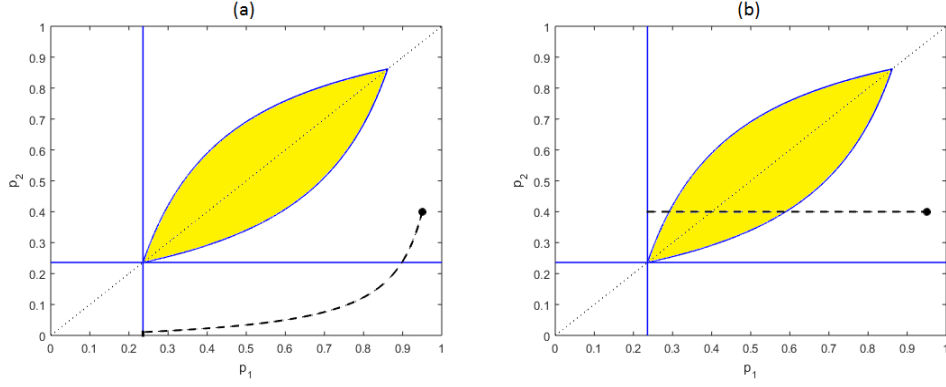


Figure 3: Regular equilibrium when  $\alpha < \bar{\alpha}$  (scenario 1).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.53, 0.95, 0.4)$ .

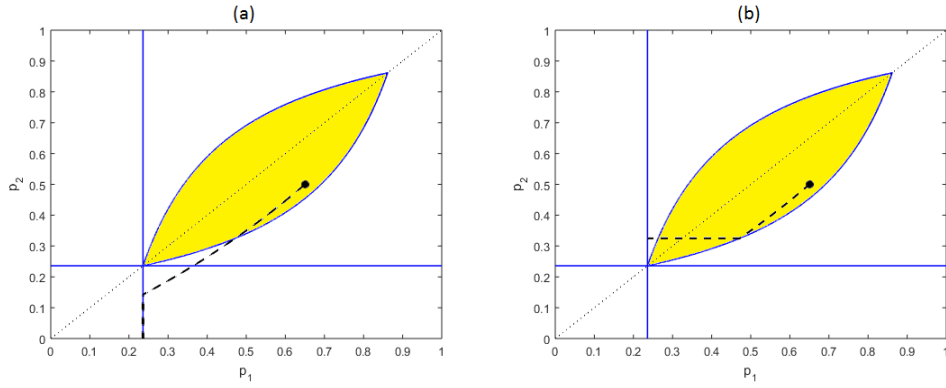


Figure 4: Regular equilibrium when  $\alpha < \bar{\alpha}$  (scenario 2).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.53, 0.65, 0.5)$ .

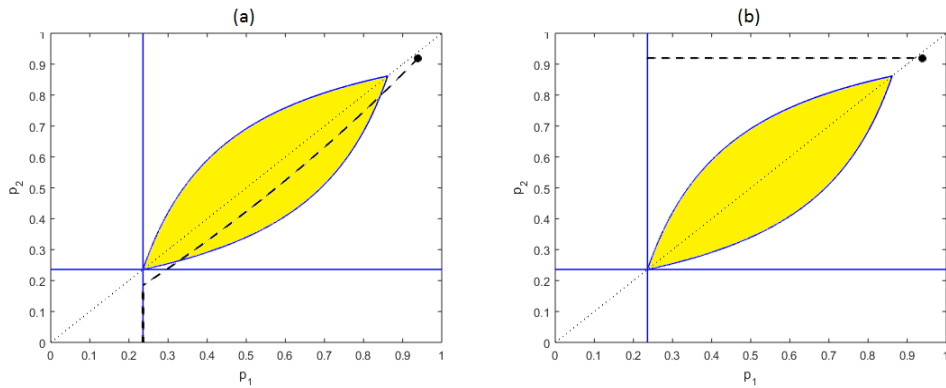


Figure 5: Regular equilibrium when  $\alpha < \bar{\alpha}$  (scenario 3(a)).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.53, 0.94, 0.92)$ .

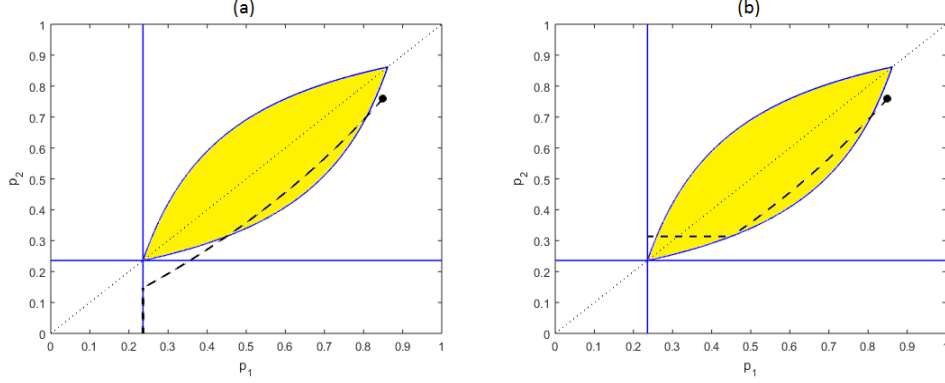


Figure 6: Regular equilibrium when  $\alpha < \bar{\alpha}$  (scenario 3(b)).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.53, 0.85, 0.76)$ .

As a quadratic function, the  $MNB$  has at most two roots. This means that  $\mu(p_{1,0}, p_{2,0})$  can intersect with  $\partial E(\alpha)$  at most twice. This leads to the three different scenarios listed in Proposition 5, which we will discuss in turn.

**Scenario 1 (Figure 3):** If the belief path  $\mu(p_{1,0}, p_{2,0})$  never reaches the set  $E(\alpha)$ , the weak firm's  $MNB$  is nonpositive for all  $t \geq 0$ . It is thus optimal for the weak firm to exit at time 0 and the posterior belief for the weak firm's project stays at the prior probability thereafter. After the weak firm exits, the strong firm continues to experiment until the posterior belief for her project reaches the single firm threshold belief. This leads to the horizontal belief path in Figure 3(b).

**Scenario 2 (Figure 4):** If the prior  $(p_{1,0}, p_{2,0})$  is inside the set  $E(\alpha)$ ,  $\mu(p_{1,0}, p_{2,0})$  only intersects with  $\partial E(\alpha)$  at one point. This is because the posterior  $(p_{1,t}, p_{2,t})$  must eventually leave the set  $E(\alpha)$ , leaving  $E(\alpha)$  and entering it again implies the quadratic function (23) must have three or more roots, which is impossible. Therefore, as illustrated by Figure 4(a), once the posterior leaves the set  $E(\alpha)$ , it never gets back to it if the weak firm does not exit. This means that the weak firm's  $MNB$  is positive until the posterior reaches the boundary of the set  $E(\alpha)$  and is negative thereafter. Thus, exiting at the boundary is optimal.

**Scenario 3 (Figures 5 and 6):** If the firms start with a prior  $(p_{1,0}, p_{2,0})$  outside the set  $E(\alpha)$  but the belief path reaches the set  $E(\alpha)$  at some point, then (23) implies that

$\mu(p_{1,0}, p_{2,0})$  intersects with  $\partial E(\alpha)$  twice. In this case, the weak firm's expected payoff has two local maxima. If the weak firm experiments until the posterior reaches  $\partial E(\alpha)$  for the second time, the weak firm suffers losses when the posterior is outside  $E(\alpha)$  but eventually gain when the posterior gets inside  $E(\alpha)$ . On the other hand, if the weak firm exits at time 0, such losses are avoided. Depending on how far the prior  $(p_{1,0}, p_{2,0})$  is from the set  $E(\alpha)$ , either or both of these strategies can be optimal. Intuitively, when  $\alpha$  is small, the weak firm has incentives to exit at time 0 to free ride on the strong firm's effort. However, if the weak firm's project is good enough, she also has incentives to insure herself by keeping her own project; even though this is suboptimal myopically, it will pay off later if the strong firm fails to produce a success. In the numerical example illustrated by Figure 5, exiting at time 0 is optimal, while in the numerical example illustrated in Figure 6, experimenting is optimal.

When  $\alpha \geq \bar{\alpha}$ , (23) is convex in  $e^{-\lambda t}$ . This means that (23) has at most one root on  $[0, t_S^*]$ . To see that, note that the *MNB* must be negative at  $t_S^*$ , if there are two roots on  $[0, t_S^*]$ , the slope of the *MNB* function (23) must be positive at the first root and negative at the second root. This contradicts the convexity of (23). Moreover, when (23) has a root  $t'$  on  $[0, t_S^*]$ , it must be a maximum since the *MNB* is positive for any  $t < t'$  and negative for any  $t > t'$ . When (23) has no root on  $[0, t_S^*]$ , *MNB* is negative for all  $t \geq 0$ , exiting at time 0 is optimal. Thus, we have,

**Proposition 6** *Suppose  $\alpha \geq \bar{\alpha}$ , given a nonextreme prior  $(p_{1,0}, p_{2,0})$ , there are two possible scenarios in a regular equilibrium.*

1.  $(p_{1,0}, p_{2,0}) \notin E(\alpha)$ : *In this case, firm 2 exits at time 0.*
2.  $(p_{1,0}, p_{2,0}) \in E(\alpha)$ : *In this case, firm 2 exits when the posterior reaches the boundary of the set  $E(\alpha)$ .*

The scenarios in Proposition 6 are illustrated in Figures 7 and 8. As in Figures 3–6, the set  $E(\alpha)$  is represented by the yellow (shaded) area. The single firm threshold belief for firm 1 (2) is indicated by the blue vertical (horizontal) lines. The dotted lines in the left

panels represents the set  $\mu(p_{1,0}, p_{2,0})$ . Those in the right panels represent the equilibrium belief paths. In all of the examples, firm 1 is the strong firm.

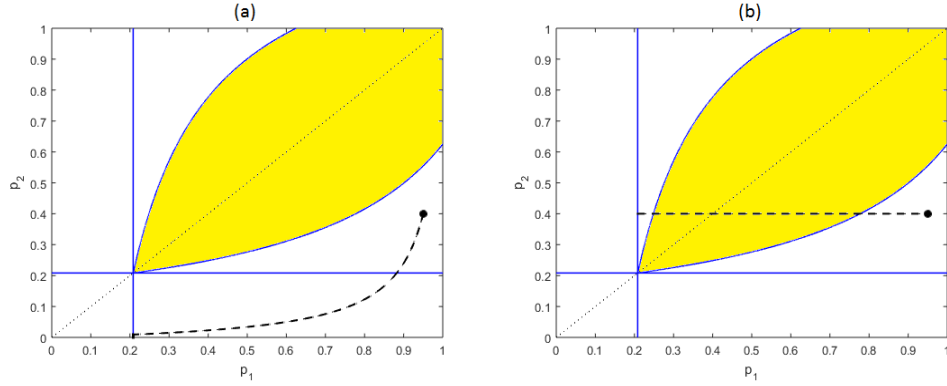


Figure 7: Regular equilibrium when  $\alpha \geq \bar{\alpha}$  (scenario 1).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.6, 0.95, 0.4)$ .

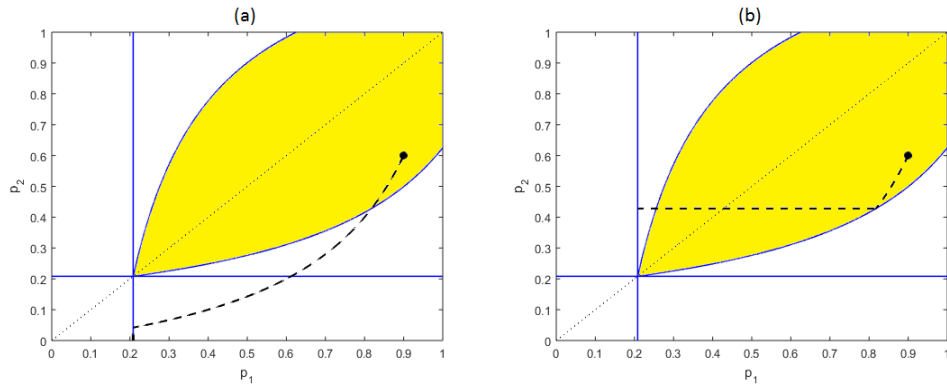


Figure 8: Regular equilibrium when  $\alpha \geq \bar{\alpha}$  (scenario 2).

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.6, 0.9, 0.6)$ .

In contrast to the case of  $\alpha < \bar{\alpha}$ , if the prior  $(p_{1,0}, p_{2,0})$  is outside the set  $E(\alpha)$ , the belief path  $\mu(p_{1,0}, p_{2,0})$  could never intersect with  $E(\alpha)$  (Figure 7(a)), the weak firm's *MNB* is negative for all  $t \geq 0$ , it is thus optimal for the weak firm to exit at time 0 (Figure 7(b)). If the prior  $(p_{1,0}, p_{2,0})$  is inside the set  $E(\alpha)$ , knowing that once the posterior leaves, it will never enter the set  $E(\alpha)$  again (Figure 8(a)), the weak firm will exit at the boundary (Figure 8(b)).

Proposition 7 is a simple consequence of the previous results.

**Proposition 7** *Consider the weak firm's regular equilibrium exit time correspondence  $T_w^* : [\frac{1}{2}, 1] \Rightarrow \mathbb{R}_+$ .  $T_w^*$  has more than one element in at most one  $\alpha' \in (\underline{\alpha}, \bar{\alpha})$ . Moreover, if such an  $\alpha'$  exists, for all  $\alpha \in [\frac{1}{2}, \alpha']$ ,  $0 \in T_w^*(\alpha)$ .*

Suppose that two regular equilibria coexist at some  $\alpha' \in [\frac{1}{2}, 1]$ . By Propositions 3, 5 and 6,  $\alpha' \in (\underline{\alpha}, \bar{\alpha})$  and one of the equilibria must have the weak firm exiting at time 0. Propositions 2 and 4 then imply that for all  $\alpha > \alpha'$ , the weak firm's equilibrium exit time must be at least as high as the higher equilibrium exit time at  $\alpha'$ . Similarly, for all  $\alpha < \alpha'$ , there is a unique regular equilibria in which the weak firm exits at time 0. Otherwise, an increase of the patent strength to  $\alpha + \Delta < \alpha'$  can decrease the weak firm's equilibrium exit time to 0, contradicting Proposition 4. Thus, the correspondence  $T_w^*$  has more than one element in at most one point.

## 4 Welfare Analysis

In this section, we analyze the welfare implications of our model. To provide a benchmark of comparison, we first define and solve the planner's problem. Then, we define the concepts of under-experimentation and over-experimentation relative to the efficiency benchmark. In the end of this section, we apply these results to consider some comparative statics of the total welfare in our model.

### 4.1 Planner's Problem

The objective of the planner is to maximize the sum of the expected payoffs of the two firms. As such, how the prize  $v$  is allocated among the firms is irrelevant. Therefore, we define,

**Definition 2 (The planner's problem)** *A pair of exit times  $(t_1^{**}, t_2^{**})$  for the two firms solves the planner's problem if and only if it maximizes the sum of the expected payoffs of the two firms.*



Given a pair of exit times  $(t_1, t_2)$  with  $t_i \leq t_j$ , the planner's expected payoff is given by

$$\begin{aligned} & V_P(t_1, t_2) \\ &= \int_0^{t_i} ((p_{1,t} + p_{2,t}) \lambda v - 2c) e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} dt \\ & \quad + e^{-\int_0^{t_i} (p_{1,s} + p_{2,s}) \lambda ds} \left( \int_{t_i}^{t_j} (p_{j,t} \lambda v - c) e^{-\int_{t_i}^t p_{j,s} \lambda ds} dt \right). \end{aligned} \quad (25)$$

The expression for the case when  $t_i > t_j$  can be obtained easily by interchanging the roles of  $i$  and  $j$ . Differentiating (25) and the corresponding expression for  $t_i > t_j$ , we obtain

$$\frac{\partial V_P(t_1, t_2)}{\partial t_i} = e^{-\int_0^{t_i} p_{i,s} \lambda ds - \int_0^{t_i \wedge t_j} p_{j,s} \lambda ds} \{ p_{i,t_i} \lambda (v - U_P^i(t_i, t_j)) - c \}, \quad (26)$$

where

$$U_P^i(t, t') \equiv \begin{cases} \int_t^{t'} (p_{j,s} \lambda v - c) e^{-\int_t^s p_{j,z} \lambda dz} ds & \text{if } t \leq t', \\ 0 & \text{if } t > t'. \end{cases} \quad (27)$$

The function  $U_P^i(t, t')$  is the continuation payoff of the planner after the planner retires firm  $i$  at time  $t$  given that firm  $j$  exits at time  $t'$ . It depends on the identity of the exited firm (firm  $i$ ), the current time  $t$  and the exit time  $t'$  for the remaining firm (firm  $j$ ). The continuation payoff is zero if firm  $i$  is last firm to exit (i.e.,  $t > t'$ ). If  $t < t'$ , firm  $j$  continues to experiment until  $t'$ , conditional on the absence of a breakthrough. With probability  $\exp(-\int_t^s p_{j,z} \lambda dz)$  no breakthrough has occurred between times  $t$  and  $s$ , the planner incurs instantaneous cost  $c$  to experiment; with instantaneous probability  $p_{j,s} \lambda$ , a breakthrough occurs and the planner receives  $v$  from a breakthrough. The continuation payoff  $U_P^i(t, t')$  is thus obtained by integrating from  $t$  to  $t'$ .

The derivative  $\frac{\partial V_P(t_1, t_2)}{\partial t_i}$  in (26) represents the marginal net benefit of experimentation of firm  $i$  at time  $t_i$  to the planner. From the viewpoint of the planner, the instantaneous benefit from firm  $i$ 's experimentation can be calculated as follows. With instantaneous probability  $p_{i,t_i} \lambda$ , a breakthrough occurs and the value is  $v$ . If firm  $i$  exits, the planner's continuation value becomes  $U_P^i(t_i, t_j)$ . The term  $p_{i,t_i} \lambda (v - U_P^i(t_i, t_j))$  is thus the instantaneous benefit. The instantaneous *net* benefit of experimentation,  $p_{i,t_i} \lambda (v - U_P^i(t_i, t_j)) - c$ ,

is then weighted by the probability that no breakthrough has occurred until time  $t_i$ ,  $\exp\left\{-\int_0^{t_i} p_{i,s}\lambda ds - \int_0^{t_i \wedge t_j} p_{j,s}\lambda ds\right\}$ . Since firm  $j$  stops experimentation at time  $t_j$ , the probability that firm  $j$  has not achieved a breakthrough at time  $t_i$  depends on whether  $t_i > t_j$ .

Our next lemma establishes that the planner's marginal net benefit of experimentation for a particular firm (26) satisfies the single-crossing property. As a result, the socially efficient exit time  $t_i^\dagger$  for firm  $i$  given firm  $j$ 's exit time  $t_j'$  is unique and pinned down by the first order condition. Moreover, it also establishes a range for exit time  $t_j'$ , so that firm  $i$ 's R&D effort is a substitute of firm  $j$ 's. In this range, as firm  $j$ 's amount of experimentation decreases, the *socially efficient* amount of experimentation for firm  $i$  increases. Notice that any equilibrium exit time  $t_j^*$  must be inside this range.

**Lemma 7** *Given an exit time  $t_j' \geq 0$  for firm  $j$ , there exists an exit time  $t_i^\dagger \geq 0$  such that for all  $t \in [0, t_i^\dagger)$ ,  $\frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t, t_j=t_j'} > 0$  and for all  $t \in (t_i^\dagger, \infty)$ ,  $\frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t, t_j=t_j'} < 0$ . Moreover, the exit time  $t_i^\dagger$  is decreasing in  $t_j'$  for  $t_j' \in [0, \bar{t}_j^\dagger]$ , where  $\bar{t}_j^\dagger \equiv -\frac{1}{\lambda} \log\left(\frac{1-p_{j,0}}{p_{j,0}} \frac{c}{\lambda v - c}\right)$ .*

Lemma 7 implies that for each  $i \in \{1, 2\}$ , the function  $t_i^\dagger : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is well defined. A solution to the planner problem  $(t_1^{**}, t_2^{**})$  must satisfy

$$t_i^\dagger(t_j^{**}) = t_i^{**},$$

for  $i \in \{1, 2\}$ . Next, it is easy to see that the weak firm must exit weakly before the strong firm under the planner's optimal policy.

**Lemma 8** *It is never optimal for the planner to retire the weak firm strictly later than the strong firm.*

With Lemmas 7 and 8 at hand, we can characterize the planner's solution easily. By Lemma 8, the strong firm exits later in the planner's solution, thus,  $U_P^S(t_S^{**}, t_w^{**}) = 0$ , the first order condition implies that  $t_S^{**}$  must solve  $p_{S, t_S^{**}} \lambda v - c = 0$ , which, by Lemma 7, is unique. Unlike the noncooperative problem, the corner solution is ruled out by the

assumption of nonextreme prior, which asserts that  $p_{S,0} > \frac{c}{\lambda v}$ . Then, Lemma 1 and (27) imply that

$$U_P^w(t, t_S^{**}) = \hat{W}(p_{S,t}; 1).$$

In other words, after the weak firm has exited, the continuation value to the planner at time  $t$  is the same as the continuation value of the strong firm at time  $t$  in the single firm problem when  $\alpha = 1$ . Intuitively, after the weak firm has exited, the planner's optimal action and value must be identical to the strong firm's when the payoff externality is internalized. Define the instantaneous net benefit of experimentation *of the weak firm to the planner*  $INB_P^w$  by

$$INB_P^w(p, p') \equiv p\lambda \left( v - \hat{W}(p'; 1) \right) - c, \quad (28)$$

where  $p$  is the probability for the weak firm's project and  $p'$  is the probability for the strong firm's project. The weak firm's first order condition implies that

$$INB_P^w(p_{w,t_w^{**}}, p_{S,t_w^{**}}) \leq 0.$$

By Lemma 7, the first order condition is also sufficient. Thus, we have,

**Proposition 8** *The solution to the planner's problem  $(t_1^{**}, t_2^{**})$  is given by:*

1.  $t_S^{**}$  is the unique solution to the equation  $p_{S,t_S^{**}}\lambda v = c$  and has the expression

$$t_S^{**} = \bar{t}_S^\dagger \equiv -\frac{1}{\lambda} \log \left( \frac{1 - p_{S,0}}{p_{S,0}} \frac{c}{\lambda v - c} \right). \quad (29)$$

2. If  $INB_P^w(p_{w,0}, p_{S,0}) \leq 0$ , then  $t_w^{**} = 0$ . Otherwise,  $t_w^{**}$  is the unique solution to

$$INB_P^w(p_{w,t_w^{**}}, p_{S,t_w^{**}}) = 0. \quad (30)$$

Moreover,  $t_w^{**}$  is increasing in  $p_{w,0}$  and decreasing in  $p_{S,0}$ .

As in the case of the noncooperative problem, we can illustrate the characterization of the planner's solution and the resulting dynamics visually by 2-dimensional figures in the belief space. Let  $E^{**}$  be the set of beliefs at which the weak firm's  $INB$  to the planner is strictly positive. That is

$$E^{**} \equiv \bigcup_{i=1,2} \left\{ (p_1, p_2) \in [0, 1]^2 : p_i \geq p_j, INB_P^w(p_j, p_i) > 0 \right\}, \quad (31)$$

Let  $\partial E^{**}$  denote the boundary of  $E^{**}$ . Proposition 8 implies the set  $E^{**}$  is also the set of beliefs at which the planner finds it optimal to keep both firms experimenting. In Figure 9, the set  $E^{**}$  is represented by the green (shaded) areas. If the prior  $(p_{1,0}, p_{2,0})$  is inside the set  $E^{**}$ , the weak firm exits once the posterior reaches the boundary of  $E^{**}$ , while the strong firm continues to experiment until the posterior belief for her project reaches the single firm threshold belief  $\frac{c}{\lambda v}$ , as illustrated by Figure 9(a). If the firms start with a prior  $(p_{1,0}, p_{2,0})$  outside the set  $E^{**}$ , the weak firm exits immediately at time 0 and the strong firm again experiments until her posterior belief reaches the single firm threshold belief, as illustrated by Figure 9(b).

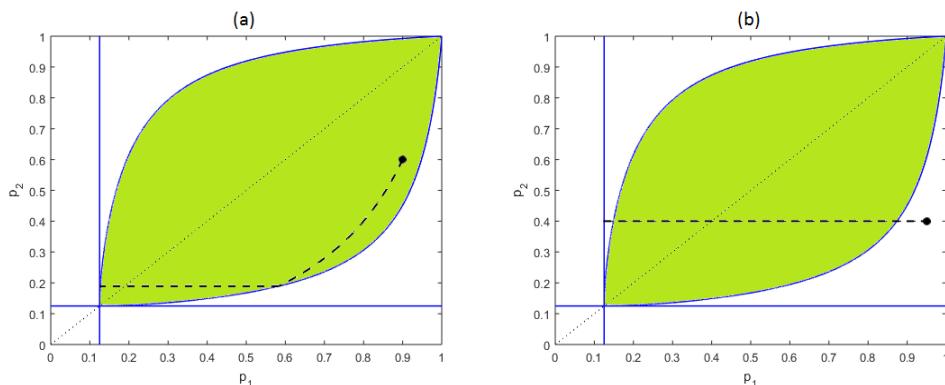


Figure 9: The planner's solution. Parameter Values:  $(v, \lambda, c) = (8, 1, 1)$

(a)  $(p_{1,0}, p_{2,0}) = (0.9, 0.6)$ . (b)  $(p_{1,0}, p_{2,0}) = (0.95, 0.4)$ .

The planner's solution provides a useful benchmark for welfare comparisons. Comparing the first parts of Propositions 1 and 8, we find that the noncooperative exit time  $t_S^*$  is always less than the socially optimal exit time  $t_S^{**}$  and strictly so if  $\alpha < 1$ . This is intuitive. Unless  $\alpha = 1$ , the strong firm does not internalize the weak firm's gain from a breakthrough and

under-invests in R&D. As a result, the strong firm exits too early relative to the planner's solution. Thus, we introduce the following definition of under-experimentation for the strong firm.

**Definition 3** *The strong firm under-experiments in a regular equilibrium  $(t_1^*, t_2^*)$  if and only if  $t_S^* < t_S^{**}$ .*

Since  $t_S^* \leq t_S^{**}$ , the strong firm never over-experiments in our model. This is, however, not true for the weak firm. On the one hand, the social value of a breakthrough from the weak firm  $v$  is larger than its private value  $\alpha v$ . On the other hand, the “outside option”, i.e., the value of exiting, at time  $t$  is  $\hat{W}(p_{S,t}; 1)$  to the planner and  $\check{W}(p_{S,t}; \alpha)$  to the weak firm. Since  $\hat{W}(p_{S,t}; 1) \geq \check{W}(p_{S,t}; \alpha)$ , comparing (13) and (28), the weak firm's incentives to experiment in the noncooperative problem may exceed or fall short of the planner's. Therefore, the weak firm's equilibrium exit time may be higher or lower than the planner's exit time. However, having a higher equilibrium exit time does not necessarily mean that the weak firm over-experiments. This is because the socially optimal amount of experimentation by the weak firm  $t_w^\dagger$  also depends on the strong firm's equilibrium exit time  $t_S^*$ . Thus, if the strong firm exits prematurely, Lemma 7 implies that it would be socially optimal for the weak firm to experiment beyond the planner's exit time to compensate the loss. To deal with this issue, we provide a formal definition of under-experimentation by comparing the weak firm's equilibrium exit time  $t_w^*$  and the socially optimal exit time  $t_w^\dagger$  given the strong firm's equilibrium exit time  $t_S^*$ .

**Definition 4** *Given the strong firm's equilibrium exit time  $t_S^* \in [0, t_S^{**}]$ , the weak firm under(over)-experiments in a regular equilibrium  $(t_1^*, t_2^*)$  if and only if  $t_w^* < (>) t_w^\dagger(t_S^*)$ .*

Note that when  $t_S^* = t_S^{**}$ ,  $t_w^\dagger(t_S^*) = t_w^{**}$ . In this case, the weak firm under-experiments if and only if  $t_w^* < t_w^{**}$ . However, if the strong firm experiments less than the amount in the planner's solution, i.e.,  $t_S^* < t_S^{**}$ , by Lemma 7, the benchmark for the weak firm's exit time  $t_w^\dagger$  is in general larger than the weak firm's exit time  $t_w^{**}$  in the planner's solution. We have,

**Lemma 9** *If the weak firm's exit time in a regular equilibrium  $(t_1^*, t_2^*)$  is strictly smaller than the planner's exit time for the weak firm, i.e.,  $t_w^* < t_w^{**}$ , then the weak firm under-experiments in equilibrium, i.e.,  $t_w^* < t_w^\dagger(t_S^*)$ .*

Lemma 9 is a useful shortcut in showing that the weak firm under-experiments in equilibrium. Without finding the socially efficient amount of experimentation  $t_w^\dagger$ , one can simply check the sufficient condition by comparing the weak firm's equilibrium exit time with the planner's. Together with the next lemma, it implies that the weak firm cannot over-experiment when  $\alpha \in [\frac{1}{2}, \bar{\alpha})$  (Lemma 11).

**Lemma 10** *For all  $\alpha \in [\frac{1}{2}, \bar{\alpha})$ ,  $(E(\alpha) \cup \partial E(\alpha)) \setminus \{(1, 1)\} \subseteq E^{**}$ .*

To see that the weak firm cannot over-experiment when  $\alpha \in [\frac{1}{2}, \bar{\alpha})$ , note that, by Proposition 5, the weak firm either exits at time 0 or when the posterior reaches  $\partial E(\alpha) \setminus \{(1, 1)\} \subseteq E^{**}$ . Since  $t_w^\dagger(t) \geq 0$  for all  $t \geq 0$ , if the weak firm exits at time 0, she cannot over-experiment. If the weak firm exits at  $\partial E(\alpha)$ , then  $(p_{1,t_w^*}, p_{2,t_w^*}) \in E^{**}$ . Thus, the weak firm's *INB* to the planner is strictly positive. Lemma 7 implies that  $t_w^* < t_w^\dagger(t_S^{**}) = t_w^{**}$ . By Lemma 9, the weak firm under-experiments. In either case, the weak firm does not over-experiment. Thus, we have,

**Lemma 11** *Suppose  $\alpha \in [\frac{1}{2}, \bar{\alpha})$ , the weak firm cannot over-experiment in a regular equilibrium.*

Lemma 11 allows us to select the welfare superior equilibrium when there are multiple equilibria.

**Proposition 9** *Whenever two regular equilibria coexist, the equilibrium in which the weak firm exits at a later time dominates the equilibrium with early exit in terms of total welfare.*

By Proposition 7, two regular equilibria coexist only when  $\alpha \in (\underline{\alpha}, \bar{\alpha})$ . By Proposition 1, the amount of experimentation by the strong firm is the same in both regular equilibria. By Lemma 11, the weak firm cannot over-experiment in equilibrium. Thus, the equilibrium in which the weak firm exits at a later time must dominate the other equilibrium in terms of total welfare.

## 4.2 Comparative Statics

In this subsection, we consider the comparative statics of increasing the prior probabilities. As illustrated by Examples 1 and 2, it turns out that increasing the prior probabilities can have an adverse effect on the total welfare. In Example 1, an increase in the prior probability about the strong firm's project leads to more severe free riding by the weak firm. In Example 2, an increase in the prior probability about the weak firm's project results in more severe duplication. In both cases, the total welfare decreases.

**Example 1 (Adverse effect of increasing the strong firm's probability)** *For each  $\alpha \in (\underline{\alpha}, \bar{\alpha})$ , we can construct an example in which an infinitesimal increase of the prior probability about the strong firm's project leads to a downward jump in the total welfare. Suppose  $\alpha \in (\underline{\alpha}, \bar{\alpha})$ . By Lemma 5,  $E(\alpha)$  is nonempty. By the definition of  $\bar{\alpha}$ ,  $E(\alpha) \cup \partial E(\alpha)$  does not contain the point  $(1, 1)$ . As illustrated by Figure 10, this implies that there is a segment of the  $45^\circ$  line laying between  $E(\alpha)$  and  $(1, 1)$ . If the prior  $(p_{1,0}, p_{2,0})$  is on the segment of the  $45^\circ$  line that is inside the set  $E(\alpha)$ , both firms will experiment until the beliefs reach  $\frac{c}{\lambda\alpha v}$  on the lower boundary of the set  $E(\alpha)$ . However, as the prior moves outside the set  $E(\alpha)$  and up the  $45^\circ$  line, one of the firms must exit at time 0. This is because the time taken to reach  $E(\alpha)$  becomes arbitrarily large as the prior gets close to  $(1, 1)$ .<sup>9</sup> The continuity of the firms' payoff functions implies that there must exist a point on the  $45^\circ$  line at which firm 2 is indifferent between exiting at time 0 and experimenting until  $p_{2,t} = \frac{c}{\lambda\alpha v}$  given that firm 1 experiments until  $p_{1,t} = \frac{c}{\lambda\alpha v}$ . At such a point, there are two regular equilibria as given by firm 2's two best responses. By Lemma 10, the set  $E(\alpha) \cup \partial E(\alpha)$  is contained in the set  $E^{**}$ . ( $\partial E^{**}$  is illustrated by the blue dotted line in Figures 10(a) and 10(b).) By Proposition 9, the equilibrium with immediate exit is dominated by the experimentation equilibrium in terms of total welfare. Now, if the prior probability for firm 1's project increases for an arbitrarily small amount, firm 2 strictly prefers to exit at time 0 and the welfare superior equilibrium disappears. It is easy to see that the same construction also works by moving up the belief path from any point  $(p_1, p_2) \in E(\alpha)$  and then increasing the prior probability for the strong firm at the point of*

<sup>9</sup>If the prior is  $(1, 1)$ , the posterior never enters  $E(\alpha)$ .

indifference. In Figure 10, we identify such a point of indifference numerically. The two equilibrium belief paths are illustrated in the two figures, respectively.

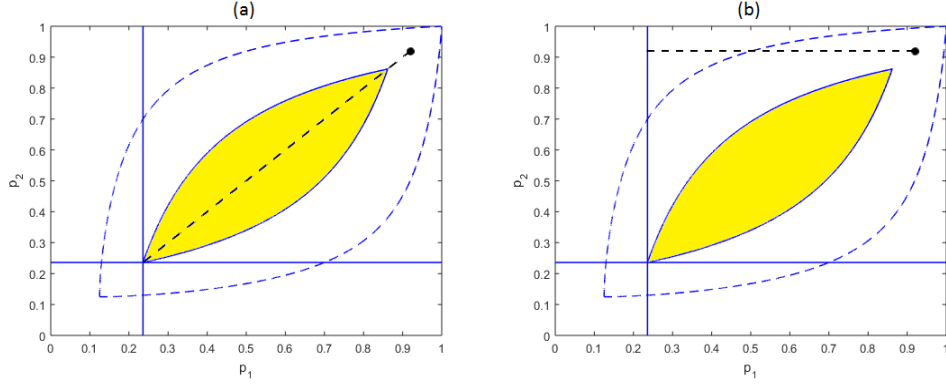


Figure 10: Welfare comparison of two equilibria.

Parameter Values:  $(v, \lambda, c, \alpha, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.53, 0.9202, 0.9202)$ .

**Example 2 (Adverse effect of increasing the weak firm's probability)** *The adverse effect of increasing the prior probability about the weak firm's project is best illustrated by the extreme case when the strong firm's project is known to be good. As a result, we will depart briefly from the assumption of nonextreme prior in this example and assume that  $p_{1,0} = 1$ . The continuity of the total payoff function implies that the same conclusion holds when  $p_{S,0}$  is close but not equal to 1. Notice that, without discounting, there is no point to have firm 2 engaging in R&D at all in this case. This is because, even if  $p_{2,0} = 1$ , using firm 2 only speeds up the arrival of a breakthrough and has no effect on the total welfare. With only firm 1 engaging in R&D, the total welfare is simply  $v - \frac{c}{\lambda}$ , since a breakthrough arrives with probability 1 and its expected arrival time is  $\frac{1}{\lambda}$ . (If both firms' projects are good and both firms experiment, the expected arrival time is  $\frac{1}{2\lambda}$  and the flow cost is  $2c$ , resulting in the same expected cost of  $\frac{c}{\lambda}$ .) Suppose  $\alpha > \bar{\alpha}$  and  $p_{2,0} \leq \frac{c}{\lambda(2\alpha-1)v}$ . By (6),  $\check{W}(1; \alpha) = (1 - \alpha)v$ , Proposition 1 implies that  $t_2^* = 0$  and the equilibrium welfare is given by  $v - \frac{c}{\lambda}$ . On the other hand, if  $\frac{c}{\lambda(2\alpha-1)v} < p_{2,0} < 1$ , Proposition 1 implies that  $t_2^* > 0$ , the equilibrium welfare is given by*

$$v - \frac{c}{\lambda} - (1 - p_{2,0}) \left( 1 - \frac{1 - p_{2,0}}{p_{2,0}} \frac{c}{\lambda(2\alpha - 1)v - c} \right) \frac{c}{\lambda}.$$



This is because, with probability  $1 - p_{2,0}$ , firm 2's R&D process is bad and the expected cost incurred is  $(1 - e^{-\lambda t_2^*}) \frac{c}{\lambda}$ , where  $t_2^* = -\frac{1}{\lambda} \log\left(\frac{1-p_{2,0}}{p_{2,0}} \frac{c}{\lambda(2\alpha-1)v-c}\right)$ . Therefore, as illustrated by Figure 11, the total welfare decreases as  $p_{2,0}$  increases from  $p_{2,0} < \frac{c}{\lambda(2\alpha-1)v}$  (point B) to  $\frac{c}{\lambda(2\alpha-1)v} < p_{2,0} < 1$  (point A). Notice that both points are outside the set  $E^{**}$ , whose boundary is illustrated by the blue dotted line. This indicates that the planner finds it optimal to retire firm 2 at time 0 in both cases. However, in equilibrium, firm 2 exits at point B, but not at point A.

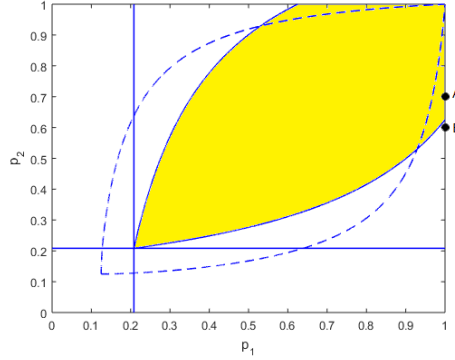


Figure 11: Increasing  $p_{w,0}$  decreases welfare.  $A = (1,0.7)$ ,  $B = (1,0.6)$ .

Parameter Values:  $(v, \lambda, c, \alpha) = (8, 1, 1, 0.6)$ .

## 5 Optimal Patent

In this section, we characterize the optimal patent, which we define below.

**Definition 5 (Optimal Patent)** *A patent system represented by a patent strength  $\alpha^* \in [\frac{1}{2}, 1]$  is optimal if and only if  $\alpha^*$  maximizes the sum of the expected payoffs of the two firms in the welfare maximizing equilibrium.*

In the definition of the optimal patent, the welfare maximizing equilibrium is selected whenever there are multiple equilibria. This choice makes sure that the total welfare function is upper-semicontinuous. By Proposition 2 and Lemma 3, the welfare maximizing equilibrium must be regular. Our first lemma in this section rules out patent systems that are too weak so that none of the firms experiments in equilibrium.

**Lemma 12** *Under the optimal patent  $\alpha^*$ , the strong firm performs a positive amount of experimentation in equilibrium, i.e.,  $t_S^* > 0$ .*

Suppose that, under the optimal patent  $\alpha^*$ , both firms exit at time 0. Consequently, the total welfare is zero. Such a patent system is strictly dominated by the strict patent system. When  $\alpha = 1$ , (6) implies that for all  $t \geq 0$ ,  $\check{W}(p_{S,t}; \alpha) = 0$ . The assumption of nonextreme prior and the characterizations in Proposition 1 then imply that both firms exit when the posterior belief for her project reaches the single firm threshold belief  $\frac{c}{\lambda v}$ . That is, for  $i = 1, 2$ ,  $p_{i,t_i^*} \lambda v = c$ . Under the strict patent system, the total payoff  $V_P(t_1^*, t_2^*)$  is strictly positive, since all the integrands in (25) is positive and strictly so on a set of positive measure.

Lemma 13 provides the basic characterizations of the optimal patent that drive lots of the results in this section.

**Lemma 13** *Under the optimal patent  $\alpha^*$ , the welfare maximizing equilibrium  $(t_1^*, t_2^*)$  satisfies*

1.  $\frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_w=t_w^*, t_S=t_S^*} \leq 0$ .
2.  $INB(p_{w,t_w^*}, p_{S,t_S^*}; \alpha) = 0$ .

The first characterization of the optimal patent  $\alpha^*$  in Lemma 13 states that the *MNB* of the weak firm's experimentation to the planner must be nonpositive at  $t_w^*$ . Intuitively, if this is violated at the optimum and  $\alpha^* \in [\frac{1}{2}, 1)$ , one can strengthen the patent system for a small amount and increase the total welfare. By Proposition 4 and Lemma 12, this will weakly increase the exit time of the weak firm and strictly increase the exit time of the strong firm. Since the strong firm never over-experiments in equilibrium, the total welfare must increase when the increase in patent strength is small enough.<sup>10</sup> If  $\alpha^* = 1$ , one can check that the characterization is satisfied.

<sup>10</sup>Although Proposition 7 implies that the weak firm's exit time in the welfare maximizing equilibrium is in general discontinuous in  $\alpha$ , it is discontinuous at only one point and Proposition 9 implies that it is upper-semicontinuous. Thus, for any  $\alpha \in [\frac{1}{2}, 1]$  and  $\varepsilon > 0$ , we can find  $\Delta > 0$  small enough so that  $t_w^*(\alpha) \leq t_w^*(\alpha + \Delta) < t_w^*(\alpha) + \varepsilon$ .

The second characterization of the optimal patent  $\alpha^*$  in Lemma 13 states that the weak firm's  $INB$  must be zero at her time of exit. This is trivially true when  $t_w^* > 0$ . Suppose the weak firm's exit time in the welfare maximizing equilibrium is 0 under the optimal patent  $\alpha^*$  and  $INB(p_{w,0}, p_{S,0}; \alpha^*) < 0$ . In this case, the regular equilibrium must be unique. Otherwise, by Proposition 9, the weak firm's exit time in the welfare maximizing equilibrium would not be 0. The continuity of the equilibrium exit time  $t_w^*$  at  $\alpha^*$  then implies one can find  $\Delta > 0$  small enough so that the weak firm's equilibrium exit time remains at 0 under the patent  $\alpha^* + \Delta$ . By Proposition 4 and Lemma 12, this strictly increases the exit time of the strong firm. Since the strong firm always under-experiments in equilibrium, the total welfare must increase as a result.

With Lemma 13 at hand, we can derive Propositions 10 and 11 easily.

**Proposition 10** *Under the optimal patent  $\alpha^*$ , the weak firm never under-experiments.*

To prove Proposition 10, suppose the weak firm under-experiments under the optimal patent  $\alpha^*$ . By Lemma 7,  $t_w^* < t_w^\dagger(t_S^*)$  implies that  $\frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_w=t_w^*, t_S=t_S^*} > 0$ , which contradicts Lemma 13.

**Proposition 11** *Any patent  $\alpha \in [\frac{1}{2}, \bar{\alpha})$  is suboptimal. In particular, it is never optimal to impose no patent protection. i.e.,  $\alpha^* \neq \frac{1}{2}$ .*

To prove Proposition 11, suppose by way of contradiction that  $\alpha^* \in [\frac{1}{2}, \bar{\alpha})$ . By the assumption of nonextreme prior,  $(p_{w, t_w^*}, p_{S, t_w^*}) \neq (1, 1)$ . By Lemma 13, the weak firm exits when  $INB(p_{w, t_w^*}, p_{S, t_w^*}; \alpha^*) = 0$ . Thus,  $(p_{1, t_w^*}, p_{2, t_w^*}) \in \partial E(\alpha^*)$ . By Lemma 10,  $\partial E(\alpha^*) \setminus \{(1, 1)\} \subseteq E^{**}$ , which, by Lemma 7, implies that  $t_w^* < t_w^{**}$ . Lemma 9 then implies that the weak firm under-experiments in equilibrium, which contradicts Proposition 10. Notice that the fact that  $\alpha^* \neq \frac{1}{2}$  does not follow from Lemma 12, as the strong firm may still perform a positive amount of experimentation in the absence of patent protection.

Proposition 11 shows that some patent protection is always desirable. Strict patent, however, is also suboptimal in general.

**Proposition 12** *Given a nonextreme prior  $(p_{1,0}, p_{2,0})$ , the strict patent system, i.e.,  $\alpha = 1$ , is optimal if and only if the firms are symmetric, i.e.,  $p_{1,0} = p_{2,0}$ .*

To see why the strict patent system is in general suboptimal, note that when  $\alpha = 1$ , the strong firm's equilibrium exit time is socially optimal, i.e.,  $t_S^* = t_S^\dagger(t_w^*)$ . Therefore, the strong firm's *MNB* at time  $t_S^*$  to the planner is 0, i.e.,  $\frac{\partial V_P(t_1, t_2)}{\partial t_S} \Big|_{t_w=t_w^*, t_S=t_S^*} = 0$ . Thus, a small departure of the strong firm's exit time from the optimum has a vanishing effect on the total welfare. While lowering the patent strength distorts the strong firm's exit time, it also discourages the weak firm from over-experimenting, which does occur in equilibrium when  $p_{1,0} \neq p_{2,0}$  and  $\alpha = 1$ . If the decrease in  $\alpha$  is small enough, the gain from the discouragement effect outweighs the distortion effect. Thus,  $\alpha = 1$  is not optimal.

An immediate consequence of Proposition 12 is that no patent system can implement the planner's solution unless  $p_{1,0} = p_{2,0}$ . This is because, by Propositions 1 and 8, strict patent protection is necessary to induce the strong firm to exist at the socially optimal time. But, by Proposition 12, this is suboptimal if  $p_{1,0} \neq p_{2,0}$ . Conversely, if  $p_{1,0} = p_{2,0}$ ,  $\alpha^* = 1$ , both firms exit when  $p_{1,t} = p_{2,t} = \frac{c}{\lambda v}$ , which also coincides with the planner's exit time in Proposition 8. Thus, we have,

**Proposition 13** *The optimal patent  $\alpha^*$  implements the planner's solution if and only if the firms are symmetric, i.e.,  $p_{1,0} = p_{2,0}$ .*

The optimal patent in general does not vary monotonically with the prior probabilities. By Proposition 12, for any  $p_{w,0} \in (\frac{c}{\lambda v}, p_{S,0})$ ,  $\alpha^* < 1$  and when  $p_{w,0} = p_{S,0}$ ,  $\alpha^* = 1$ . Moreover, suppose that  $p_{w,0} \leq \frac{c}{\lambda v}$ , so that the assumption of nonextreme prior is violated, then it must be optimal to set  $\alpha^* = 1$  as the weak firms will exit at time 0 regardless of the patent policy. Thus, as  $p_{w,0} \rightarrow \frac{c}{\lambda v}$ ,  $\alpha^* \rightarrow 1$ . This is illustrated in Figure 12(a). In this numerical example, the optimal patent  $\alpha^*$  decreases as  $p_{w,0}$  decreases from  $p_{S,0} = 0.95$  and reaches its minimum when  $p_{w,0}$  is about 0.6. Around that point,  $t_w^*$  reaches 0 and the optimal patent  $\alpha^*$  increases as  $p_{w,0}$  decreases further because a decrease in the weak firm's ability at this point means that one can set a stricter patent to encourage the strong firm's effort while still keeping the weak firm from experimenting.

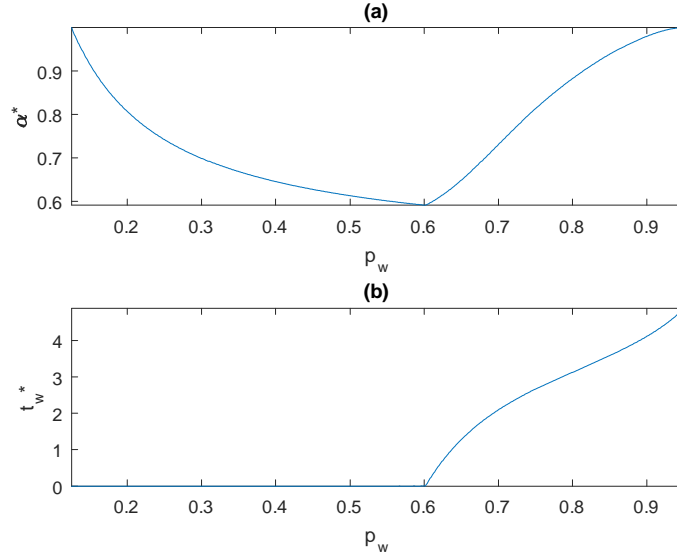


Figure 12: Optimal patent and the weak firm's exit time.  $(v, \lambda, c, p_{S,0}) = (8, 1, 1, 0.95)$ .

Next, we move to consider some comparative statics of the total welfare. Example 1 in Section 4.2 suggests that an increase in the prior probability about the strong firm's project may aggravate the free riding behavior of the weak firm and have an adverse effect on the total welfare. This, however, cannot occur when the optimal patent is used. This is because the weak firm never under-experiments under the optimal patent. Thus, encouraging the weak firm to free ride and reduce experimentation is actually beneficial at the optimum.

**Proposition 14** *Under the optimal patent, an increase in the prior probability about the strong firm's project always increases total welfare.*

To understand Proposition 14, consider  $p_{S,0}$  as a parameter and the optimal patent  $\alpha^*$  a function of  $p_{S,0}$ , denote the equilibrium exit times by  $t_1^*(\alpha^*, p_{S,0})$  and  $t_2^*(\alpha^*, p_{S,0})$  and the total payoff by  $V_P(t_1^*, t_2^*; p_{S,0})$ , we have

$$\begin{aligned} & \frac{dV_P(t_1^*(\alpha^*, p_{S,0}), t_2^*(\alpha^*, p_{S,0}); p_{S,0})}{dp_{S,0}} \\ = & \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial p_{S,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_S^*} \frac{\partial t_S^*}{\partial p_{S,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_w^*} \frac{\partial t_w^*}{\partial p_{S,0}}. \end{aligned}$$

The first term  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial p_{S,0}}$  represents the change in the total payoff due to an increase in  $p_{S,0}$  holding the firms' exit times fixed. This term is obviously positive, as the probability of achieving a breakthrough by the strong firm at each time  $t$  increases. The second term represents the change in the total payoff through the change in the strong firm's exit time. Since the strong firm never over-experiments in equilibrium,  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_S^*} \geq 0$ . Moreover, by (8),  $\frac{\partial t_S^*}{\partial p_{S,0}} > 0$ . The second term is also positive. The third term represents the change in the total payoff through the change in the weak firm's exit time. By the first part of Lemma 13,  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_w^*} \leq 0$ . By the second part of Lemma 13, (9) holds at equality. Direct differentiation yields  $\frac{\partial t_w^*}{\partial p_{S,0}} < 0$ . The third term is also positive. The desired conclusion then follows from an application of the Envelope Theorem (e.g., Corollary 4 of Milgrom and Segal (2002)).

The optimal patent, however, is not immune from the adverse effect of increasing the prior probability about the *weak* firm's project. This is illustrated by Example 3. Intuitively, when the weak firm becomes more confident, it becomes more difficult to discourage her from experimenting. But this is exactly what is called for when the strong firm's project is "too good" relative to the weak firm's.

**Example 3 (Increasing the weak firm's probability under optimal patent)** *In this example, we show that an increase in the prior probability about the weak firm's project can have a negative effect on the total welfare even under the optimal patent. (In the Appendix, we demonstrate this possibility analytically by taking derivative of the total payoff function with respect to  $p_{w,0}$ .) In the situation illustrated by Figure 13(a), the strong firm's prior probability is much larger than the weak firm's. Under the optimal patent, the weak firm exits at time 0. By Lemma 13,  $(p_{1,0}, p_{2,0}) \in \partial E(\alpha^*)$ . In Figure 13(b), the weak firm's probability increases, yet the weak firm is still too inefficient to experiment under the optimal patent. In this case, the optimal patent is weakened by  $\Delta > 0$  so that  $(p_{1,0}, p_{2,0}) \in \partial E(\alpha^* - \Delta)$ . As a consequence of this, the single firm belief threshold increases from  $\frac{c}{\lambda\alpha^*v}$  to  $\frac{c}{\lambda(\alpha^* - \Delta)v}$  and moves away from the single firm efficient experimentation frontier  $\frac{c}{\lambda v}$ , which is illustrated by vertical blue dotted line. Thus, the strong firm's equilibrium exit time further moves away from the social efficient level after the increase in*

$p_{w,0}$ . Since the weak firm never experiments in both cases, the total welfare decreases.

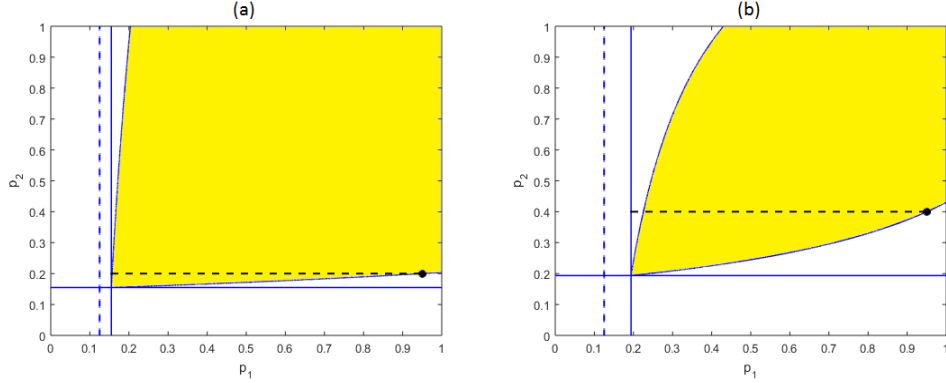


Figure 13: Increasing  $p_{w,0}$  decreases welfare under optimal patent.

$$(a) (v, \lambda, c, \alpha^*, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.8068, 0.95, 0.2),$$

$$(b) (v, \lambda, c, \alpha^*, p_{1,0}, p_{2,0}) = (8, 1, 1, 0.6453, 0.95, 0.4).$$

## 6 Optimal Distribution Policy

We have shown in the previous section that unless the prior probabilities are equal, no patent system can achieve efficiency. This calls for a search for more general policies that may improve social welfare beyond the optimal patent. Obviously, the more general the class of policies we consider, the better the outcomes that we can implement. In this section, we show that a simple class of policies that we call distribution policy is sufficient to implement the planner's solution.

**Definition 6** A distribution policy is a pair  $(\alpha_1, \alpha_2) \in [0, 1]^2$  so that if firm  $i$  is the first to achieve a breakthrough, the rewards to firm  $i$  and firm  $j$  are given by  $\alpha_i v$  and  $(1 - \alpha_i) v$ , respectively.

Thus, a distribution policy resembles the patent system we consider previously in that it specifies only how the total surplus  $v$  is distributed among the firms. However, the sharing scheme can depend on the identity of the firm that made the discovery. In Holmström's (1982) terminology, distribution policies satisfy budget balance. At first glance, it is not clear how much can be achieved by distribution policies since any attempt to correct one firm's exit time with such a policy will inevitably change the other firm's incentive to

experiment as well. Nevertheless, we identify in this section a distribution policy that implements the planner's solution.

It should be emphasized that while differential policies could be efficient, we realize the difficulties of implementing such a policy in practice. As many works in the literature of theoretical mechanism design, the optimal policy that we propose depends on the details of the model. In particular, the optimal policy is sensitive to the prior probabilities, which may be difficult to obtain by the government. Differential policy may also be prohibited by laws or simply impossible to be carried out politically. Nevertheless, we suggest that the study of such policies could be interesting from a theoretic perspective as it demonstrates what is lacking in a patent system.

**Proposition 15** *The distribution policy described by*

$$\alpha_S^{**} = 1 \text{ and} \tag{32}$$

$$\alpha_w^{**} = 1 - \frac{\hat{W}(p_{S,t_w^{**}}; 1)}{v}, \tag{33}$$

*implements the planner's solution. Moreover,  $\alpha_w^{**}$  is increasing in  $p_{w,0}$  and decreasing in  $p_{S,0}$ .*

Notice that if  $p_{1,0} = p_{2,0}$ , by Proposition 8, the two firms exit at the same time  $t_w^{**} = t_S^{**}$  in the planner's solution. (5) and (29) imply  $\hat{W}(p_{S,t_w^{**}}; 1) = 0$  and thus  $\alpha_w^{**} = 1$ . The optimal distribution policy in this case is simply the strict patent system, as we found in Propositions 12 and 13. Unless  $p_{1,0} = p_{2,0}$ , the optimal distribution policy is not anonymous. Moreover, intuitively, as the difference between the abilities of the two firms increases, less is granted to the weak firm upon a breakthrough.

To see why the equilibrium exit times of the firms coincide with those of the planner, consider the weak firm's problem. Since  $\alpha_S^{**} = 1$ , the weak firm never benefits from the strong firm's innovation. The best response of the weak firm is thus to follow the optimal policy in the single firm problem and exit when the posterior belief falls below the single



firm threshold belief  $\frac{c}{\lambda\alpha_w^{**}v}$ . Thus, the weak firm exits at the smallest  $t$  that solves

$$p_{w,t}\lambda\left(v - \hat{W}(p_{S,t^{**}}; 1)\right) - c \leq 0.$$

The choice of  $\alpha_w^{**}$  in (33) implies that the weak firm's exit time must be identical to  $t_w^{**}$  in equilibrium. Next, we would like to show that  $t_S^{**}$  is a best response for the strong firm. First, observe that any  $t \in [t_w^{**}, t_S^{**}) \cup (t_S^{**}, \infty)$  cannot be optimal, since the strong firm can improve by exiting at  $t_S^{**}$  instead. Suppose  $t_w^{**} > 0$ , to show that exiting at any  $t \in [0, t_w^{**})$  is suboptimal, consider the strong firm's *MNB* at each time  $t \in [0, t_w^{**})$ , we have

$$\begin{aligned} & \frac{\partial V_S(t, t_w^{**})}{\partial t} \\ &= e^{-\int_0^t (p_{1,s} + p_{2,s})\lambda ds} \left\{ p_{S,t}\lambda \left( v - \check{W}(p_{w,t}; \alpha_w^{**}) \right) - c \right\} \\ &\geq e^{-\int_0^t (p_{1,s} + p_{2,s})\lambda ds} \left\{ p_{S,t}\lambda \left( v - \check{W}(p_{w,t}; \alpha_w^{**}) - \hat{W}(p_{w,t}; \alpha_w^{**}) \right) - c \right\} \\ &\geq e^{-\int_0^t (p_{1,s} + p_{2,s})\lambda ds} \left\{ p_{S,t}\lambda \left( v - \hat{W}(p_{w,t}; 1) \right) - c \right\} \\ &\geq e^{-\int_0^t (p_{1,s} + p_{2,s})\lambda ds} \left\{ p_{w,t}\lambda \left( v - \hat{W}(p_{S,t}; 1) \right) - c \right\} \\ &> e^{-\int_0^t (p_{1,s} + p_{2,s})\lambda ds} \left\{ p_{w,t^{**}}\lambda \left( v - \hat{W}(p_{S,t^{**}}; 1) \right) - c \right\} \\ &= 0 \end{aligned}$$

The first inequality follows from the fact that  $\hat{W}(p_{w,t}; \alpha_w^{**}) \geq 0$ , the second from the fact the total welfare must be maximized when all the externality is internalized (i.e., when the full award  $v$  is given to the remaining firm upon a breakthrough), the third from the facts that  $p_{S,t} \geq p_{w,t}$  and  $\hat{W}(\cdot; 1)$  is increasing and the fourth from the single-crossing property of the weak firm's *MNB* to the planner (Lemma 7). The expression in the last inequality is simply the weak firm's *MNB* to the planner at time  $t_w^{**}$ , which must be 0 if  $t_w^{**} > 0$ , by Proposition 8. Thus, exiting at any  $t \in [0, t_w^{**})$  is suboptimal as  $\frac{\partial V_S(t, t_w^{**})}{\partial t} > 0$ .

## 7 Discussion

In this section, we discuss our modelling assumptions and some possible extensions.

**The model as a static game.** In our model, each firm chooses an exit time. As

such, the game is a special case of a two-player static game, in which the players' actions are strategic substitutes, the policy parameter  $\alpha$  is strategic complementary to the actions, and the total welfare depends on the actions but not on  $\alpha$ . In the general model, however, varying the parameter  $\alpha$  has an ambiguous effect on both the equilibrium actions and welfare. In this paper, we explore the implications of the specific (dynamic) payoff structure of our game and show that in a regular equilibrium, the strong firm's exit time is independent of the weak firm's action (Proposition 1) and demonstrate how the weak firm's conflicted incentives to free ride and to insure herself by maintaining her own project lead to a non-concave maximization problem (Propositions 5 and 6) and, for some parameters, multiple equilibria (Proposition 7). To evaluate the total welfare, we study the relative position of the competitive outcome to the planner's solution. By making use of the dynamic structure of the game, our analysis differs from the traditional approach also in the use of diagrams on the posterior space  $(p_1, p_2)$ , which produces not only visualizations of the equilibria on the parameter space  $(p_{1,0}, p_{2,0})$ , but also geometric proofs. Moreover, since a change in the parameters  $(p_{1,0}, p_{2,0})$  can be visualized as the moving the starting point of the equilibrium belief path on the belief space, the posterior diagram is also a useful tool to investigate the model's comparative statics, as demonstrated by Examples 1, 2, and 3.

**Discounting.** For tractability, we have assumed no discounting in this paper. However, many of the results, including the basic equilibrium characterization (Proposition 1), the characterization of the planner's solution (Proposition 8), and the optimal distribution policy (Proposition 15), do not rely on this assumption. These results remain unchanged after the value functions (5) and (6) are replaced by their counterparts under discounting. Moreover, from a robustness perspective, small discounting would not qualitatively alter our results. The main difficulty in analyzing the high discounting case is that the weak firm's *MNB* (12) can no longer be expressed as a quadratic function as in (23). This makes a detailed characterization of the weak firm's exit time difficult.

**Consumer welfare.** Our analysis in this paper has abstracted away from the consideration of consumer welfare. To incorporate this aspect in the model, suppose an innovation leads to producer surplus  $v$  and consumer surplus  $C > 0$  in the post-innovation market

regardless of the patent system used. Since consumer surplus is irrelevant to the firms' decision, the characterizations of equilibrium as presented in Section 3 remain unchanged. The welfare consideration in Section 4 and the characterizations of the optimal patent in Section 5, however, should be adjusted accordingly. Since the value of an innovation to the planner is larger than that in the benchmark scenario as presented in Section 4, the planner will prefer both firms to experiment more and the optimal patent will be stricter. In particular, the strict patent system, i.e.,  $\alpha = 1$ , may be optimal if the consumer surplus  $C$  is large enough. When  $\alpha^* = 1$ , it is also possible that both firms under-experiment. This is because strict patent protection is already in use and cannot be strengthened further to incentivize experimentation. However, if an interior  $\alpha^*$  is optimal, it will still feature too much experimentation by the weak firm and too little by the strong firm. Finally, the optimal distribution policy no longer implements the planner's solution. This is because even if  $\alpha_S^{**} = 1$ , the strong firm still does not internalize the consumer's gain from an innovation.

**Post-innovation market efficiency.** Classical analyses of the optimal patent (e.g., Nordhaus 1969; Denicolò 1999) focus on the trade-off between ex ante incentives to innovate and ex post allocative inefficiency. That is, a strict patent system results in monopoly in the post-innovation market, which is socially inefficient. However, such an ex post inefficient arrangement may improve the ex ante incentives to innovate. In such an environment, optimal patent should be set to balance the two forces. In our model, the post-innovation market efficiency is assumed to be constant across patent systems. The total producer surplus  $v$  remains unchanged. Incorporating post-innovation market efficiency consideration would make the trade-off between duplication and under-experimentation that we highlighted less transparent. In analyzing such a model, the planner's problem as presented in Section 4.1 is no longer a useful benchmark, as allocative efficiency in the firms' efforts is no longer the only concern.

**Reversible exit.** In our model, we assume that the exit decision of a firm is irrevocable. It is easy to see, for example, from Figure 4(b), that, in general, equilibria identified under the irrevocability assumption do not survive when the assumption is removed, as the

weak firm would like to re-enter once the posterior re-enters the set  $E(\alpha)$ . To relax this assumption, the standard approach in the literature is to convexify the action space so that each firm can choose to invest  $e \in [0, 1]$  in R&D at each point in time. A firm's strategy in this alternative model is then a time-dependent path of effort, which allows for more complex dynamics than the exit time in our model. We suggest the alternative model here as a direction for future research.

**Unobservable exit.** In our model, we assume that the exit decision of a firm is unobservable to the other firm. Rosenberg et al. (2013) show in a related model of strategic experimentation that when one's exit decision is observable to the other player, the players can internalize payoff externalities by conditioning one's exit decision on the other's and decrease the exiting belief threshold (Theorem 1). In our model, such behaviors lead not only to multiple equilibria but it also means that, by mutual monitoring, it is possible to reduce, and sometimes eliminate entirely, the incentives to free ride, even when there is little or no patent protection. Such cooperative behaviors are implicitly ruled out when actions are not observed.

## 8 Conclusion

In this paper, we analyze how a patent system may affect the amount of experimentation in an R&D race. We find that the optimal patent is sensitive to the qualities of the R&D projects. With heterogeneous firms, a strict patent system is not optimal and the choice of optimal patent system involves a trade-off between duplication in the early stage of R&D and under-experimentation in the latter stage. An asymmetric distribution policy that implements the social optimum is proposed.

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## 9 Appendix

**Proof of Lemma 1.** Given an exit time  $t_i \geq 0$ , firm  $i$ 's expected payoff at time 0 is given by

$$\begin{aligned} & p_{i,0} \left( \int_0^{t_i} \lambda (\alpha v - ct) e^{-\lambda t} dt - e^{-\lambda t_i} ct_i \right) - (1 - p_{i,0}) ct_i \\ = & p_{i,0} \left( 1 - e^{-\lambda t_i} \right) \left( \alpha v - \frac{c}{\lambda} \right) - (1 - p_{i,0}) ct_i. \end{aligned} \quad (34)$$

The first order condition for optimality is thus

$$p_{i,0} \lambda e^{-\lambda t_i^\dagger} \left( \alpha v - \frac{c}{\lambda} \right) - (1 - p_{i,0}) c \begin{cases} = 0 & \text{if } t_i^\dagger > 0, \\ \leq 0 & \text{if } t_i^\dagger = 0. \end{cases}$$

Differentiate the objective function (34) twice with respect to  $t_i$  shows that it is strictly concave. Thus, firm  $i$ 's expected payoff has a unique maximizer given by (4). Moreover, the continuation values of firms  $i$  and  $j$  at time  $t$  are given by

$$\check{W}(p_{i,t}; \alpha) = p_{i,t} \left( 1 - e^{-\lambda \max\{t_i^\dagger - t, 0\}} \right) \left( \alpha v - \frac{c}{\lambda} \right) - (1 - p_{i,t}) c \max\{t_i^\dagger - t, 0\}, \quad (35)$$

and

$$\begin{aligned} \check{W}(p_{i,t}; \alpha) &= p_{i,t} \int_t^{t_i^\dagger \wedge t} \lambda (1 - \alpha) v e^{-\lambda(s-t)} ds \\ &= p_{i,t} \left( 1 - e^{-\lambda \max\{t_i^\dagger - t, 0\}} \right) (1 - \alpha) v, \end{aligned} \quad (36)$$

respectively. Plugging (4) into (35) and (36) results in (5) and (6). ■

Fact 1 gives the expressions for firm  $i$ 's expected payoff and the first derivative of the expected payoff given that firm  $i$  uses the exit time  $t_i$  and firm  $j$  uses the exit time  $t_j$ , where  $t_i$  and  $t_j$  are not necessarily equilibrium choices. These expressions are useful in the proof of Proposition 2 and are provided here for easy reference.

**Fact 1.** Let  $i, j \in \{1, 2\}$ , given that firm  $j$  uses the exit time  $t_j$ , firm  $i$ 's expected payoff

from using an exit time  $t_i$  is given by

$$V_i(t_i, t_j) = \begin{cases} \int_0^{t_j} \{(p_{i,t}\alpha + p_{j,t}(1-\alpha))\lambda v - c\} e^{-\int_0^t (p_{1,s}+p_{2,s})\lambda ds} dt & \text{if } t_i \geq t_j \\ +e^{-\int_0^{t_j} (p_{1,s}+p_{2,s})\lambda ds} \left( \int_{t_j}^{t_i} (p_{i,t}\lambda\alpha v - c) e^{-\int_{t_j}^t p_{i,s}\lambda ds} dt \right) & \\ \int_0^{t_i} \{(p_{i,t}\alpha + p_{j,t}(1-\alpha))\lambda v - c\} e^{-\int_0^t (p_{1,s}+p_{2,s})\lambda ds} dt & \text{if } t_i < t_j \\ +e^{-\int_0^{t_i} (p_{1,s}+p_{2,s})\lambda ds} \left( \int_{t_i}^{t_j} p_{j,t}\lambda(1-\alpha) v e^{-\int_{t_i}^t p_{j,s}\lambda ds} dt \right) & \end{cases} \quad (37)$$

Its first derivative is given by

$$\frac{\partial V_i(t_i, t_j)}{\partial t_i} = \begin{cases} e^{-\int_0^{t_i} p_{i,s}\lambda ds - \int_0^{t_j} p_{j,s}\lambda ds} (p_{i,t_i}\lambda\alpha v - c) & \text{if } t_i \geq t_j, \\ e^{-\int_0^{t_i} (p_{1,s}+p_{2,s})\lambda ds} \{p_{i,t_i}\lambda(\alpha v & \text{if } t_i < t_j. \\ -\int_{t_i}^{t_j} p_{j,t}\lambda(1-\alpha) v e^{-\int_{t_i}^t p_{j,s}\lambda ds} dt) - c\} & \end{cases} \quad (38)$$

which is continuous at  $t_i = t_j$ .

**Proof of Proposition 1.** In text. ■

**Proof of Proposition 2.** Without loss of generality, suppose  $p_{1,0} \geq p_{2,0}$ . We would like to show that an equilibrium  $(t_1^*, t_2^*)$  with  $t_1^* \geq t_2^*$  exists. Suppose  $p_{1,0}\lambda\alpha v \leq c$ , we claim that  $t_1^* = t_2^* = 0$  is an equilibrium. Since  $p_{1,0} \geq p_{2,0}$ , Lemma 1 implies that both firms find it optimal to exit at time 0 given that the opponent exits at time 0. Suppose  $p_{1,0}\lambda\alpha v > c$ , we claim that there exists an equilibrium  $(t_1^*, t_2^*)$  with  $t_1^* \geq t_2^*$ , where  $t_1^* > 0$  is the unique solution to  $p_{1,t_1^*}\lambda\alpha v = c$  and pinned down by (8). This is done in two steps. 1. Given that firm 1 exits at time  $t_1^*$ , we show that firm 2's best response is to exit at some  $t_2^* \in [0, t_1^*]$ . 2. Given  $t_2^*$ , we show that  $t_1^*$  is indeed firm 1's best response.

1. Since  $p_{1,0} \geq p_{2,0}$ , we have for all  $t > t_1^*$ ,

$$p_{2,t}\lambda\alpha v \leq p_{1,t}\lambda\alpha v < p_{1,t_1^*}\lambda\alpha v = c.$$

Thus, by (38), the first derivative of firm 2's payoff function  $\frac{\partial V_2(t, t_1^*)}{\partial t}$  is strictly negative for all  $t > t_1^*$ . Since the payoff function is continuous, a best response  $t_2^*$  exists but  $t_2^* \notin (t_1^*, \infty)$ .

2. Given  $t_2^* \in [0, t_1^*]$ , we need to show that  $t_1^*$  is indeed a best response for firm 1. Differentiate (38), we have, for all  $t \in [t_2^*, \infty)$ ,

$$\frac{\partial^2 V_1(t, t_2^*)}{\partial t^2} = -p_{1,t} \lambda e^{-\int_0^t p_{1,s} \lambda ds - \int_0^{t_2^*} p_{2,s} \lambda ds} (\lambda \alpha v - c) < 0,$$

as  $\lambda \alpha v > p_{1,t_1^*} \lambda \alpha v = c$ . Since  $V_1(\cdot, t_2^*)$  is strictly concave on  $[t_2^*, \infty)$  and  $\frac{\partial V_1(t, t_2^*)}{\partial t} \Big|_{t=t_1^*} = 0$ ,  $t_1^*$  maximizes  $V_1(\cdot, t_2^*)$  on  $[t_2^*, \infty)$ . Next, consider the derivative  $\frac{\partial V_1(t, t_2^*)}{\partial t}$  for  $t \in [0, t_2^*]$ , we have

$$\begin{aligned} \frac{\partial V_1(t, t_2^*)}{\partial t} &= e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} \{p_{1,t} \lambda (\alpha v - p_{2,t} (1 - e^{-\lambda(t_2^* - t)})) (1 - \alpha) v - c\} \\ &\geq e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} \{p_{2,t} \lambda (\alpha v - p_{1,t} (1 - e^{-\lambda(t_2^* - t)})) (1 - \alpha) v - c\} \\ &\geq e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} \{p_{2,t} \lambda (\alpha v - p_{1,t} (1 - e^{-\lambda(t_1^* - t)})) (1 - \alpha) v - c\} \\ &= \frac{\partial V_2(t, t_1^*)}{\partial t}, \end{aligned}$$

where the first inequality follows from the fact that  $p_{1,0} \geq p_{2,0}$  and the second from the fact that  $t_1^* \geq t_2^*$ . This implies that the exit time  $t_2^*$  is at least as good as any  $t \in [0, t_2^*)$  for firm 1. To see this, consider any  $t \in [0, t_2^*)$ , then

$$V_1(t_2^*, t_2^*) - V_1(t, t_2^*) = \int_t^{t_2^*} \frac{\partial V_1(s, t_2^*)}{\partial s} ds \geq \int_t^{t_2^*} \frac{\partial V_2(s, t_1^*)}{\partial s} ds = V_2(t_2^*, t_1^*) - V_2(t, t_1^*) \geq 0,$$

where the last inequality follows from the fact that  $t_2^*$  maximizes  $V_2(\cdot, t_1^*)$ . Since  $V_1(t_1^*, t_2^*) \geq V_1(t_2^*, t_2^*)$ ,  $t_1^*$  is indeed a best response for firm 1.

■

**Definition 7** The function  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\begin{aligned} f_i(x) &\equiv p_{1,0} p_{2,0} (\lambda v (2\alpha - 1) - c) x^2 \\ &+ \left( p_{i,0} (1 - p_{j,0}) \left( \frac{\lambda (\lambda \alpha^2 v + (1 - 2\alpha) c) v}{\lambda \alpha v - c} - c \right) - c p_{j,0} (1 - p_{i,0}) \right) x \\ &- (1 - p_{1,0}) (1 - p_{2,0}) c \end{aligned} \tag{39}$$

which coincides with  $\frac{\partial V_i(t, t_j^\dagger)}{\partial t} \Big|_{t=-\frac{1}{\lambda} \log x}$  on  $[e^{-\lambda t_j^\dagger}, 1]$ . By Proposition 1, the composite function  $f_i(e^{-\lambda t})$  represents the MNB of firm  $i$  for  $t \in [0, t_j^\dagger]$  in an equilibrium in which firm  $i$  exits earlier than firm  $j$ . Notice, however, that for  $t \in (t_j^\dagger, \infty)$ ,  $f_i(e^{-\lambda t})$  does not represent the MNB of firm  $i$  in equilibrium.

**Proof of Lemma 2.** Without loss of generality, suppose  $p_{1,0} \geq p_{2,0}$ . Since  $t_2^* > t_1^* \geq 0$ , it is impossible to have  $t_2^* = 0$ . Since  $t_2^* > 0$ , by Proposition 1,

$$e^{-\lambda t_2^*} = \frac{1 - p_{2,0}}{p_{2,0}} \frac{c}{\lambda \alpha v - c}.$$

Since  $p_{1,t_2^*} \geq p_{2,t_2^*}$ , we have

$$\begin{aligned} f_1(e^{-\lambda t_2^*}) &= (p_{1,0} - p_{2,0}) \lambda \alpha v \left( \frac{1 - p_{2,0}}{p_{2,0}} \frac{c}{\lambda \alpha v - c} \right) \\ &= e^{-\int_0^{t_2^*} (p_{1,s} + p_{2,s}) \lambda ds} (p_{1,t_2^*} \lambda \alpha v - c) \\ &= \frac{\partial V_1(t, t_2^*)}{\partial t} \Big|_{t=t_2^*} \geq 0. \end{aligned}$$

We consider the two cases,  $\alpha \leq \bar{\alpha}$  and  $\alpha > \bar{\alpha}$ , separately.

1. Suppose  $\alpha \leq \bar{\alpha} \equiv \frac{\lambda v + c}{2\lambda v}$ , by (39),  $f_1(x)$  is concave.  $f_1(e^{-\lambda t_2^*}) \geq 0$  imply that there is at most one  $x' \in (e^{-\lambda t_2^*}, 1)$  such that  $f_1(x') = 0$ . (Otherwise, the derivative  $\frac{\partial f_1(x)}{\partial x}$  must change from negative to positive between two roots, violating the concavity of  $f_1(x)$ .) Moreover, for such  $x'$ , we must have

$$\frac{\partial^2 V_1(t, t_2^*)}{\partial t^2} \Big|_{t=t'} = -\lambda e^{-\lambda t} \frac{\partial f_1(x)}{\partial x} \Big|_{x=x'} > 0,$$

where  $t' \equiv -\frac{1}{\lambda} \log x'$ . Therefore,  $t'$ , if exists, is a local minimum of firm 1's payoff function. Therefore, any  $t \in (0, t_2^*)$  fails to satisfy either the first order condition for optimality or the second order condition. We must have  $t_1^* = 0$ .

2. Suppose  $\alpha > \bar{\alpha} \equiv \frac{\lambda v + c}{2\lambda v}$ . By (39),  $f_1(x)$  is strictly convex in  $x$ . Since

$$f_1(e^{-\lambda t_2^*}) \geq 0 > -c(1 - p_{1,0})(1 - p_{2,0}) = f_1(0),$$

$\frac{\partial f_1(x)}{\partial x}|_{x=x'} > 0$  for some  $x' \in (0, e^{-\lambda t_2^*})$ . Strict convexity means that the slope of  $f_1(x)$  must increase with  $x$ . Thus, for all  $x \in [e^{-\lambda t_2^*}, 1]$ , we must have  $\frac{\partial f_1(x)}{\partial x} > 0$ . Thus, for all  $x \in (e^{-\lambda t_2^*}, 1]$ ,  $f_1(x) > f_1(e^{-\lambda t_2^*}) \geq 0$ . Since  $f_1(e^{-\lambda t}) = \frac{\partial V_1(t, t_2^*)}{\partial t}$  for  $t \in [0, t_2^*]$ . Any  $t \in [0, t_2^*)$  fails to satisfy firm 1's first order condition (9), which contradicts the assumption of the nonregular equilibrium.<sup>11</sup>

■

**Proof of Lemma 3.** Without loss of generality, suppose  $p_{1,0} \geq p_{2,0}$  and a nonregular equilibrium  $(0, t_2^*)$  exists, we would like to show that a regular equilibrium  $(t_1^*, 0)$  exists. Notice that both  $t_1^*$  and  $t_2^*$  are uniquely pinned down by (8). Compare the first derivative of firm 2's payoff function in the regular equilibrium (i.e., the *MNB*) and the first derivative of firm 1's payoff function in the nonregular equilibrium, by (38), we have, for all  $t \in [0, t_2^*]$ ,

$$\begin{aligned} \frac{\partial V_2(t, t_1^*)}{\partial t} &= e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} \{p_{2,t} \lambda (\alpha v - \check{W}(p_{1,t}; \alpha)) - c\} \\ &\leq e^{-\int_0^t (p_{1,s} + p_{2,s}) \lambda ds} \{p_{1,t} \lambda (\alpha v - \check{W}(p_{2,t}; \alpha)) - c\} \\ &= \frac{\partial V_1(t, t_2^*)}{\partial t}. \end{aligned}$$

Since  $(0, t_2^*)$  is an equilibrium, the exit time 0 maximizes firm 1's payoff. This implies that the exit time 0 maximizes firm 2's payoff in the regular equilibrium. To see this, consider any exit time  $t \in [0, t_2^*]$ , then

$$V_2(t, t_1^*) - V_2(0, t_1^*) = \int_0^t \frac{\partial V_2(s, t_1^*)}{\partial s} ds \leq \int_0^t \frac{\partial V_1(s, t_2^*)}{\partial s} ds = V_1(t, t_2^*) - V_1(0, t_2^*) \leq 0.$$

---

<sup>11</sup>Notice that, in this case, a nonregular equilibrium fails to exist.

Thus, the exit time 0 weakly dominates any exit time  $t \in (0, t_2^*]$ . Next, for all  $t > t_2^*$ ,

$$\begin{aligned} \frac{\partial V_2(t, t_1^*)}{\partial t} &= e^{-\int_0^t p_{2,s} \lambda ds - \int_0^{t_1^* \wedge t} p_{1,s} \lambda ds} \{p_{2,t} \lambda (\alpha v - \check{W}(p_{1,t}; \alpha)) - c\} \\ &\leq e^{-\int_0^t p_{2,s} \lambda ds - \int_0^{t_1^* \wedge t} p_{1,s} \lambda ds} \{p_{2,t} \lambda \alpha v - c\} \\ &< 0, \end{aligned}$$

which implies that the exit time 0 weakly dominates any exit time  $t \in (0, \infty)$ . Thus, the exit time 0 is a best response for firm 2. By Proposition 1,  $t_1^* = t_1^\dagger(\alpha)$  is a best response for firm 1. ■

**Proof of Lemma 4.** We prove the lemma by showing that  $E(\alpha)$  is the intersection of two convex sets. First, we claim that

$$\begin{aligned} E(\alpha) &= \bigcap_{i \in \{1,2\}} \left\{ (p_1, p_2) \in [0, 1]^2 : INB(p_i, p_j; \alpha) > 0 \right\} \\ &= \bigcap_{i \in \{1,2\}} \left\{ (p_1, p_2) \in [0, 1]^2 : p_i > \frac{c}{\lambda (\alpha v - \check{W}(p_j; \alpha))} \right\}. \end{aligned}$$

To see that, consider  $(p_1, p_2) \in E(\alpha)$ . Then, by definition of  $E(\alpha)$ , if  $p_i \geq p_j$ , then  $INB(p_j, p_i; \alpha) > 0$ . But this also implies

$$\begin{aligned} &INB(p_i, p_j; \alpha) \\ &= p_i \lambda (\alpha v - \check{W}(p_j; \alpha)) - c \\ &\geq p_j \lambda (\alpha v - \check{W}(p_i; \alpha)) - c \\ &= INB(p_j, p_i; \alpha) \\ &> 0. \end{aligned}$$

Conversely, any element  $(p_1, p_2)$  in the intersection must have either  $p_1 \geq p_2$  or  $p_1 < p_2$ , which means that  $(p_1, p_2) \in E(\alpha)$ . For all  $p \in [0, 1]$ , let

$$G(p) = \frac{c}{\lambda (\alpha v - \check{W}(p; \alpha))}.$$

We have

$$G''(p) = \begin{cases} \frac{2\left(\frac{\lambda\alpha v}{\lambda\alpha v - c}\right)^2(1-\alpha)^2v^2c}{\lambda\left(\alpha v - \left(\frac{p\lambda\alpha v - c}{\lambda\alpha v - c}\right)(1-\alpha)v\right)^3} & \text{if } p > \frac{c}{\lambda\alpha v}, \\ 0 & \text{if } p < \frac{c}{\lambda\alpha v}. \end{cases}$$

The convexity of  $E(\alpha)$  then follows from the fact that for all  $p \in (0, 1) \setminus \left\{\frac{c}{\lambda\alpha v}\right\}$ ,  $G''(p) \geq 0$ .

■

**Proof of Lemma 5.** Suppose the set  $E(\alpha)$  is nonempty, there exists  $(p'_1, p'_2) \in E(\alpha)$ . If  $p'_1 > p'_2$ , then  $(p'_1, p'_1) \in E(\alpha)$ . Similarly, if  $p'_1 < p'_2$ , then  $(p'_2, p'_2) \in E(\alpha)$ . This analysis shows that the 45° line must intersect with  $E(\alpha)$  if it is nonempty. We would like to show that  $\alpha \leq \underline{\alpha}$  if and only if for all  $p \in [0, 1]$ ,  $(p, p) \notin E(\alpha)$ . This is done in 2 parts.

1. Suppose  $\alpha \leq \underline{\alpha}$ . Then, for all  $p \leq \frac{c}{\lambda\alpha v}$ ,  $\tilde{W}(p; \alpha) = 0$ , so

$$INB(p, p; \alpha) = p\lambda\alpha v - c \leq 0.$$

Thus,  $(p, p) \notin E(\alpha)$ . If  $\alpha \leq \frac{c}{\lambda\alpha v}$ , we are done. If  $\alpha > \frac{c}{\lambda\alpha v}$ , then, for all  $p \in \left[\frac{c}{\lambda\alpha v}, 1\right]$ , we have

$$INB(p, p; \alpha) = p\lambda \left( \alpha v - \left( \frac{p\lambda\alpha v - c}{\lambda\alpha v - c} \right) (1 - \alpha) v \right) - c$$

which is a quadratic function of  $p$ . Moreover,

$$INB\left(\frac{c}{\lambda\alpha v}, \frac{c}{\lambda\alpha v}; \alpha\right) = 0,$$

and, for all  $p \in \left(\frac{c}{\lambda\alpha v}, 1\right)$ ,

$$\frac{\partial^2 INB(p, p; \alpha)}{\partial^2 p} = -\frac{2\lambda^2\alpha(1-\alpha)v^2}{\lambda\alpha v - c} < 0.$$

Thus, if  $\lim_{p \rightarrow \frac{c}{\lambda\alpha v}^+} \frac{\partial INB(p, p; \alpha)}{\partial p} \leq 0$ , then, for all  $p \in \left(\frac{c}{\lambda\alpha v}, 1\right)$ ,  $\frac{\partial INB(p, p; \alpha)}{\partial p} < 0$  and  $INB(p, p; \alpha) < 0$ . Since

$$\lim_{p \rightarrow \frac{c}{\lambda\alpha v}^+} \frac{\partial INB(p, p; \alpha)}{\partial p} = \lambda \left( \alpha v - \frac{c}{\lambda\alpha v - c} (1 - \alpha) v \right)$$

is strictly increasing in  $\alpha$  and equals to 0 when  $\alpha = \underline{\alpha}$ .  $E(\alpha)$  must be empty.

2. Suppose  $\alpha > \underline{\alpha}$ , then,  $\alpha > \frac{c}{\lambda v}$ . Consider  $(\frac{c}{\lambda \alpha v}, \frac{c}{\lambda \alpha v}) \in (0, 1)^2$ , then

$$INB\left(\frac{c}{\lambda \alpha v}, \frac{c}{\lambda \alpha v}; \alpha\right) = 0,$$

If  $\lim_{p \rightarrow \frac{c}{\lambda \alpha v} +} \frac{\partial INB(p, p; \alpha)}{\partial p} > 0$ , there is a  $p' \in (\frac{c}{\lambda \alpha v}, 1)$  such that for all  $p \in (\frac{c}{\lambda \alpha v}, p')$ ,  $(p, p) \in E(\alpha)$ . Since

$$\lim_{p \rightarrow \frac{c}{\lambda \alpha v} +} \frac{\partial INB(p, p; \alpha)}{\partial p} = \lambda \left( \alpha v - \frac{c}{\lambda \alpha v - c} (1 - \alpha) v \right)$$

is strictly increasing in  $\alpha$  and equals to 0 when  $\alpha = \underline{\alpha}$ .  $E(\alpha)$  is nonempty.

■

**Proof of Proposition 3.** Consider a regular equilibrium  $(t_1^*, t_2^*)$ . The strong firm's equilibrium exit times is characterized by (8). Since  $E(\alpha) = \phi$ , for all  $t \geq 0$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} \leq 0$ . Since (23) is a quadratic function with nonzero coefficients,  $\frac{\partial V_w(t, t_S^*)}{\partial t} = 0$  in at most one point. Thus,  $t_w^* = 0$ . ■

**Proof of Lemma 6.** Without loss of generality, suppose  $p_1 \geq p_2 > 0$ , plugging (6) into (13), we have

$$INB(p_2, p_1; \alpha) = \begin{cases} p_2 \lambda \left( \alpha v - \left( \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} \right) (1 - \alpha) v \right) - c & \text{if } p_1 > \frac{c}{\lambda \alpha v}, \\ p_2 \lambda \alpha v - c & \text{if } p_1 \leq \frac{c}{\lambda \alpha v}. \end{cases}$$

Suppose  $p_1 < \frac{c}{\lambda \alpha v}$ , then

$$\frac{\partial INB(p_2, p_1; \alpha)}{\partial \alpha} = p_2 \lambda v > 0.$$

Suppose  $p_1 > \frac{c}{\lambda \alpha v}$ , we have



$$\begin{aligned}
\frac{\partial INB(p_2, p_1; \alpha)}{\partial \alpha} &= p_2 \lambda \left( v - (1 - p_1) \frac{\lambda v^2 (1 - \alpha) c}{(\lambda \alpha v - c)^2} + \left( \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} \right) v \right) \\
&> p_2 \lambda \left( v - \left( 1 - \frac{c}{\lambda \alpha v} \right) \frac{\lambda v^2 (1 - \alpha) c}{(\lambda \alpha v - c)^2} + \left( \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} \right) v \right) \\
&= p_2 \lambda \left( v - \frac{vc}{\lambda \alpha v - c} \frac{1 - \alpha}{\alpha} + \left( \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} \right) v \right) \\
&\geq p_2 \lambda \left( v - \frac{vc}{\lambda \left( \sqrt{\frac{c}{\lambda v}} \right) v - c} \frac{1 - \sqrt{\frac{c}{\lambda v}}}{\sqrt{\frac{c}{\lambda v}}} + \left( \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} \right) v \right) \\
&= p_2 \lambda \frac{p_1 \lambda \alpha v - c}{\lambda \alpha v - c} v \\
&\geq 0
\end{aligned}$$

The first inequality follows from the fact that  $p_1 > \frac{c}{\lambda \alpha v}$  and the second inequality follows from the fact that the second term in the bracket is decreasing in  $\alpha$  and our premise that  $\alpha \geq \bar{\alpha} \equiv \sqrt{\frac{c}{\lambda v}}$ . ■

**Proof of Proposition 4.** Consider a regular equilibrium  $(t_1^*, t_2^*)$ , suppose  $t_S^* > 0$  under the patent system  $\alpha$ . If  $t_S^* > 0$ , (8) implies

$$\frac{\partial t_S^*}{\partial \alpha} = \frac{v}{\lambda \alpha v - c} > 0. \quad (40)$$

Thus,  $t_S^*$  increases with  $\alpha$ . If  $t_w^* > 0$ , then, by Proposition 3,  $\alpha > \underline{\alpha}$ . Consider an increase of the patent strength from  $\alpha$  to  $\alpha'$  and let  $t_i^*(\alpha)$  and  $t_i^*(\alpha')$  denote the equilibrium exit times of firm  $i$ . We claim that  $t_w^*(\alpha) < t_w^*(\alpha')$ . Since the weak firm's  $MNB$  at time  $t$  is simply the  $INB$  multiplied by the term  $\exp\{-\lambda \int_0^t p_{w,z} dz + \lambda \int_0^{t \wedge t_S^*} p_{S,z} dz\}$  and  $t_w^*(\alpha) \leq t_S^*(\alpha) < t_S^*(\alpha')$ , by Lemma 6,  $\frac{\partial V_w(t, t_S^*)}{\partial t}$  strictly increases for all  $t \leq t_w^*(\alpha)$  under  $\alpha'$ .

Thus, the first order condition cannot be satisfied at  $t_w^*(\alpha)$ . Moreover, for any  $t < t_w^*(\alpha)$ ,

$$\begin{aligned}
& V_w(t_w^*(\alpha), t_S^*(\alpha'); \alpha') - V_w(t, t_S^*(\alpha'); \alpha') \\
&= \int_t^{t_w^*(\alpha)} \frac{\partial V_w(t, t_S^*(\alpha'); \alpha')}{\partial t} dt \\
&> \int_t^{t_w^*(\alpha)} \frac{\partial V_w(t, t_S^*(\alpha); \alpha)}{\partial t} dt \\
&= V_w(t_w^*(\alpha), t_S^*(\alpha); \alpha) - V_w(t, t_S^*(\alpha); \alpha) \\
&\geq 0.
\end{aligned}$$

Thus, any  $t \leq t_w^*(\alpha)$  cannot be optimal under the new patent system  $\alpha'$ . We must have  $t_w^*(\alpha) < t_w^*(\alpha')$ . ■

**Proof of Proposition 5.** Consider a regular equilibrium  $(t_1^*, t_2^*)$ , since  $p_{w,t_S^*} \leq p_{S,t_S^*}$ , by (38), for all  $t > t_S^*$ , we have  $\frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ . By (39),  $f_w(e^{-\lambda t_S^*}) = \frac{\partial V_w(t, t_S^*)}{\partial t}|_{t=t_S^*} \leq 0$ . Suppose  $\alpha < \bar{\alpha}$ ,  $f_w$  is strictly concave. Consider the three cases in Proposition 5.

1. Suppose  $\mu(p_{1,0}, p_{2,0}) \cap E(\alpha) = \emptyset$ , so that for all  $t \in [0, t_S^*]$ ,  $f_w(e^{-\lambda t}) = \frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ .

The weak firm must exit at time 0.

2. Suppose  $(p_{1,0}, p_{2,0}) \in E(\alpha)$ , so  $\frac{\partial V_w(t, t_S^*)}{\partial t}|_{t=0} = f_w(1) > 0$ . Since  $f_w$  is quadratic,  $f_w(e^{-\lambda t_S^*}) \leq 0$  implies that it has a unique root on  $[e^{-\lambda t_S^*}, 1]$ . Let  $e^{-\lambda t_r}$  be the root so that  $f_w(e^{-\lambda t_r}) = 0$ . We must have, for all  $x \in (e^{-\lambda t_r}, 1]$ ,  $f_w(x) > 0$  and for all  $x \in [e^{-\lambda t_S^*}, e^{-\lambda t_r})$ ,  $f_w(x) < 0$ . This means that for all  $t < t_r$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} > 0$  and for all  $t > t_r$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ . Thus, the weak firm must exit at time  $t_r$  when the posterior reaches the set  $\partial E(\alpha)$ .

3. Suppose  $(p_{1,0}, p_{2,0}) \notin E(\alpha)$  and  $\mu(p_{1,0}, p_{2,0}) \cap E(\alpha) \neq \emptyset$ . Thus,  $f_w(e^{-\lambda t_S^*}) \leq 0$ ,  $f_w(1) \leq 0$  and but  $f_w(x) > 0$  for some  $x \in (e^{-\lambda t_S^*}, 1)$ . Since  $f_w$  is quadratic, there must be exactly two roots on  $[e^{-\lambda t_S^*}, 1]$ . Let  $e^{-\lambda t_{r1}}$  and  $e^{-\lambda t_{r2}}$  be the two roots and

$t_{r_1} < t_{r_2}$ . We must have,

$$\frac{\partial V_w(t, t_S^*)}{\partial t} \begin{cases} < 0 & t < t_{r_1}, \\ = 0 & t = t_{r_1}, \\ > 0 & t_{r_1} < t < t_{r_2}, \\ = 0 & t = t_{r_2}, \\ < 0 & t > t_{r_2}. \end{cases}$$

Thus,  $V_w(\cdot, t_S^*)$  has two local maxima, 0 and  $t_{r_2}$ . If 0 is the global maximum, the weak firm exits at time 0. If  $t_{r_2}$  is the global maximum, the weak firm exits when the posterior reaches the set  $\partial E(\alpha)$  for the second time.

■

**Proof of Proposition 6.** Consider a regular equilibrium  $(t_1^*, t_2^*)$ , since  $p_{w, t_S^*} \leq p_{S, t_S^*}$ , by (38), for all  $t > t_S^*$ , we have  $\frac{\partial V_w(t, t_S^*)}{\partial t} < f_w(e^{-\lambda t_S^*}) \leq 0$ . Suppose  $\alpha \geq \bar{\alpha}$ , by (39),  $f_w$  is convex. Consider the two cases in Proposition 6.

1. Suppose  $(p_{1,0}, p_{2,0}) \notin E(\alpha)$ , so  $f_w(1) \leq 0$ . Suppose  $\alpha > \bar{\alpha}$ , strict convexity implies that for all  $x \in (e^{-\lambda t_S^*}, 1)$ ,  $f_w(x) < \max\{f_w(e^{-\lambda t_S^*}), f_w(1)\} \leq 0$ . Thus, for all  $t \in (0, t_S^*) \cup (t_S^*, \infty)$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ . Suppose  $\alpha = \bar{\alpha}$ ,  $f_w$  is linear,  $f_w(0) = -(1 - p_{1,0})(1 - p_{2,0})c < 0$  implies that for all  $t \in (0, \infty)$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ . In both cases, we must have  $t_w^* = 0$ .
2. Suppose  $(p_{1,0}, p_{2,0}) \in E(\alpha)$ , so  $f_w(1) > 0$ . Since  $f_w$  is quadratic,  $f_w(e^{-\lambda t_S^*}) \leq 0$  implies that it has a unique root on  $[e^{-\lambda t_S^*}, 1)$ . Let  $e^{-\lambda t_r}$  be the root so that  $f_w(e^{-\lambda t_r}) = 0$ . This means that for all  $t < t_r$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} > 0$  and for all  $t > t_r$ ,  $\frac{\partial V_w(t, t_S^*)}{\partial t} < 0$ . Thus, the weak firm must exit at time  $t_r$  when the posterior reaches the set  $\partial E(\alpha)$ .

■

**Proof of Proposition 7.** In text. ■

**Fact 2.** Given a pair of exit times  $(t_1, t_2)$ , where  $t_i \geq t_j$ , the planner's payoff is given

by

$$\begin{aligned}
& V_P(t_1, t_2) \\
= & p_{1,0}p_{2,0} \left( \int_0^{t_j} 2\lambda(v - 2ct) e^{-2\lambda t} dt + \int_{t_j}^{t_i} \lambda(v - c(t + t_j)) e^{-\lambda(t+t_j)} dt - e^{-\lambda(t_i+t_j)} (t_i + t_j) c \right) \\
& + p_{i,0}(1 - p_{j,0}) \left( \int_0^{t_j} \lambda(v - 2ct) e^{-\lambda t} dt + \int_{t_j}^{t_i} \lambda(v - c(t + t_j)) e^{-\lambda t} dt - e^{-\lambda t_i} (t_i + t_j) c \right) \\
& + (1 - p_{i,0})p_{j,0} \left( \int_0^{t_j} \lambda(v - 2ct) e^{-\lambda t} dt - e^{-\lambda t_j} (t_i + t_j) c \right) - (1 - p_{1,0})(1 - p_{2,0})(t_1 + t_2) c \\
= & p_{1,0}p_{2,0} \left( 1 - e^{-\lambda(t_1+t_2)} \right) \left( v - \frac{c}{\lambda} \right) \\
& + p_{i,0}(1 - p_{j,0}) \left( \left( 1 - e^{-\lambda t_j} \right) \left( v - \frac{2c}{\lambda} \right) + \left( e^{-\lambda t_j} - e^{-\lambda t_i} \right) \left( v - \frac{c}{\lambda} \right) \right) \\
& + (1 - p_{i,0})p_{j,0} \left( \left( 1 - e^{-\lambda t_j} \right) \left( v - \frac{2c}{\lambda} \right) - e^{-\lambda t_j} (t_i - t_j) c \right) \\
& - (1 - p_{1,0})(1 - p_{2,0})(t_1 + t_2) c
\end{aligned}$$

Its first derivative is given by

$$\frac{\partial V_P(t_1, t_2)}{\partial t_i} = \begin{cases} p_{i,0}e^{-\lambda t_i} \lambda \left( (p_{j,0}e^{-\lambda t_j} + 1 - p_{j,0}) \left( v - \frac{c}{\lambda} \right) + (1 - p_{j,0}) c (t_j - t_i) \right) & \text{if } t_i \leq t_j \\ - (1 - p_{i,0}) (p_{j,0}e^{-\lambda t_i} + 1 - p_{j,0}) c & \\ (p_{j,0}e^{-\lambda t_j} + 1 - p_{j,0}) (p_{i,0}e^{-\lambda t_i} (\lambda v - c) - (1 - p_{i,0}) c) & \text{if } t_i > t_j \end{cases} \quad (41)$$

Notice also that  $\frac{\partial V_P(t_1, t_2)}{\partial t_i}$  is continuous at  $t_1 = t_2$ .

**Proof of Lemma 7.** Given  $t'_j \geq 0$ , we first show that there is at most one  $t_i^\ddagger \geq 0$  that satisfies

$$\frac{\partial V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t_i^\ddagger, t_j=t'_j} = 0. \quad (42)$$

Suppose such a  $t_i^\ddagger$  exists, if  $t_i^\ddagger \in [0, t'_j)$ , differentiate (41), we have

$$\begin{aligned}
& \frac{\partial^2 V_P(t_1, t_2)}{\partial t_i^2} \Big|_{t_i=t_i^\ddagger, t_j=t'_j} \\
= & -\lambda \frac{\partial V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t_i^\ddagger, t_j=t'_j} - \lambda (1 - p_{1,0})(1 - p_{2,0}) c - p_{i,0} e^{-\lambda t_i^\ddagger} \lambda (1 - p_{j,0}) c \\
= & -\lambda (1 - p_{1,0})(1 - p_{2,0}) c - p_{i,0} e^{-\lambda t_i^\ddagger} \lambda (1 - p_{j,0}) c < 0. \quad (43)
\end{aligned}$$

If  $t_i^\dagger \in (t'_j, \infty)$ , then

$$\frac{\partial^2 V_P(t_1, t_2)}{\partial t_i^2} \Big|_{t_i=t_i^\dagger, t_j=t'_j} = -\lambda \left( p_{j,0} e^{-\lambda t'_j} + 1 - p_{j,0} \right) p_{i,0} e^{-\lambda t_i^\dagger} (\lambda v - c) < 0.$$

Thus, for all  $t \in [0, t_i^\dagger)$ ,  $\frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t, t_j=t'_j} > 0$  and for all  $t \in (t_i^\dagger, \infty)$ ,  $\frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t, t_j=t'_j} < 0$ . If (42) has no solution, then,  $\lim_{t_i \rightarrow \infty} \frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_j=t'_j} < 0$  implies that for all  $t \geq 0$ ,  $\frac{V_P(t_1, t_2)}{\partial t_i} \Big|_{t_i=t, t_j=t'_j} < 0$ . Thus,  $t_i^\dagger = 0$  satisfies the requirement in this case.

Therefore,  $t_i^\dagger(t_j)$  is well-defined for all  $t_j \geq 0$ . Berge maximum theorem implies that it is continuous. Consider  $t_j \in [0, \bar{t}_j^\dagger]$ . If  $t_i^\dagger(t_j) \in (t_j, \infty)$ , then  $\frac{dt_i^\dagger(t_j)}{dt_j} = 0$ . If  $t_i^\dagger(t_j) \in (0, t_j)$ , then the implicit function theorem implies

$$\frac{dt_i^\dagger(t_j)}{dt_j} = - \frac{\frac{\partial^2 V_P(t_1, t_2)}{\partial t_j \partial t_i} \Big|_{t_i=t_i^\dagger(t_j), t_j=t'_j}}{\frac{\partial^2 V_P(t_1, t_2)}{\partial t_i^2} \Big|_{t_i=t_i^\dagger(t_j), t_j=t'_j}} \leq 0,$$

which follows from (43) and

$$\frac{\partial}{\partial t_j} \left\{ \frac{\partial V_P(t_1, t_2)}{\partial t_i} \right\} = -p_{i,0} e^{-\lambda t_i} \lambda \left( \left( p_{j,0} e^{-\lambda t_j} \right) (\lambda v - c) - (1 - p_{j,0}) c \right) \leq 0. \quad (44)$$

as  $p_{j,0} e^{-\lambda \bar{t}_j^\dagger} (\lambda v - c) = (1 - p_{j,0}) c$ . Thus, since  $t_i^\dagger$  is continuous, it is decreasing for all  $t_j \in [0, \bar{t}_j^\dagger]$ . ■

**Proof of Lemma 8.** Without loss of generality, suppose  $p_{1,0} \geq p_{2,0}$  and suppose by way of contradiction that  $t_2^{**} > t_1^{**}$ , then consider an alternative strategy with  $(t'_1, t'_2) = (t_2^{**}, t_1^{**})$ , then

$$\begin{aligned} & V_P(t_2^{**}, t_1^{**}) - V_P(t_1^{**}, t_2^{**}) \\ &= (p_{1,0} - p_{2,0}) \left( \left( e^{-\lambda t_1^{**}} - e^{-\lambda t_2^{**}} \right) \left( v - \frac{c}{\lambda} \right) + e^{-\lambda t_1^{**}} (t_2^{**} - t_1^{**}) c \right) \\ &> 0 \end{aligned}$$

Thus, it is without loss to consider only exit times with which the strong firm exits weakly later. ■

**Proof of Proposition 8.** The characterizations of the exit times follow immediately from Lemma 7 and the discussion in the text. To show the dependence of  $t_w^{**}$  on  $p_{w,0}$  and  $p_{S,0}$ , substitute (29) into (41),

$$\begin{aligned} & \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} \\ = & p_{w,0} e^{-\lambda t_w^{**}} \lambda (1 - p_{S,0}) \left( v - \frac{c}{\lambda} \log \left( \frac{1 - p_{S,0}}{p_{S,0}} \frac{c}{\lambda v - c} \right) - c t_w^{**} \right) \\ & - (1 - p_{w,0}) \left( p_{S,0} e^{-\lambda t_w^{**}} + 1 - p_{S,0} \right) c \end{aligned} \quad (45)$$

Suppose  $t_w^{**} > 0$ , the planner's first order condition is

$$\frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} = 0. \quad (46)$$

Total differentiate (46), we have

$$\frac{dt_w^{**}}{dp_{i,0}} = - \frac{\frac{\partial}{\partial p_{i,0}} \left( \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}(p_{i,0}), t_w=t_w^{**}} \right)}{\frac{\partial^2 V_P(t_1, t_2)}{\partial t_w^2} \Big|_{t_S=t_S^{**}(p_{i,0}), t_w=t_w^{**}}}. \quad (47)$$

The denominator in (47) is strictly negative by (43) (Lemma 7). Differentiate (45), we have,

$$\begin{aligned} & \frac{\partial}{\partial p_{w,0}} \left( \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} \right) \\ = & \frac{1}{p_{w,0}} \left( \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} + \left( p_{S,0} e^{-\lambda t_w^{**}} + 1 - p_{S,0} \right) c \right) \\ > & 0. \end{aligned}$$

Therefore,  $t_w^{**}$  is increasing with  $p_{w,0}$ . Moreover,

$$\begin{aligned} & \frac{\partial}{\partial p_{S,0}} \left( \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} \right) \\ = & - \frac{1}{1 - p_{S,0}} \frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_S=t_S^{**}, t_w=t_w^{**}} + \left( \frac{p_{w,0}}{p_{S,0}} - \frac{1 - p_{w,0}}{1 - p_{S,0}} \right) e^{-\lambda t_w^{**}} c \\ \leq & 0. \end{aligned}$$

Thus,  $\frac{dt_w^{**}}{dp_{S,0}} \leq 0$ .  $t_w^{**}$  is decreasing with  $p_{S,0}$ . ■

**Proof of Lemma 9.** By Proposition 1 and Proposition 8, the strong firm in a regular equilibrium  $(t_1^*, t_2^*)$  never over-experiments, i.e.,  $t_S^* < t_S^{**} = \bar{t}_S^\dagger$ . By Lemma 7,  $t_w^\dagger(t_S^*) \geq t_w^\dagger(t_S^{**}) = t_w^{**}$ . Thus,  $t_w^* < t_w^{**}$  implies  $t_w^* < t_w^\dagger(t_S^*)$ . The weak firm under-experiments. ■

**Proof of Lemma 10.** Suppose  $\alpha \in [\frac{1}{2}, \underline{\alpha}]$ , by Lemma 5,  $E(\alpha) = \phi$ . It is easy to see that, in this case,  $\partial E(\alpha) = \{(\frac{c}{\lambda\bar{\alpha}v}, \frac{c}{\lambda\bar{\alpha}v})\} \cap [0, 1]^2$ . Moreover, since  $INB(\frac{c}{\lambda\bar{\alpha}v}, \frac{c}{\lambda\bar{\alpha}v}; \bar{\alpha}) = INB(1, 1; \bar{\alpha}) = 0$ , and  $INB(p, p; \bar{\alpha})$  is strictly concave in  $p$ , for all  $p \in (\frac{c}{\lambda\bar{\alpha}v}, 1)$ ,  $INB(p, p; \bar{\alpha}) > 0$ . Thus,  $(E(\alpha) \cup \partial E(\alpha)) \setminus \{(1, 1)\} \in E(\bar{\alpha})$ . Moreover, by Lemmas 4 and 6, for all  $\alpha, \alpha' \in (\underline{\alpha}, \bar{\alpha}]$ ,  $\alpha' > \alpha$  implies that  $E(\alpha) \cup \partial E(\alpha) \subseteq E(\alpha')$ . Thus, we only need to show that  $E(\bar{\alpha}) \subseteq E^{**}$ . Consider the difference

$$D(p) = v - \hat{W}(p; 1) - (\bar{\alpha}v - \check{W}(p; \bar{\alpha})).$$

Comparing (13) and (28), it's easy to see that we only need to show that for all  $p \in (\frac{c}{\lambda\bar{\alpha}v}, 1)$ ,  $D(p) > 0$ . Using (5) and (6), direct differentiation yields

$$D''(p) = -\frac{c}{p^2(1-p)\lambda} < 0.$$

Strict concavity of  $D$  then implies for all  $p \in (\frac{c}{\lambda\bar{\alpha}v}, 1)$ ,  $D(p) > \min\{D(\frac{c}{\lambda\bar{\alpha}v}), D(1)\}$ . Since

$$\begin{aligned} D(1) &= v - \left(v - \frac{c}{\lambda}\right) - (\bar{\alpha}v - (1 - \bar{\alpha})v) \\ &= \frac{c}{\lambda} - (2\bar{\alpha} - 1)v = 0, \end{aligned}$$

if  $D\left(\frac{c}{\lambda\bar{\alpha}v}\right) > 0$ , then for all  $p \in \left(\frac{c}{\lambda\bar{\alpha}v}, 1\right)$ ,  $D(p) > 0$ . Indeed, we have,

$$\begin{aligned}
& D\left(\frac{c}{\lambda\bar{\alpha}v}\right) \\
&= v - \hat{W}\left(\frac{c}{\lambda\bar{\alpha}v}; 1\right) - \left(\bar{\alpha}v - \check{W}\left(\frac{c}{\lambda\bar{\alpha}v}; \bar{\alpha}\right)\right) \\
&= v - \left(\frac{2c}{\lambda v + c}v - \frac{c}{\lambda} + \left(1 - \frac{2c}{\lambda v + c}\right) \frac{c}{\lambda} \log\left(\frac{1 - \frac{2c}{\lambda v + c}}{\frac{2c}{\lambda v + c}} \frac{c}{\lambda v - c}\right)\right) - \frac{\lambda v + c}{2\lambda v}v \\
&= \left(\frac{1}{2} - \frac{c}{\lambda v + c}\right)v + \frac{c}{\lambda} \left(\frac{\lambda v - c}{\lambda v + c}\right) \left(\log 2 - \frac{1}{2}\right) \\
&> 0.
\end{aligned}$$

■

**Proof of Lemma 11.** In text. ■

**Proof of Proposition 9.** In text. ■

**Proof of Lemma 12.** In text. ■

**Proof of Lemma 13.** Consider each part of Lemma 13 separately.

1. To prove the first part of Lemma 13, suppose  $\alpha^* = 1$ , then

$$\begin{aligned}
\frac{\partial V_P(t_1^*, t_2^*)}{\partial t_w^*} &= e^{-\int_0^{t_w^*} (p_{1,s} + p_{2,s})\lambda ds} \{p_{w,t_w^*} \lambda (v - U_P^w(t_w^*, t_S^*)) - c\} \\
&\leq e^{-\int_0^{t_w^*} (p_{1,s} + p_{2,s})\lambda ds} \{p_{w,t_w^*} \lambda v - c\} \\
&= 0.
\end{aligned}$$

where the inequality follows from (27) and the fact that the welfare-maximizing equilibrium must be regular, i.e.,  $t_w^* \leq t_S^* \leq t_S^{**}$ . Next, suppose  $\alpha^* < 1$  and

$\frac{\partial V_P(t_1, t_2)}{\partial t_w} \Big|_{t_w=t_w^*, t_S=t_S^*} > 0$ , consider the derivative

$$\frac{dV_P(t_1^*, t_2^*)}{d\alpha} = \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} \frac{\partial t_S^*}{\partial \alpha} + \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_w^*} \frac{\partial t_w^*}{\partial \alpha},^{12}$$

---

<sup>12</sup>If the weak firm's exit time in the welfare maximizing equilibrium is discontinuous at  $\alpha^*$ , we replace  $\frac{\partial t_w^*}{\partial \alpha}$  by its right hand limit, which always exists. The same argument suggests that a small increase in  $\alpha$  strictly increases the total welfare.



we claim that it must be strictly positive, so that a small increase in  $\alpha$  strictly increases the total welfare. Since  $\alpha^* < 1$ ,  $t_S^* < t_S^\dagger(t_w^*) = t_S^{**}$ . By Lemma 7, this means  $\frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} > 0$ . Moreover, by Lemma 12,  $t_S^* > 0$ . Proposition 4 then implies that  $\frac{\partial t_S^*}{\partial \alpha} > 0$  and  $\frac{\partial t_w^*}{\partial \alpha} \geq 0$ . Thus,  $\frac{dV_P(t_1^*, t_2^*)}{d\alpha} > 0$ , contradicting the assumption that  $\alpha^*$  is optimal.

2. To prove the second part of Lemma 13, suppose  $\alpha^* = 1$ , then the assumption of nonextreme prior and Proposition 1 implies that  $INB(p_w, t_w^*, p_S, t_w^*; \alpha^*) = 0$ . Next, suppose  $\alpha^* < 1$  and  $INB(p_w, t_w^*, p_S, t_w^*; \alpha^*) < 0$ . Proposition 1 implies that we must have  $t_w^* = 0$ . If there exists an  $\alpha' \in (\underline{\alpha}, \bar{\alpha})$  such that under  $\alpha'$ , two regular equilibria coexist, then, Propositions 7 and 9 imply that  $\alpha^* < \alpha'$  and that there exists a  $\Delta > 0$  such that  $t_w^* = 0$  for  $\alpha \in [\frac{1}{2}, \alpha^* + \Delta)$ . Thus,  $\frac{\partial t_w^*}{\partial \alpha} = 0$ . Moreover,  $\alpha^* < 1$  implies that  $t_S^* < t_S^\dagger(t_w^*) = t_S^{**}$ . By Lemma 7, this means  $\frac{\partial V_P(t_1^*, t_2^*; p_S, 0)}{\partial t_S^*} > 0$ . Moreover, by Lemma 12,  $t_S^* > 0$ , Proposition 4 implies that  $\frac{\partial t_S^*}{\partial \alpha} > 0$ . As a result,

$$\frac{dV_P(t_1^*, t_2^*)}{d\alpha} = \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} \frac{\partial t_S^*}{\partial \alpha} > 0.$$

Thus,  $\alpha^*$  cannot be optimal. Next, suppose the regular equilibrium is unique for all  $\alpha \in [\frac{1}{2}, 1]$ . Then, Berge maximum theorem implies that the equilibrium exit time is continuous in  $\alpha$ . The fact that  $t_w^* = 0$  and  $INB(p_w, t_w^*, p_S, t_w^*; \alpha^*) < 0$  implies that 0 must remain an equilibrium exit time for the weak firm after a small increase in  $\alpha$ . Using the same argument, we conclude that  $\frac{dV_P(t_1^*, t_2^*)}{d\alpha} = \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} \frac{\partial t_S^*}{\partial \alpha} > 0$  and  $\alpha^*$  cannot be optimal. Thus,  $INB(p_w, t_w^*, p_S, t_w^*; \alpha^*) = 0$  must hold under the optimal patent  $\alpha^*$ .

■

**Proof of Proposition 10.** In text. ■

**Proof of Proposition 11.** In text. ■

**Proof of Proposition 12.** Suppose  $p_{S,0} > p_{w,0}$ , we claim that the derivative

$$\frac{dV_P(t_1^*, t_2^*)}{d\alpha} = \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} \frac{\partial t_S^*}{\partial \alpha} + \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_w^*} \frac{\partial t_w^*}{\partial \alpha} \quad (48)$$

is strictly negative at  $\alpha = 1$ . It follows that  $\alpha = 1$  is suboptimal. By Proposition 6, for all  $\alpha \in [\bar{\alpha}, 1]$ , the regular equilibrium is unique. By the implicit function theorem,  $\frac{\partial t_i^*}{\partial \alpha}$  exists for all  $\alpha > \bar{\alpha}$  and  $i \in \{1, 2\}$ . Thus, the derivative (48) exists at  $\alpha = 1$ . Moreover, the assumption of nonextreme prior and Proposition 1 imply that for  $i \in \{1, 2\}$ ,  $t_i^* > 0$  when  $\alpha = 1$ . Proposition 4 then implies that

$$\frac{\partial t_i^*}{\partial \alpha} \Big|_{\alpha=1} > 0.$$

Thus, the derivative (48) exists at  $\alpha = 1$ . Since  $t_S^* = t_S^{**} = t_S^\dagger(t_w^*) > 0$ ,  $\frac{\partial V_P(t_1^*, t_2^*)}{\partial t_S^*} = 0$ . Moreover,

$$\begin{aligned} \frac{\partial V_P(t_1^*, t_2^*)}{\partial t_w^*} &= e^{-\int_0^{t_w^*} (p_{1,s} + p_{2,s}) \lambda ds} \{p_{w,t_w^*} \lambda (v - U_P^w(t_w^*, t_S^*)) - c\} \\ &< e^{-\int_0^{t_w^*} (p_{1,s} + p_{2,s}) \lambda ds} \{p_{w,t_w^*} \lambda v - c\} \\ &= 0. \end{aligned}$$

where the inequality follows from the fact that  $t_S^{**} = t_S^* > t_w^*$ . Thus,  $\frac{dV_P(t_1^*, t_2^*)}{d\alpha} < 0$ . The designer gains by reducing  $\alpha$  for a small amount from 1. ■

**Proof of Proposition 13.** In text. ■

**Proof of Proposition 14.** As pointed out in the main text, we only need to show that the derivative

$$\begin{aligned} &\frac{dV_P(t_1^*(\alpha^*, p_{S,0}), t_2^*(\alpha^*, p_{S,0}); p_{S,0})}{dp_{S,0}} \\ &= \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial p_{S,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_S^*} \frac{\partial t_S^*}{\partial p_{S,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_w^*} \frac{\partial t_w^*}{\partial p_{S,0}} \end{aligned}$$

is strictly positive for all  $p_{S,0} > p_{w,0}$ . Consider the first term, we have

$$\begin{aligned}
& \frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial p_{S,0}} \\
&= p_{w,0} \left\{ e^{-\lambda t_w^*} \left( 1 - e^{-\lambda t_S^*} \right) \left( v - \frac{c}{\lambda} \right) + \left( 1 - e^{-\lambda t_w^*} \right) \frac{c}{\lambda} \right\} + p_{w,0} e^{-\lambda t_w^*} (t_S^* - t_w^*) c \\
& \quad + (1 - p_{w,0}) \left\{ \left( 1 - e^{-\lambda t_S^*} \right) \left( v - \frac{c}{\lambda} \right) + c t_S^* + c \left( e^{-\lambda t_w^*} - (1 - \lambda t_w^*) \right) \right\} \\
&> 0,
\end{aligned}$$

where we have used the fact that for all  $x \in \mathbb{R}$ ,  $e^x \geq 1+x$  and the fact that  $t_S^* > 0$ . Consider the second term, since  $t_S^* \leq t_S^\dagger(t_w^*) = t_S^{**}$ , by Lemma 7,  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_S^*} \geq 0$ . Moreover, By Lemma 12,  $t_S^* > 0$ , (8) implies

$$\frac{\partial t_S^*}{\partial p_{S,0}} = \frac{1}{\lambda p_{S,0} (1 - p_{S,0})} > 0.$$

The second term is also positive. Consider the third term, by the first part of Lemma 13,  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_w^*} \leq 0$ . By the second part of Lemma 13, (9) holds at equality. By Propositions 6 and 11, the regular equilibrium is unique under the optimal patent. The implicit function theorem implies that

$$\frac{\partial t_w^*}{\partial p_{S,0}} = - \frac{\frac{\partial INB(p_{w,t}, p_{S,t}; \alpha) \big|_{t=t_w^*}}{\partial p_{S,t}} \frac{\partial p_{S,t_w^*}}{\partial p_{S,0}}}{\frac{\partial INB(p_{w,t}, p_{S,t}; \alpha) \big|_{t=t_w^*}}{\partial t}}.$$

The weak firm's second order condition implies that  $\frac{\partial INB(p_{w,t}, p_{S,t}; \alpha^*) \big|_{t=t_w^*}}{\partial t} < 0$ . Moreover,

$$\frac{\partial INB(p_{w,t}, p_{S,t}; \alpha^*) \big|_{t=t_w^*}}{\partial p_{S,t}} = -p_{w,t_w^*} \lambda \left( \frac{\lambda \alpha^* v}{\lambda \alpha^* v - c} \right) (1 - \alpha) v < 0,$$

where we have used the fact that  $t_S^* > 0$  so that  $\lambda \alpha^* v > p_{S,t_S^*} \lambda \alpha^* v = c$ . Moreover, differentiating (1) yields

$$\frac{\partial p_{S,t_w^*}}{\partial p_{S,0}} = \frac{e^{-\lambda t_w^*}}{(p_{S,0} e^{-\lambda t_w^*} + 1 - p_{S,0})^2} > 0.$$

The third term is also positive. ■

**Proof for Example 3.** As in the proof of Proposition 14, we can show that the derivative

$$\begin{aligned} & \frac{dV_P(t_1^*(\alpha^*, p_{w,0}), t_2^*(\alpha^*, p_{w,0}); p_{w,0})}{dp_{w,0}} \\ = & \frac{\partial V_P(t_1^*, t_2^*; p_{w,0})}{\partial p_{w,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{w,0})}{\partial t_S^*} \frac{\partial t_S^*}{\partial p_{w,0}} + \frac{\partial V_P(t_1^*, t_2^*; p_{w,0})}{\partial t_w^*} \frac{\partial t_w^*}{\partial p_{w,0}} \end{aligned} \quad (49)$$

is strictly negative for some  $(p_{1,0}, p_{2,0})$  and apply the Envelope Theorem to obtain the desired conclusion. Consider a sequence of nonextreme priors  $\{(p_{1,0}^n, p_{2,0}^n)\}$  with  $p_{1,0}^n \rightarrow 1$  and  $p_{2,0}^n \rightarrow \frac{c}{\lambda v}$ , we claim that (49) is strictly negative for  $n$  large enough. To do so, we first show that for  $n$  large enough,  $t_2^{*n} = 0$ .

**Claim 1** *For  $n$  large enough, the weak firm exits at time 0 in the regular equilibrium under the optimal patent  $\alpha_n^*$ , i.e.,  $\exists N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $t_2^{*n} = 0$ .*

Suppose not. Take a subsequence if necessary, we have, for each  $n \in \mathbb{N}$ ,  $t_2^{*n} > 0$ . By Propositions 6 and 11, for each  $n \in \mathbb{N}$ , the optimal patent  $\alpha_n^*$  is larger than  $\bar{\alpha}$  and the regular equilibrium is unique. For each  $n \in \mathbb{N}$ , the optimal patent  $\alpha_n^*$  satisfies the first order condition

$$\frac{dV_P(t_1^{*n}, t_2^{*n})}{d\alpha} = \frac{\partial V_P(t_1^{*n}, t_2^{*n})}{\partial t_1^{*n}} \frac{\partial t_1^{*n}}{\partial \alpha} + \frac{\partial V_P(t_1^{*n}, t_2^{*n})}{\partial t_2^{*n}} \frac{\partial t_2^{*n}}{\partial \alpha} \geq 0. \quad (50)$$

Since for all  $n \in \mathbb{N}$ ,  $INB(p_{2,t}^n, p_{1,t}^n; \alpha_n^*) \leq p_{2,t}^n \lambda v - c$  for all  $t \geq 0$ , the fact that  $p_{2,0}^n \rightarrow \frac{c}{\lambda v}$  and  $t_2^{*n} > 0$  implies that  $\alpha_n^* \rightarrow 1$ . Direct differentiating (8) yields,

$$\lim_{n \rightarrow \infty} \frac{\partial t_1^{*n}}{\partial \alpha} \Big|_{\alpha = \alpha_n^*} = \frac{v}{\lambda v - c} > 0.$$

(9) and the implicit function theorem imply

$$\lim_{n \rightarrow \infty} \frac{\partial t_2^{*n}}{\partial \alpha} \Big|_{\alpha = \alpha_n^*} = - \lim_{n \rightarrow \infty} \frac{\frac{\partial INB(p_{2,t}^n, p_{1,t}^n; \alpha)}{\partial \alpha} \Big|_{t=t_w^{*n}, \alpha = \alpha_n^*}}{\frac{\partial INB(p_{2,t}^n, p_{1,t}^n; \alpha_n^*)}{\partial t} \Big|_{t=t_w^{*n}, \alpha = \alpha_n^*}} = \frac{2v}{\lambda v - c} > 0.$$

Since  $\alpha_n^* \rightarrow 1$ , we have

$$\lim_{n \rightarrow \infty} \frac{\partial V_P(t, t_2^{*n})}{\partial t} \Big|_{t=t_1^{*n}} = \lim_{n \rightarrow \infty} \frac{\partial V_1(t, t_2^{*n})}{\partial t} \Big|_{t=t_1^{*n}} = 0,$$

Moreover,  $p_{2,0}^n \rightarrow \frac{c}{\lambda v}$  implies that  $t_2^{*n} \rightarrow 0$ . As  $p_{1,0}^n \rightarrow 1$ ,  $t_1^{*n} \rightarrow \infty$ , we have, by (26),

$$\lim_{n \rightarrow \infty} \frac{\partial V_P(t_1^{*n}, t)}{\partial t} \Big|_{t=t_2^{*n}} = - \left( \frac{\lambda v - c}{\lambda v} \right) c < 0. \quad (51)$$

Thus, (50) cannot be satisfied for  $n$  large enough, we have must have  $t_2^{*n} = 0$  instead.

**Claim 2** For  $n$  large enough, the derivative  $\frac{dV_P(t_1^*, t_2^*; p_{2,0})}{dp_{2,0}} \Big|_{(p_{1,0}, p_{2,0}) = (p_{1,0}^n, p_{2,0}^n)}$  (49) is strictly negative under the optimal patent  $\alpha_n^*$ .

By Claim 1, for  $n$  large enough,  $t_2^{*n} = 0$  under the optimal patent  $\alpha^*$ . Take any such  $n$  and drop the superscript/subscript from this point on. Consider the first term in (49), we have

$$\begin{aligned} & \frac{\partial V_P(t_1^*, t_2^*; p_{w,0})}{\partial p_{w,0}} \\ &= p_{S,0} \left( 1 - e^{-\lambda(t_1^* + t_2^*)} \right) \left( v - \frac{c}{\lambda} \right) \\ & \quad - p_{S,0} \left( \left( 1 - e^{-\lambda t_w^*} \right) \left( v - \frac{2c}{\lambda} \right) + \left( e^{-\lambda t_w^*} - e^{-\lambda t_S^*} \right) \left( v - \frac{c}{\lambda} \right) \right) \\ & \quad + (1 - p_{S,0}) \left( \left( 1 - e^{-\lambda t_w^*} \right) \left( v - \frac{2c}{\lambda} \right) - e^{-\lambda t_w^*} (t_S^* - t_w^*) c \right) \\ & \quad + (1 - p_{S,0}) (t_1^* + t_2^*) c \\ &= 0. \end{aligned}$$

Moreover, by (8), the strong firm's equilibrium exit time is independent of  $p_{w,0}$ , so  $\frac{\partial t_1^*}{\partial p_{2,0}} = 0$ . The second term is also zero. Consider the third term, by (51),  $\frac{\partial V_P(t_1^*, t_2^*; p_{S,0})}{\partial t_2^*} < 0$ . By the second part of Lemma 13, (9) holds at equality. By Propositions 6 and 11, the equilibrium is unique under the optimal patent. The implicit function theorem implies that

$$\frac{\partial t_2^*}{\partial p_{2,0}} = - \frac{\frac{\partial INB(p_{2,t}, p_{1,t}; \alpha^*)}{\partial p_{2,t}} \Big|_{t=t_2^*} \frac{\partial p_{2,t_2^*}}{\partial p_{2,0}}}{\frac{\partial INB(p_{2,t}, p_{1,t}; \alpha^*)}{\partial t} \Big|_{t=t_2^*}}.$$

The weak firm's second order condition implies that  $\frac{\partial INB(p_{2,t}, p_{1,t}; \alpha^*)}{\partial t} \Big|_{t=t_2^*} < 0$ . Moreover,

$$\frac{\partial INB(p_{2,t}, p_{1,t}; \alpha^*)}{\partial p_{2,t}} \Big|_{t=t_2^*} = \lambda (\alpha^* v - \check{W}(p_{1,t_2^*}; \alpha^*)) > 0,$$

as  $\check{W}(p_{1,t_2^*}; \alpha) \leq (1 - \alpha^*) v \leq \alpha^* v$ . Moreover, since  $t_2^* = 0$ ,

$$\frac{\partial p_{2,t_2^*}}{\partial p_{2,0}} = 1 > 0.$$

The third term is *strictly negative*. ■

**Proof of Proposition 15.** We have already shown in the main text that the distribution policy given by (32) and (33) implements the planner's solution. Suppose  $INB_P^w(p_{w,t^{**}}, p_{S,t^{**}}) < 0$ , by Proposition 8,  $t_w^{**} = 0$ . There exists a open ball  $B$  around  $(p_{1,0}, p_{2,0})$  such that for all  $(p'_{1,0}, p'_{2,0}) \in B$ ,

$$\alpha_w^{**} = 1 - \frac{\hat{W}(p'_{S,0}; 1)}{v}$$

Therefore,

$$\frac{d\alpha_w^{**}}{dp_{w,0}} = 0,$$

and

$$\frac{d\alpha_w^{**}}{dp_{S,0}} = -\frac{1}{v} \frac{d\hat{W}(p_{S,0}; 1)}{dp_{S,0}} \leq 0.$$

Next, suppose  $INB_P^w(p_{w,t^{**}}, p_{S,t^{**}}) = 0$ , total differentiate (33), we have,

$$\frac{d\alpha_w^{**}}{dp_{w,0}} = \frac{\partial \alpha_w^{**}}{\partial t_w^{**}} \frac{dt_w^{**}}{dp_{w,0}}.$$

By Proposition 8,  $\frac{dt_w^{**}}{dp_{w,0}} \geq 0$ . Moreover,

$$\begin{aligned} \frac{\partial \alpha_w^{**}}{\partial t_w^{**}} &= -\frac{1}{v} \frac{d\hat{W}(p_{S,t^{**}}; 1)}{dp_{S,t^{**}}} \frac{dp_{S,t^{**}}}{dt_w^{**}} \\ &= \frac{1}{v} \frac{d\hat{W}(p_{S,t^{**}}; 1)}{dp_{S,t^{**}}} \lambda p_{S,t^{**}} (1 - p_{S,t^{**}}) \\ &\geq 0. \end{aligned}$$

Therefore,  $\alpha_w^{**}$  increases with  $p_{w,0}$ . Next, using (30) to rewrite (33) into

$$\alpha_w^{**} = \frac{c}{\lambda v p_{w,t_w^{**}}}. \quad (52)$$

Total differentiate (52), we have,

$$\begin{aligned} \frac{d\alpha_w^{**}}{dp_{S,0}} &= -\frac{c}{\lambda v p_{w,t_w^{**}}^2} \frac{dp_{w,t_w^{**}}}{dt_w^{**}} \frac{dt_w^{**}}{dp_{S,0}} \\ &= \frac{1 - p_{w,t_w^{**}}}{p_{w,t_w^{**}}} \frac{c}{v} \frac{dt_w^{**}}{dp_{S,0}}, \end{aligned}$$

which is negative by Proposition 8. ■