OPTIMAL DYNAMIC MECHANISM DESIGN WITH DEADLINES

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Abstract. A dynamic mechanism design problem with multi-dimensional private information is studied. There is one object and two buyers who arrive in two different periods. In addition to his privately known valuation, the first buyer also has a privately known deadline for purchasing the object. The seller wants to maximize revenue.

Depending on the type distribution, the incentive compatibility constraint for the deadline may or may not be binding in the optimal mechanism. Sufficient conditions on the type distribution and examples are given for either case.

An optimal mechanism for the binding case is derived. It can be implemented by a fixed price in period one and an asymmetric auction in period two. The asymmetry prevails even if the valuations of both buyers are identically distributed. In order to prevent buyer one from buying in the first period when his deadline is two, the seller sets a reserve price that is lower than in the classical (Myerson, 1981) optimal auction and gives him a (non-linear) bonus. The bonus leads to robust bunching at the top of the type-space. The optimal mechanism can be characterized in terms of generalized virtual valuations which depend endogenously on the allocation rule.
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1. Introduction

This paper analyzes the problem of a seller facing two buyers in a private values environment, who arrive in two different time periods. The seller has a single indivisible object that can be allocated only once. He wants to maximize his expected revenue. Both buyers have private information about their valuations. In addition, the first buyer has a privately known deadline. The deadline determines whether he will still be interested in buying the object in the second period.

In many cases, buyers have deadlines that are imposed by third parties. Consider for example a company that needs to buy a good from a seller in order to enter a

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contractual relationship with a third party. The good could be a physical object, an option contract, a license, a patent etc. It is conceivable that the third party sets a deadline after which the contractual relationship is no longer available. Therefore, the object becomes worthless for the company if it is purchased after the deadline. Other examples of dynamic allocation problems in which buyers can have deadlines are online auctions (think for example of buying a birthday present at eBay), the sale of airline tickets, hotel reservations or the sale of houses and real estate.

So far, most of the literature on dynamic mechanism design has abstracted from private information about deadlines (see section 1.3). This abstraction leads to mechanism design problems with one-dimensional private information which are tractable under quite general assumptions about the dynamic arrival of new bidders and new objects. In this paper, we take a different direction and allow for private information about the deadline. This yields a two-dimensional adverse selection problem. To keep the analysis tractable, a simple model with two buyers and two periods is studied.

1.1. Summary of Results

As a benchmark, consider the *relaxed problem* of maximizing revenue when the deadline is commonly known. If the deadline is one, the seller faces one short-lived buyer in each period. The optimal mechanism is to set fixed prices in both periods. If the deadline is two, the seller can wait until the second buyer arrives without losing the opportunity to sell to the first buyer. Hence, his revenue is maximized if he conducts an optimal static auction with both buyers in the second period.

We show that for a class of type distributions, the relaxed solution is not incentive compatible. We use the virtual valuation function of buyer one to identify two effects that lead to a violation of incentive compatibility.[1] Firstly, the deadline determines the mode of competition between buyer one and buyer two. For deadline one, the seller sets a fixed price that reflects his opportunity cost of selling to buyer one. In this sense, buyer one competes indirectly with buyer two through the fixed price. For deadline two, competition is direct because both buyers participate in an auction. We show that buyer one has a high valuation, a concave virtual valuation leads to higher average prices in the case of direct competition.[2] To avoid higher average prices, a buyer with deadline two should lie about his deadline in this case. The converse effect arises for convex virtual valuations. Several examples of distributions with convex and concave virtual valuations, respectively, are provided.

[1]The virtual valuation expresses the marginal expected revenue from selling to a buyer.
[2]In static models (e.g. Iyengar and Kumar (2008)) this effect does not arise because the second dimension of private information does not affect the mode of competition.

The second effect arises if deadline and valuation are not independently distributed. In this case, buyer one provides information about the virtual valuation by reporting his deadline. If the virtual valuation decreases in the deadline, the seller uses this information to set on average higher prices in the second period. Again, a buyer with deadline two should lie and report deadline one to avoid higher prices.[3]

Now we turn to the main result of the paper—the optimal mechanism for the case that the relaxed solution is not incentive compatible. One instrument to achieve incentive compatibility is to change the auction format in the second period. The seller must take into account that buyer one has the *outside option* to buy in period one. This leads to an asymmetric auction that favors buyer one in order to make the auction more attractive. Since the asymmetry results from the buyer's outside option, it prevails even if both buyers have identically distributed valuations. A second instrument to achieve incentive compatibility, is an increase of the fixed price in period one. This makes the outside option relatively less attractive. We show that in the optimal solution, the seller employs both instruments.

In the auction, the distortion has several consequences. Firstly, the reserve price for buyer one is lower than in the relaxed solution. Secondly, for intermediate valuations, winning probabilities are higher than in the relaxed solution. Finally, for valuations close to the top, buyer one always wins the auction and the expected price he has to pay equals the fixed price in the first period. For these valuations, it is not optimal for the seller to discriminate between buyers with different deadlines and valuations. This is in contrast to the efficient allocation rule and the relaxed solution. The introduction of private information about the deadline leads to bunching at the top of the type-space that cannot be found in dynamic mechanism design models with one-dimensional private information. Bunching occurs in all cases where the relaxed solution is not incentive compatible.

Albeit the distortions, the optimal allocation rule in period two has a similar structure as in the classical auction model. For each type, there is a (generalized) virtual valuation that can be interpreted as the marginal expected revenue from raising the winning probability of this type. A buyer wins if and only if his (generalized) virtual valuation is non-negative and higher than that of his opponent. In contrast to the classical model, the generalized virtual valuation has a parameter that depends endogenously on the allocation rule. This parameter determines the magnitude of

[3]A related effect arises in models of dynamic learning (Gershkov and Moldovanu, 2009a,b). There, incentive compatibility of the efficient allocation rule is destroyed, because of an informational externality that arises when a buyer reveals information about other buyers' type distributions. In the present paper, the effect is more direct. Buyer one reveals information about his *own* type distribution which is used by the revenue maximizing allocation rule. The efficient allocation rule, on the other hand, does not depend on this information and remains incentive compatible.

the distortion compared to the relaxed solution (where the parameter vanishes). A simple procedure to compute the optimal distortion is provided. Using the generalized virtual valuation function, it is straight forward to define an ascending clock auction that implements the optimal mechanism in the second period.

1.2. Methods

In the auction problem, we have to deal with a type-dependent participation constraint for buyer one, because he can choose to buy in the first period. The participation constraint is defined in terms of the interim expected utility. Therefore, Myerson's (1981) classical approach to solve the optimal auction problem by point-wise maximization is not applicable. Instead, the feasibility constraint, i.e. the condition that the object be allocated only once, is formulated in terms of interim winning probabilities. We use the characterization of feasibility of asymmetric reduced form allocation rules in Mierendorff (2008) to solve the resulting control problem.[4]

If the interim winning probability of buyer one is absolutely continuous, the feasibility constraint can be substituted into the objective function. We get a standard control problem in which the winning probability is a state variable and its derivative is the control. (See Guesnerie and Laffont (1984) for an early application of this method). To allow for jumps in the winning probability, the control problem is first solved under the assumption that the winning probability is Lipschitz continuous (and hence absolutely continuous). The Lipschitz solutions converge to an optimal solution of the general problem if the Lipschitz constant approaches infinity. This method was pioneered by Reid (1968) and seems to be new to the mechanism design literature. It may be useful in other auction models where continuity of the winning probability is not guaranteed.[5]

Reid also provides a method to show that Myerson's ironing procedure can be applied to ensure monotonicity of the winning probability. This is important in the context of this paper because the usual hazard rate assumption on the type distribution which ensures a non-decreasing virtual valuation does not guarantee monotonicity of the *generalized* virtual valuation.[6]

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[4]The characterization is a generalization of Border (1991) who studies symmetric allocation rules. Matthews (1984) conjectured the result proven by Border (see also Chen, 1986). For an early application of a special case of the result see Maskin and Riley (1984). Che, Condorelli and Kim (private communication) independently derived the characterization of feasibility in Mierendorff (2008).

[5]Recently, Hellwig (2008) has derived a version of Pontriyagin's maximum principle that allows for a monotonicity constraint on the control variable without requiring absolute continuity. This is not applicable here, however, as we have to deal with the non-standard feasibility constraint.

[6]Except for Myerson's (1981) paper, which does not use control theory, there does not seem to be a full-fledged solution technique for the (valuation-)bunching case. Guesnerie and Laffont (1984) and earlier Mussa and Rosen (1978) derive necessary conditions for bunches, but do not give precise conditions on the location of bunches.

1.3. Related Literature

The literature on dynamic revenue maximization emerged from the literature on dynamic pricing and revenue management. For a survey, see for example Elmaghraby and Keskinocak (2003). McAfee and te Velde (2007) survey airline pricing. This literature typically assumes stochastic demand and abstracts from strategic buyers. If buyers are short-lived and only one buyer is present at the same time, this is a reasonable assumption and the optimal mechanism is a sequence of posted prices (Das Varma and Vettas, 2001). Gallien (2006) shows that a sequence of posted prices is the optimal strategy-proof mechanism for the sale of an inventory of identical objects to long-lived buyers with a commonly known discount factor. He gives conditions on the arrival time distribution that ensure that buyers are served only upon arrival, providing some justification for this assumption in the revenue management literature. More recently, Gershkov and Moldovanu (2008) derived the revenue maximizing policy for an inventory of heterogeneous but commonly ranked objects.

Another strand of literature considers discrete time models in which several buyers can arrive simultaneously. Vulcano, van Ryzin, and Maglaras (2002) show that a sequence of auctions with appropriately chosen reserve prices maximizes revenue if bidders are short-lived. Said (2008) considers a model with stochastic arrival and exit of bidders that have unit demand and a commonly known discount factor. A random number of perishable objects is available in each period. The optimal allocation rule awards the objects to the bidders with the highest virtual valuations. It can be implemented by a sequence of open auctions with suitable reserve prices.

Several papers have shown the implementability of the efficient allocation rule in dynamic settings (Parkes and Singh (2003), Bergemann and Välimäki (2008), Athey and Segal (2007)). Said (2008) also considers the efficient allocation rule for the model described above and shows that the mechanism derived in Bergemann and Välimäki (2008) can be implemented by a sequence of open auctions. Mierendorff (2009) shows that the efficient allocation of a single object with stochastically arriving long-lived bidders with privately known time preferences only requires transfers between the seller and the winning bidder.

Pavan, Segal, and Toikka (2008) consider a very general dynamic mechanism design model with a fixed set of agents who receive one-dimensional private information in every period. For a special case, the revenue-maximizing allocation rule can be described in terms of virtual surplus in this model.

While general multi-dimensional mechanism design models are very complex to analyze (see e.g. Armstrong (1996), Rochet and Choné (1998) and Jehiel, Moldovanu, and Stacchetti (1999)), several authors have analyzed two-dimensional models with additional structure on the second dimension of private information (the deadline in the present case). Firstly, in these models, agents only have feasible deviations in one direction of the second dimension (i.e. report earlier deadlines but not later deadlines). Secondly, the second parameter does not directly enter the expected utility of an agent (i.e. the true deadline is immaterial as long as the agent receives the object before the deadline.) See Beaudry, Blackorby, and Szalay (2009) for an analysis of optimal taxation; Blackorby and Szalay (2008) and Szalay (2009) for regulation; Iyengar and Kumar (2008) for a static auction model with capacitated bidders; Dizdar, Gershkov, and Moldovanu (2009) for a dynamic model with capacitated bidders; and Che and Gale (2000) and Malakhov and Vohra (2005) for static models with budget constrained buyers.

Closest to the present paper is Pai and Vohra (2008), who consider a more general dynamic allocation problem with buyers who have privately known deadlines. They show that the relaxed solution is incentive compatible if the virtual valuation is "sufficiently monotone" in the deadline. Their condition, however, cannot be applied directly to the primitives of the model (i.e. the type distribution).

Szalay (2009) is the only other paper that derives the optimal mechanism for the case that the incentive compatibility constraint in the second dimension is binding. All other papers make assumptions that guarantee that the relaxed solution solves the general problem. Jullien (2000) studies a principal-agent problem with type dependent participation constraints. The analysis of the auction model in the present paper, however, requires different solution techniques than the models of Szalay (2009) and Jullien (2000) because of the (non-standard) feasibility constraint and the possibility of discontinuities in the optimal solution.

1.4. Organization of the paper

Section 2 describes the model. Section 3 states and solves the sellers problem. Section 3.1 presents a characterization of incentive compatibility. Section 3.2 decomposes the seller's problem into two subproblems which are linked by the incentive constraint for the deadline. In the first problem, revenue for deadline one is maximized. This is solved in section 3.3. Section 3.4 gives conditions under which the relaxed problem is (not) incentive compatible. Section 3.5 presents the general solution of the second maximization problem (revenue maximization for deadline two). Section 3.6 combines the two solutions and discusses the optimal mechanism. Section 4 concludes with a discussion of generalizations of the model. The formal derivation of the optimal mechanism is developed in Appendix A.

2. The Model

A seller wants to maximize the revenue from selling a single indivisible object. The seller's valuation is normalized to zero. There are two time periods and two potential buyers $i = 1, 2$.

Buyer 1 arrives in the first period. He has valuation $v_1 \in [0, \overline{v}]$, $\overline{v} > 0$, and a deadline $d \in \{1, 2\}$. Both variables are private information. If he gets the object in the first period and has to pay p, then his payoff is $v_1 - p$. If he gets the object in the second period and has to pay p, then his payoff is $v_1 - p$ if $d = 2$, and $-p$ if $d = 1$. Buyer 2 and the seller regard v_1 and d as jointly distributed random variables. The distribution function of v_1, conditional on d, is $F_1(.|d)$ with density $f_1(.|d)$. The probability that $d = 1$ is denoted by ρ.

Buyer 2 arrives in the second period. This means that the object cannot be sold to him in the first period and that his valuation, $v_2 \in [0, \overline{v}]$, is not known to anybody in the first period. In the second period, v_2 is private information of buyer two. Buyer 1 and the seller regard v_2 as a random variable with distribution function F_2 and density f_2.

Both buyers are risk neutral and their payoffs are zero if they do not receive the object and their payments are zero. Buyers and sellers do *not* discount future payoffs.[7] (v_1, d) and v_2 are stochastically independent. F_i and ρ are commonly known. We assume that $f_1(v|d)$ and $f_2(v)$ are continuous in v and strictly positive for all $v \in [0, \overline{v}]$. Furthermore they are continuously differentiable in v for $v \in (0, \overline{v})$, and $f_1'(.|d)$ and f_2' can be extended continuously to $[0, \overline{v}]$. To avoid additional technicalities, *virtual valuations* of both buyers are assumed to be strictly increasing.

Assumption 1. *For both buyers, the* virtual valuations $J_1(v|d) := v - \frac{1-F_1(v|d)}{f_1(v|d)}$ *and* $J_2(v) := v - \frac{1-F_2(v)}{f_2(v)}$ *are strictly increasing in* v.

The zeros of $J_1(.|d)$ and J_2 are denoted $v_1^0|d$ and v_2^0, respectively. In the case that the relaxed solution is not incentive compatible, the following additional assumption on the type distribution of buyer one will be used to show that the seller uses a fixed price in the first period rather than a stochastic mechanism.

Assumption 2. $J_1(v|1)f_1(v|1)$ *is strictly increasing for* $v \in [v_1^0|1, 1]$.

A sufficient condition for Assumption 2 to be fulfilled is that $F_1(.|1)$ satisfies Assumption 1 and has a non-decreasing density $f_1(.|1)$.

[7] If buyer one and the seller only discount payments and have a common discount factor, the results in this paper do not change. See also section 4 for a discussion of discounting.

2.1. Mechanisms

The seller's goal is to design a mechanism that has a Bayes-Nash-Equilibrium which maximizes his expected revenue. In general, a mechanism can be any game form with two stages, such that only buyer one is active in the first stage. The second stage is only reached if the object has not been allocated in the first stage. We assume that the mechanism designer can choose to conceal any information about the first stage from buyer two, except the allocation decision.[8,9] We also assume that the seller can commit ex-ante to a mechanism.

By the revelation principle, the seller can restrict attention to incentive compatible and individually rational direct mechanisms in which no information is revealed to buyer two.

Definition 1. A *direct mechanism* consists of message spaces $S_1 = [0, \overline{v}] \times \{1, 2\}$ and $S_2 = [0, \overline{v}]$, an *allocation rule* x which consists of three functions $x_1^1(v_1, d) \in [0, 1]$, $x_1^2(v_1, d, v_2) \in [0, 1]$, and $x_2(v_1, d, v_2) \in [0, 1]$, that satisfy the feasibility constraint

$$\forall (v_1, d) \in S_1,\ \forall v_2 \in S_2: \qquad x_1^1(v_1, d) + x_1^2(v_1, d, v_2) + x_2(v_1, d, v_2) \leq 1, \tag{F}$$

and a *payment rule* y that consists of three functions $y_1^1(v_1, d) \in \mathbb{R}$, $y_1^2(v_1, d, v_2) \in \mathbb{R}$, and $y_2(v_1, d, v_2) \in \mathbb{R}$.

In this definition, x_i^t denotes the probability that the object is allocated to buyer i in period t, for a given type profile. y_i^t are expected payments made by player i in period t, for a given type profile. Note that x_1^1 and y_1^1 only depend on (v_1, d) because v_2 is not known to anybody in period one.

Assuming that the respective other buyer reports his type truthfully, buyers have the following interim winning probabilities if they participate in a direct mechanism (x, y):

$$\begin{aligned}
q_1^1(v_1, d) &= x_1^1(v_1, d), \\
q_1^2(v_1, d) &= \int_0^{\overline{v}} x_1^2(v_1, d, v_2) f_2(v_2) dv_2 \\
q_2(v_2) &= \frac{1}{\nu} \int_0^{\overline{v}} x_2(v_1, 1, v_2)\, \rho f_1(v_1|1) + x_2(v_1, 2, v_2)\, (1-\rho) f_1(v_1|2) dv_1, \\
\text{where} \quad \nu &= \int_0^{\overline{v}} (1 - x_1^1(s, 1))\, \rho f_1(s|1) + (1 - x_1^1(s, 2))\, (1-\rho) f_1(s|2) ds.
\end{aligned}$$

[8]The allocation decision is observed by buyer two because the game reaches stage two if and only if the object was not allocated in the first stage.

[9]The results of this paper do not change if buyer two receives a signal about buyer one's deadline before choosing his actions in stage two.

Interim expected payments are given by

$$p_1(v_1,d) = y_1^1(v_1,d) + \int_0^{\overline{v}} y_1^2(v_1,2,v_2)\, f_2(v_2)dv_2,$$

$$p_2(v_2) = \frac{1}{\nu}\int_0^{\overline{v}} y_2(v_1,1,v_2)\,\rho f_1(v_1|1) + y_2(v_1,2,v_2)\,(1-\rho)f_1(v_1|2)dv_1.$$

We call (q,p) the *reduced form* of a direct mechanism (x,y). ν denotes the probability that the object has not been allocated in the first period. The interim winning probabilities and payments of buyer two are defined conditional on this event because buyer two observes the allocation decision in period one. The interim expected payments of buyer one have been aggregated over periods because neither the seller nor buyer one discounts payments.

Expected utilities from participating in the mechanism (x,y), with reduced form (q,p), if the respective other buyer reports his type truthfully, are given by

$$U_1(v,d,v',d') = \left[q_1^1(v',d') + q_1^2(v',d')\mathbf{1}_{\{d=2\}}\right] v - p_1(v',d'),$$

$$\text{and} \quad U_2(v,v') = q_2(v')v - p_2(v'),$$

where v,d denote true types and v',d' denote reported types. $\mathbf{1}_X$ denotes the indicator function of a set X. The indirect utilities from truth telling are abbreviated as $U_1(v_1,d) = U_1(v_1,d,v_1,d)$, and $U_2(v_2) = U_2(v_2,v_2)$, respectively.

Definition 2. (i) A direct mechanism (x,y) is *(Bayesian) incentive compatible*, if for all $v,v' \in [0,\overline{v}]$ and $d,d' \in \{1,2\}$

$$U_1(v,d) \geq U_1(v,d,v',d'), \tag{$IC1$}$$

$$\text{and} \quad U_2(v) \geq U_2(v,v'). \tag{$IC2$}$$

(ii) A direct mechanism is *individually rational*, if for all $v \in [0,\overline{v}]$, $d \in \{1,2\}$,

$$U_1(v,d) \geq 0, \quad \text{and} \quad U_2(v) \geq 0. \tag{IR}$$

3. Revenue Maximization

By the revelation principle, the seller's maximization problem (denoted $\mathcal{P}$) is given by

$$\max_{(q,p)} \int_0^{\overline{v}} p_1(v_1,1)\,\rho f_1(v_1|1) + p_1(v_1,2)\,(1-\rho)f_1(v_1|2)dv_1 + \nu\int_0^{\overline{v}} p_2(v_2)f_2(v_2)dv_2, \tag{$\mathcal{P}$}$$

where (q,p) must be the reduced form of an incentive compatible and individually rational direct mechanism.

We will solve the seller's problem in several steps. First, it is shown that the multi-dimensional incentive compatibility constraint of buyer one, $(IC1)$, can be replaced by two one-dimensional constraints. Then, the seller's problem is decomposed into two sub-problems. In the first problem, revenue is maximized conditional on $d = 1$. In the second problem, revenue is maximized for $d = 2$. The two problems are only linked by the incentive compatibility constraint for the deadline. Both subproblems are solved independently and the solutions are connected to a solution of the original problem.

3.1. Constraint Simplification

Without loss of generality, the seller can restrict attention to direct mechanisms with an allocation rule that only allocates at the deadline.

Definition 3. An allocation rule x *allocates only at the deadline* if

$$x_1^1(v_1, 2) = 0 \text{ and } x_1^2(v_1, 1) = 0, \quad \forall v_1 \in [0, \overline{v}].$$

Lemma 1. *Let (x, y) be a direct mechanism that satisfies $(IC1)$–$(IC2)$ and (IR). Then, there exists an allocation rule $\hat{x}$ that allocates only at the deadline, such that the direct mechanism $(\hat{x}, y)$ also satisfies $(IC1)$–$(IC2)$ and (IR). (x, y) and $(\hat{x}, y)$ yield the same expected revenue.*

Proof. See Appendix B. □

In the rest of the paper, only mechanisms that allocate at the deadline will be considered. Hence, we simplify notation and write $q_1(v, 1)$ instead of $q_1^1(v, 1)$, and $q_1(v, 2)$ instead of $q_1^2(v, 2)$ and analogously for x_1^t. The feasibility condition can now be stated separately for $(x_1(., 1), x_2(., 1, .))$,

$$x_1(v_1, 1) + x_2(v_1, 1, v_2) \leq 1, \tag{F1}$$

and for $(x_1(., 2, .), x_2(., 2, .))$,

$$x_1(v_1, 2, v_2) + x_2(v_1, 2, v_2) \leq 1. \tag{F2}$$

For an allocation rule that allocates only at the deadline, (F) is fulfilled if and only if $(F1)$ and $(F2)$ are fulfilled.

For mechanisms that allocate only at the deadline, $(IC1)$ can be replaced by one-dimensional constraints for the deadline and the valuation, respectively. To see this, suppose buyer one reports $d' = 1$. In this case, the mechanism does not allocate to him in period two. Therefore, his expected utility is independent of his true deadline, i.e. $U_1(v, 2, v', 1) = U_1(v, 1, v', 1)$. If his true deadline is $d = 1$, one-dimensional deviations of the form $(v_1', 1)$ are not optimal. Therefore, it is also not optimal to report $v_1' \neq v_1$ if the true deadline is $d = 2$. This implies that $(IC1)$ is satisfied for two-dimensional deviations of the form $(v_1', d' = 1)$ if it holds for all deviations (v_1', d) and (v_1, d').

A similar argument can be made to rule out two-dimensional deviations in which buyer one reports $d' = 2$ instead of $d = 1$. In this case, he will not derive utility from getting the object. He is only interested in the payment he has to make when reporting $d' = 2$. This payment is lowest for the lowest valuation. Therefore, only a misreport of $(0, 2)$ has to be ruled out. But reporting $(0, 2)$ is most tempting for a buyer with type $(0, 1)$ because $U_1(v, d)$ is non-decreasing in v. Therefore, it suffices to require $U_1(0, 1) \geq U_1(0, 2, 0, 1)$ which is equivalent to $-p_1(0, 1) \geq -p_1(0, 2)$.[10] The first part of the following theorem summarizes these observations and applies standard characterizations of one-dimensional incentive compatibility for the valuation. The second part strengthens the characterization of incentive compatibility for the case that a fixed price is used in the first period.

Theorem 1. *Let (x, y) be a direct mechanism with reduced form (q, p), that allocates only at the deadline.*

(i) (x, y) is incentive compatible if and only if $q_1(v, d)$ and $q_2(v)$ are non-decreasing as functions of v,

$$\forall v \in [0, \overline{v}],\, d \in \{1, 2\} : \qquad U_1(v, d) = U_1(0, d) + \int_0^v q_1(s, d)ds, \qquad (PE1)$$

$$\forall v \in [0, \overline{v}] : \qquad U_2(v) = U_2(0) + \int_0^v q_2(s)ds, \qquad (PE2)$$

$$\forall v \in [0, \overline{v}] : \qquad U_1(v, 1) \leq U_1(v, 2), \qquad (ICD^d)$$

$$\text{and} \qquad -p_1(0, 1) \geq -p_1(0, 2). \qquad (ICD^u)$$

(ii) If x_1 is deterministic, i.e. $x_1(v_1, 1) \in \{0, 1\}$, then (ICD^d) holds for any v, if it is fulfilled for $v = 0$ and $v = \overline{v}$.

Proof. See Appendix B. □

The intuition for the second part of the theorem is as follows. If the allocation in the first period is deterministic, the seller does not discriminate between buyers with valuations above a fixed price. The whole additional surplus of a higher valuation accrues to the buyer. This means that his expected utility increases one-to-one with

[10] Here, we use that the lower bound of the support of f_1 is zero. If f_1 has support $[\underline{v}, \overline{v}]$ with $\underline{v} > 0$, the upward incentive compatibility constraint for the deadline would be $q_1(\underline{v}, 1)\underline{v} - p_1(\underline{v}, 1) \geq -p_1(\underline{v}, 2)$. See also footnote 11 below.

his valuation if it is above the fixed price. If, instead, he participates in an auction that is run in the second period, the marginal rent left with the buyer cannot be greater than with the fixed price. The slope of the expected utility function cannot be steeper. Therefore, all types with deadline two and a valuation above the fixed price will prefer the auction if the highest type prefers the auction. Similarly, downward incentive compatibility for the deadline is fulfilled for all valuations below the fixed price if it is fulfilled for the lowest valuation.

This is a very powerful result. It will be shown that under Assumption 2, the optimal mechanism uses a deterministic allocation rule in period one. Furthermore, $U_1(0,d) = 0$, $d = 1, 2$, in the optimal mechanism. Hence, (ICD^d) for $v = 0$ and (ICD^u) hold with equality. But then, the incentive compatibility constraint for the deadline is reduced to the single inequality (ICD^d) with $v = \overline{v}$. This greatly simplifies the seller's optimization problem.

3.2. Decomposition of the Seller's Problem

First, we eliminate the payment rule from the seller's objective function. Inserting $(PE1)$–$(PE2)$ into $\mathcal{P}$ and integrating by parts yields

$$\max_{(q,U_1(0,d),U_2(0))} \int_0^{\overline{v}} [q_1(v_1,1)\, J_1(v_1|1)\, \rho\, f_1(v_1|1) + q_1(v_1,2)\, J_1(v_1|2)\, (1-\rho) f_1(v_1|2)]\, dv_1$$
$$+ \nu \int_0^{\overline{v}} q_2(v_2)\, J_2(v_2)\, f_2(v_2) dv_2 - \rho U_1(0,1) - (1-\rho) U_1(0,2) - \nu U_2(0).$$

In order to separate the expected revenue for $d = 1$ from the expected revenue for $d = 2$, we need to define interim winning probabilities of buyer two conditional on the deadline and the event that the object has not been sold in period one:

$$q_2(v_2,1) = \frac{\int_0^{\overline{v}} x_2(v_1,1,v_2) f_1(v_1|1) dv_1}{\int_0^{\overline{v}} (1 - x_1(s,1)) f_1(s|1) ds},$$
$$q_2(v_2,2) = \int_0^{\overline{v}} x_2(v_1,2,v_2) f_1(v_1|2) dv_1.$$

In the definition of $q_2(v_2,2)$, the denominator equals one because $x_1^1(s,2) = 0$ in a mechanism that only allocates at the deadline. With this definition, we have

$$q_2(v_2) = \frac{\rho}{\nu} \left(\int_0^{\overline{v}} (1 - x_1(s,1)) f_1(s|1) ds \right) q_2(v_2,1) + \frac{(1-\rho)}{\nu}\, q_2(v_2,2). \qquad (1)$$

Inserting this, we can restate $\mathcal{P}$ as follows

$$\max_{(q,U_1(0,d),U_2(0))} \rho \int_0^{\overline{v}} \left[q_1(v_1,1)J_1(v_1|1) + (1-q_1(v_1,1)) \int_0^{\overline{v}} q_2(v_2,1)J_2(v_2)f_2(v_2)dv_2 \right] f_1(v_1|1)dv_1$$

$$+ (1-\rho) \int_0^{\overline{v}} q_1(v,2)\, J_1(v|2)\, f_1(v|2) + q_2(v,2)\, J_2(v)\, f_2(v)\, dv$$

$$- \rho U_1(0,1) - (1-\rho)U_1(0,2) - \nu U_2(0) \qquad (\mathcal{P})$$

s.t. (1), $(PE1)$–$(PE2)$, (IR), $(F1)$–$(F2)$,

$q_1(v,1)$, $q_1(v,2)$, $q_2(v) \in [0,1]$ and non-decreasing in v, (2)

$U_1(v,1) \leq U_1(v,2)$ for all $v \in [0,\overline{v}]$, (3)

and $U_1(0,1) \geq U_1(0,2)$. (4)

Observe that $U_1(0,1) = U_1(0,2) = U_2(0) = 0$ in the optimal mechanism. For $U_2(0)$ this is obvious. (3) and (4) imply $U_1(0,1) = U_1(0,2)$. $U_1(0,2)$ could be used to relax (3) because by $(PE1)$, increasing $U_1(0,2)$ raises $U_1(v,2)$ for $v > 0$. But $U_1(0,1)$ and hence $U_1(v,1)$ would have to be increased by the same amount. Therefore it is optimal to set $U_1(0,d) = 0$.[11]

Now we introduce a function $U : [0,\overline{v}] \to [0,\overline{v}]$ that separates $U_1(.,1)$ from $U_1(.,2)$:

$$\forall v \in [0,\overline{v}] : \qquad U_1(v,1) \leq U(v) \leq U_2(v,2).$$

Using U as a parameter, the maximization problem can be rewritten as $\mathcal{P}'$:

$$\max_U \rho\, \pi_1[U] + (1-\rho)\, \pi_2[U] \qquad (\mathcal{P}')$$

$\pi_1[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is one and the utility of the first buyer is constrained by $U_1(v,1) \leq U(v)$ for all $v \in [0,\overline{v}]$. This maximization problem is called $\mathcal{P}_1$:

$$\pi_1[U] := \max_{q_i(.,1)} \int_0^{\overline{v}} \left[q_1(v_1,1)J_1(v_1|1) + (1-q_1(v_1,1)) \int_0^{\overline{v}} q_2(v_2,1)J_2(v_2)f_2(v_2)dv_2 \right] f_1(v_1|1)dv_1 \qquad (\mathcal{P}_1)$$

s.t. $(PE1)$ with $U_1(0,1) = 0$ (5)

$q_i(v,1) \in [0,1]$, (6)

$U_1(v,1) \leq U(v), \quad \forall v \in [0,\overline{v}]$, (7)

and $q_1(v,1)$ is non-decreasing in v. (8)

[11]If the support of f_1 is $[\underline{v},\overline{v}]$ with $\underline{v} > 0$, the argument is not valid. In this case, (4) would take a different form and the seller could increase $U_1(\underline{v},2)$ in order to relax (3). Such an increase is equivalent to an unconditional payment to every buyer who reports deadline two. This is relatively expensive for the seller and it can be shown that he would resort to this instrument only after having distorted the allocation rule by a significant amount (p_U must be greater or equal than one, cf. section 3.5).

The feasibility constraint $(F1)$ can be dispensed with because $q_2(v,1)$ is the winning probability conditional on the event that the object has not been sold in period one.

$\pi_2[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is two and the utility of the first buyer is constrained by $U_1(v,2) \geq U(v)$ for all $v \in [0,\overline{v}]$. This maximization problem is called $\mathcal{P}_2$:

$$\pi_2(U) := \max_{q_i(.,2)} \int_0^{\overline{v}} q_1(v,2)J_1(v|2)f_1(v|2) + q_2(v,2)J_2(v)f_2(v)dv \qquad (\mathcal{P}_2)$$

$$\text{s.t.} \quad (PE1) \text{ with } U_1(0,2)=0, \ (F2), \qquad (9)$$

$$q_i(v,2) \in [0,1], \qquad (10)$$

$$U_1(v,2) \geq U(v), \quad \forall v \in [0,\overline{v}], \qquad (11)$$

$$\text{and} \quad q_1(v,2) \text{ is non-decreasing in } v. \qquad (12)$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are solved for the same U, we get a solution for $\mathcal{P}$. Therefore, $\mathcal{P}$ can be reformulated as a problem of choosing U optimally as in $\mathcal{P}'$. In the next sections, the solutions to $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}'$ will be derived. Note that we have neglected the monotonicity constraint on q_2 in $\mathcal{P}_1$ and $\mathcal{P}_2$. As a consequence of Assumption 1, this will be fulfilled automatically at the optimal solution.

3.3. Solution to $\mathcal{P}_1$

If (7) is ignored, $\mathcal{P}_1$ is equivalent to the problem of finding the optimal selling strategy for a sequence of short-lived bidders. The optimal solution is a sequence of fixed prices (Riley and Zeckhauser, 1983). Optimal prices are determined working backwards in time. If the object is not sold in the first period, the optimal price in the second period is $r_2 = v_2^0$. This implies an option value of waiting for period two of $V_2^{\text{opt}} := \int_{v_2^0}^{\overline{v}} J_2(v_2)f_2(v_2)dv_2 = v_2^0(1-F_2(v_2^0))$. Consequently, the optimal price in the first period, r_1, is given by $J_1(r_1|1) = V_2^{\text{opt}}$. This is the relaxed solution of $\mathcal{P}_1$.

If constraint (7) is imposed, it is a priori not clear that the optimal solution to $\mathcal{P}_1$ is a sequence of fixed prices. Assumption 2 is a sufficient condition for this.[12] Theorem 1 implies that if the allocation rule is deterministic in the first period, (7) reduces to $U_1(\overline{v},1) \leq \bar{U}$, where we define $\bar{U} := U(\overline{v})$. We will therefore treat π_1 as a function of $\bar{U}$ and write $\pi_1(\bar{U})$ instead of $\pi_1[U]$. The optimal fixed price in period one is now given by the lowest price that satisfies $J_1(r_1|1) \geq V_2^{\text{opt}}$ and $\overline{v} - r_1 \leq \bar{U}$. The optimal fixed price in period two, r_2, is not affected by constraint (7).

[12] The no-haggling result of Riley and Zeckhauser (1983) is a consequence of a special structure of the feasible set of the maximization problem. Manelli and Vincent (2007) show that the set of extremal points of the feasible set, which contains the maximizers, is equal to the set of deterministic allocation rules. Due to the additional constraint (7), the set of extremal points changes. Rather than trying to extend the results of Manelli and Vincent here, we use Assumption 2 as a sufficient condition for a deterministic mechanism.

Theorem 2. *(i) If f_1 satisfies Assumption 2, then the optimal solution of $\mathcal{P}_1$ is given by*

$$q_1(v_1, 1) = \begin{cases} 0, & \text{if } J_1(v_1, 1) < \max\{V_2^{\text{opt}}, J_1(\overline{v} - \bar{U}|1)\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$q_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ 1, & \text{otherwise}. \end{cases}$$

(ii) Under Assumption 2, $\pi_1(\bar{U})$ is continuously differentiable for $\bar{U} \in (0, \overline{v})$ and strictly concave in $\bar{U}$ for $\bar{U} < \overline{v} - J_1^{-1}(V_2^{\text{opt}})$.

Proof. See Appendix B. □

To understand the role of Assumption 2, note that in the constraint $U_1(v, 1) = \int_0^v q_1(s, 1)ds \leq U(v)$, winning probabilities are not weighted in the integral because incentive compatibility constraints are independent of the buyer's own distribution function. In the objective, however, $q_1(v_1, 1)$ is weighted by $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$. Increasing the winning probability $q_1(v_1, 1)$ for valuations in $[v, v+\varepsilon]$ and decreasing it by the same amount on $[v', v' + \varepsilon]$, with $v' + \varepsilon \leq v$, weakly decreases $U_1(v_1, 1)$ for $v_1 \in [v', v + \varepsilon]$ and leaves $U_1(v_1, 1)$ unchanged otherwise. Hence, such a change in q_1 does not destroy incentive compatibility. On the other hand, this shift of winning probability increases the seller's revenue if $(J_1(v_1) - V_2^{\text{opt}})f_1(v_1)$ is increasing. Assumption 2 guarantees that $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$ is increasing whenever $J_1(v_1|1) - V_2^{\text{opt}} \geq 0$. Therefore, the winning probability must jump from zero to one at some point.

If Assumption 2 does not hold, raising the winning probability for a lower valuation may be more profitable than for a higher valuation because it is sufficiently more likely that a buyer has the low valuation. For this to be the case, the decrease in the density must outweigh the increase in expected revenue, i.e. the virtual valuation. Finally, note that Assumption 2 is a sufficient condition. Presumably, a necessary and sufficient condition cannot be stated as a local condition.

3.4. The relaxed problem

The *relaxed problem* consists of $\mathcal{P}_1$ and $\mathcal{P}_2$ where constraints (7) and (11) are ignored, respectively. In this section, we will derive a sufficient condition under which the *relaxed solution* is incentive compatible and a sufficient condition under which the incentive compatibility constraint for the deadline is violated. We have seen in the last section, that the relaxed solution of $\mathcal{P}_1$ is given by the fixed prices $r_2 = v_2^0$ and $r_1 = J_1^{-1}(V_2^{\text{opt}}|1)$.

$\mathcal{P}_2$ is equivalent to Myerson's optimal auction problem if the type-dependent participation constraint (11) is ignored. Hence, the relaxed solution of $\mathcal{P}_2$ is given by the reduced form of the following allocation rule

$$x_1(v_1, 2, v_2) = \begin{cases} 1, & \text{if } J_1(v_1|2) \geq \max\{0, J_2(v_2)\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$x_2(v_1, 2, v_2) = \begin{cases} 1, & \text{if } J_2(v_2) > \max\{0, J_1(v_1|2)\}, \\ 0, & \text{otherwise.} \end{cases}$$

It can be implemented by letting the winning bidder pay the valuation that ties with the valuation of his opponent if this is greater than $v_1^0|2$ (v_2^0), and $v_1^0|2$ (v_2^0) otherwise. The loser does not have to make any payment.[13]

$$y_1(v_1, 2, v_2) = x_1(v_1, 2, v_2)\, J_1^{-1}(\max\{0, J_2(v_2)\}|2).$$
$$y_2(v_1, 2, v_2) = x_2(v_1, 2, v_2)\, J_2^{-1}(\max\{0, J_1(v_1|2)\}).$$

By payoff equivalence, it is without loss of generality to restrict attention to this payment rule. Now we can state conditions under which the relaxed solution solves the general problem or not.

Theorem 3. *(i) If $J_1(.|d)$ is strictly concave on $[v_1^0|d, \overline{v}]$ for at least one $d \in \{1, 2\}$, and $J_1(v|d)$ is non-increasing in d for $v \in [v_1^0|1, \overline{v}]$, then the relaxed solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ violate (ICD^d).*

(ii) If $J_1(.|d)$ is weakly convex on $[v_1^0|d, \overline{v}]$ for at least one $d \in \{1, 2\}$, and $J_1(v|d)$ is non-decreasing in d for $v \in [v_1^0|2, \overline{v}]$, then the relaxed solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide with the solution to the general problem.

Proof. See Appendix B. □

In (i), weak concavity together with strict monotonicity is also sufficient for the incentive compatibility constraint to be violated.

To get an understanding of the result, suppose that buyer one has valuation $v_1 = \overline{v}$. By Theorem 1, the relaxed solution is incentive compatible, if and only if this buyer's interim utility with deadline two is at least as high as his interim utility with deadline one. Since the winning probability is one in both periods, we only have to compare the expected payment in the auction with the fixed price in period one.

The payment of buyer one is a function of the virtual valuation of the competing buyer two. The function is given by the inverse of buyer one's virtual valuation

[13]In the case of identical distributions ($f_1(.|2) = f_2$), this is the second price auction with reserve price $v_1^0|2 = v_2^0$ (Vickrey, 1961).

density (support: $[0,1]$)	$J(v)$	$J''(v)$
$2v$	$\frac{1}{2}\frac{3v^2-1}{v}$	$-\frac{1}{v^3} < 0$
$1-k+2kv$ ($k \in (0,1]$)	$\frac{2v-2kv+3kv^2-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} < 0$
$(k+1)v^k$	$\frac{vk+2v-v^{-k}}{k+1}$	$-v^{-2-k}k < 0$
$1/4+9(v-\frac{1}{2})^2$	$\frac{10v+24v^3-27v^2-2}{5+18v^2-18v}$	$\frac{18(-27-126v^2+116v+12v^3)}{(5+18v^2-18v)^3} < 0$ for $v > v_1^0$
$\frac{3}{2}-6(v-\frac{1}{2})^2$	$\frac{8v^2-v-1}{6v}$	$-\frac{1}{3v^3} < 0$
$2-2v$	$\frac{3v}{3}-\frac{1}{2}$	0
1 (uniform)	$2v-1$	0
$(1+k)(1-v)^k$	$\frac{(k+2)v-1}{k+1}$	0
$1-k+2kv$ ($k \in [-1,0)$)	$\frac{2v-2kv+3kv-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} > 0$

TABLE 1. Distributions with strictly concave, linear and strictly convex virtual valuations.

function. In the case of a concave virtual valuation the inverse is convex. In period one it is applied to the *expected* virtual valuation of buyer two to determine the fixed price. In period two, buyers compete directly, and the payment is computed from the *realized* virtual valuation of buyer two. The convexity implies that the seller charges on average higher prices in the case of direct competition. This destroys incentive compatibility because buyer one prefers to report deadline one to avoid higher rent extraction. The effect depends on the dynamic nature of the problem. In choosing the deadline, buyer one can choose whether he competes with buyer two directly or in expectation. Non-linearities in the virtual valuation lead to different average prices for different modes of competition. In a static model, where the second dimension of private information is for instance a capacity constraint, different reports would not change the mode of competition.

In addition to the mode of competition, buyer one also chooses the virtual valuation function that the seller uses to determine prices. If the virtual valuation function of buyer one is lower, types with lower valuations have smaller winning probabilities. Hence, for $v_1 = \overline{v}$, deviations to lower valuations are less attractive. Therefore, the seller can charge higher prices. This leads to a valuation of incentive compatibility if the virtual valuation is decreasing in the deadline, because in this case the seller extracts more rent in the second period. This effect also occurs in static models.

Table 1 shows several distributions. For the first group, the virtual valuation is strictly concave wherever it is non-negative. For the second group, it is linear and for the third group it is convex. If valuation and deadline of buyer one are independently distributed, the relaxed solution violates incentive compatibility for all distributions in the first group. An example for a violation of incentive compatibility for the dependent case is $f_1(v|1) = 1$ and $f_1(v|2) = 2-2v$. In this case, the virtual valuation is linear for both distributions but strictly decreasing in the deadline. Other examples are easily constructed.

3.5. Solution to $\mathcal{P}_2$

In this section, we solve $\mathcal{P}_2$, imposing (11) only for $v = \overline{v}$. By Theorems 1 and 2, this is sufficient for the general problem if Assumption 2 is fulfilled. In the derivation of the optimal solution of $\mathcal{P}_2$, however, Assumption 2 is not used. Therefore, the results of this section also apply if the mechanism designer is exogenously restricted to set a fixed price in the first period.

To state the optimal solution, we define the *generalized virtual valuation* of buyer one:

$$J_1^{p_U}(v) := J_1(v|1) + \frac{p_U}{f_1(v|1)}.$$

The parameter p_U determines the magnitude of the distortion of the allocation rule away from the Myerson solution. (p_U is the multiplier of constraint (11) in the underlying control problem.) Suppose we already know the optimal p_U. Then the optimal allocation rule is given by

$$\begin{aligned} x_1(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_1^{p_U}(v_1) < \max\{0, J_2(v_2)\} \\ 1, & \text{otherwise,} \end{cases} \\ x_2(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, J_1^{p_U}(v_1)\} \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{13}$$

For every $\bar{U} \in [0, \overline{v})$, let p_U^* be the lowest value $p_U \geq 0$, such that the reduced form of (13) satisfies $\int_0^{\overline{v}} q_1(v, 2)dv \geq \bar{U}$.

Theorem 4. *Fix $\bar{U}$ and suppose $J_1^{p_U^*}$ is strictly increasing. Then*

(i) the reduced form of (13) for $p_U = p_U^*$ is an optimal solution of $\mathcal{P}_2$ subject to (9), (10), (11) for $v = \overline{v}$, and (12),

(ii) $p_U = -\pi_2'(\bar{U})$.

(iii) Furthermore, π_2 is weakly concave.

Proof. Theorem 4 is a special case of Theorem 5 below. □

If the relaxed solution is incentive compatible, p_U is zero and valuations (v_1, v_2) tie if $J_1(v_1|2) = J_2(v_2)$ as in the Myerson solution. If the relaxed solution is not incentive compatible, p_U is strictly positive and valuations tie if $J_1^{p_U}(v_1) = J_2(v_2)$, which is equivalent to

$$(J_1(v_1|2) - J_2(v_2))f_1(v_1|2) = -p_U. \tag{14}$$

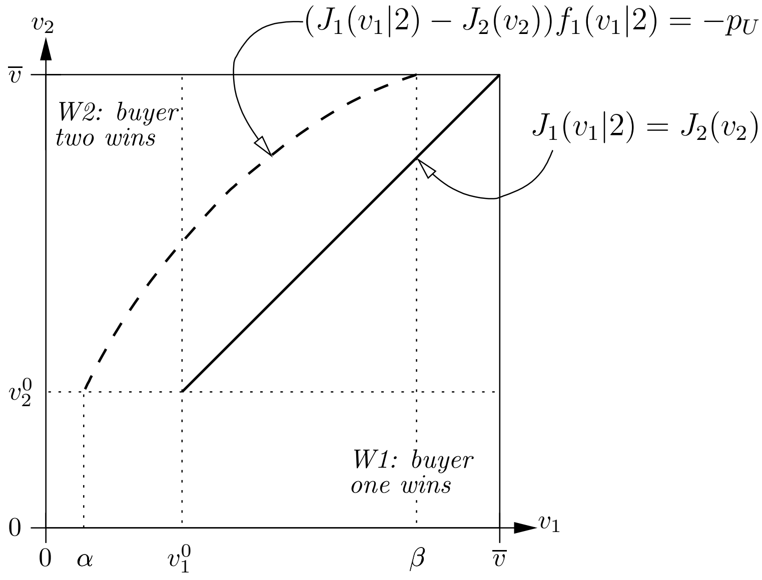

FIGURE 1. Optimal allocation rule

Figure 1 sketches both cases for identical distributions ($f_1(.|2) = f_2$). The solid line is the Myerson-line at which valuations tie in the relaxed solution. The dashed line is the distorted Myerson-line at which valuations tie in the general solution. Note that for all $p_U > 0$, valuations tie in an area where the (standard) virtual valuation of buyer one is strictly smaller than the virtual valuation of buyer two.

For an informal explanation of condition (14), consider the effect on π_2 of an increase of $q_1(.,2)$. Fix any (v_1, v_2) on the distorted Myerson-line, such that $0 \leq J_1^{p_U}(v_1) \leq 1$. In the figure, this corresponds to $\alpha \leq v_1 \leq \beta$. To increase $q_1(v_1, 2)$, the allocation has to be changed from buyer two to buyer one at (v_1, v_2). This leads to a marginal change of π_2 by $J_1(v_1|2) - J_2(v_2)$ per mass of type profiles for which the allocation is changed. This mass of type profiles is proportional to $f_1(v_1|2)$. Hence, the left-hand side of (14) quantifies the marginal cost of increasing $q_1(v_1, 2)$.

The marginal cost of increasing $q_1(v_1, 2)$ must be independent of v_1. The reason is that winning probabilities are not weighted in the constraint $\int_0^{\overline{v}} q_1(s, 2)ds \geq \bar{U}$. If the marginal cost of changing $q_1(v_1, 2)$ varies with v_1, we can increase $q_1(v_1, 2)$ where the marginal cost is small and decrease it where the marginal cost is big. If we choose this variation such that $U_1(\overline{v}, 2)$ is not changed, we increase the objective function without violating the constraints. This is a contradiction. Hence, the marginal cost of increasing $q_1(., 2)$ must be constant and equal to p_U for all $v_1 \in [\alpha, \beta]$. As $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s, 2)ds$, p_U can also be interpreted as the marginal cost of the constraint $U_1(\overline{v}, 2) \geq \bar{U}$.

(14) also implies that the distortion of the Myerson-line is bigger for types with low densities. This is intuitive because the expected cost of a distortion is lower for

types that are less frequent. Furthermore, note that the distortion is increasing in p_U, and that by Assumption 1 the marginal cost of a distortion is increasing in the distance from the Myerson solution (the LHS of (14) is decreasing in v_2). Therefore, it is optimal to choose the lowest p_U such that (11) is satisfied, and the cost of distortions is convex, which implies concavity of π_2 in $\bar{U}$.

Theorem 4 requires monotonicity of $J_1^{p_U}$ for the optimal value of p_U. Otherwise, the winning probability of buyer one would be decreasing. This is a condition on an endogenous object and Assumption 1 does not guarantee monotonicity of $J_1^{p_U}$ for all values of p_U. A decreasing density $f_1(v|2)$ together with Assumption 1 would be sufficient, but this is quite restrictive and rules out most of the examples in table 1. To give a complete solution without further assumptions, the next subsection shows how to deal with non-monotonicities of $J_1^{p_U}$.

3.5.1. **The irregular case.** If $J_1^{p_U}$ is decreasing for some values v_1, then we have to apply Myerson's ironing procedure to define the optimal solution.

Definition 4 (Ironing; Myerson, 1981). (i) For every $t \in [0,1]$, define

$$M_1^{p_U}(t) := J_1(F_1^{-1}(t|2)|2) + \frac{p_U}{f_1(F_1^{-1}(t|2)|2)},$$

as the generalized virtual valuation at the t-quantile of $F_1(.|2)$.

(ii) Integrate this function:

$$H^{p_U}(t) := \int_0^t M_1^{p_U}(s)ds.$$

(iii) Take the convex hull (i.e. the greatest convex function G such that $G(t) \leq H^{p_U}(t)$ for all t):

$$\bar{H}^{p_U}(t) := \text{conv}H^{p_U}(t).$$

(iv) Since $\bar{H}^{p_U}$ is convex, it is almost everywhere differentiable and any selection $\bar{M}_1^{p_U}(t)$ from the sub-gradient is non-decreasing.

(v) Reverse the change of variables made in (i) to obtain the *ironed generalized virtual valuation*

$$\bar{J}_1^{p_U}(v_1) := \bar{M}_1^{p_U}(F_1(v_1|2)).$$

Given the optimal p_U the optimal allocation rule is then given by

$$\bar{x}_1(v_1,2,v_2) = \begin{cases} 0, & \text{if } \bar{J}_1^{p_U}(v_1) < 0, \\ \underline{x}_1^0, & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \text{ and } J_2(v_2) \leq 0, \\ 1, & \text{if } \bar{J}_1^{p_U}(v_1) > 0 \text{ and } \bar{J}_1^{p_U}(v_1) \geq J_2(v_2), \end{cases} \tag{15}$$

$$\bar{x}_2(v_1,2,v_2) = \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, \bar{J}_1^{p_U}(v_1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

In contrast to the regular case, we need an additional parameter $\underline{x}_1^0 \in [0,1]$. This becomes relevant if $\bar{J}_1^{p_U}(v_1) = 0$ on an interval $[\underline{v}_1^0, \bar{v}_1^0]$ with $\underline{v}_1^0 < \bar{v}_1^0$. In this case, $\int_{\underline{v}_1^0}^{\bar{v}_1^0} J_1^{p_U}(v)dv = 0$ and hence, $U_1(\overline{v},2)$ can be varied at constant marginal cost p_U by changing the winning probability for all valuations in the interval $[\underline{v}_1^0, \bar{v}_1^0]$ by a constant. Therefore, the same value of p_U defines the ironed generalized virtual valuation for different values $\bar{U}$ in a non-empty interval $[a,b]$. $\underline{x}_1^0$ is varied to achieve different values of $U_1(\overline{v},2) \in [a,b]$.

The allocation rule in (15) excludes buyer one if his valuation is smaller than $\underline{v}_1^0$. With a valuation in $[\underline{v}_1^0, \bar{v}_1^0]$, he can win against buyer two if $v_2 < v_2^0$, but he gets the object only with probability $\underline{x}_1^0$.[14] To summarize, we have

Theorem 5. *(i) For each $\bar{U} \in [0, \overline{v})$, there exists a pair $(p_U, \underline{x}_1^0) \in [0,\infty) \times [0,1]$ such that the reduced form of (15) is an optimal solution of $\mathcal{P}_2$ subject to (9), (10), (11) for $v = \overline{v}$, and (12).*
p_U is unique and equal to the lowest value such that there exists a $\underline{x}_1^0 \in [0,1]$ for which the reduced form of (15) satisfies (11) and $\underline{x}_1^0$ is the lowest such value. If $\bar{J}^{p_U}(v) = 0$ for a non-empy interval, $\underline{x}_1^0$ is unique.
(ii) $p_U = -\pi_2'(\bar{U})$ for almost every $\bar{U}$.
(iii) π_2 is weakly concave in $\bar{U}$ and strictly concave if $p_U > 0$ and $\bar{J}_1^{p_U}(v) = 0$ has a unique solution.

Proof. See Appendix A. □

Note that if $J_1^{p_U}$ is increasing, $\bar{J}_1^{p_U}$ equals $J_1^{p_U}$. Therefore, Theorem 4 is a special case of Theorem 5.

[14] It is also possible to construct a deterministic allocation rule with the same reduced form. Choose $\hat{v}_2$ such that $q_1(\underline{v}_1^0, 2) = F_2(\hat{v}_2)$. For $v \in [\underline{v}_1^0, \bar{v}_1^0]$ and $v_2 \leq v_2^0$, set $x_1(v_1,2,v_2) = 1$ if $v_1 \geq \hat{v}_2$ and $x_1(v_1,2,v_2) = 0$ otherwise. This construction, however, has the disadvantage that the allocation decision for buyer one depends on the participation of types of buyer two that can never win the object.

3.6. Global solution and discussion

Under Assumption 2, or if the relaxed solution is incentive compatible, the optimal mechanism is deterministic in period one. Therefore, the optimal value of $\bar{U}$ satisfies the first order condition

$$\rho\, \pi_1'(\bar{U}) = -(1-\rho)\, \pi_2'(\bar{U}).$$

As Assumption 2 ensures concavity of π_1 this is also a sufficient condition for the global maximum. To compute the optimal distortion, it suffices to compute the unique solution $p_U \geq 0$ of

$$p_U = \frac{\rho}{1-\rho}\, \pi_1'(\bar{U}),$$

$$\text{and} \qquad \bar{U} \leq \int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1, \quad \text{with equality if } p_U > 0,$$

where q^{p_U} is the reduced form of (15) for given value of p_U. An explicit form of the solution is not available. However, for given p_U, $\int_0^{\overline{v}} q_1^{p_U}(v,2)dv_1$ is easy to calculate and an explicit expression for π_1' is given in the proof of Theorem 2. Hence, it is easy to compute the optimal p_U numerically.

We will now discuss several properties of the general solution.

Monotonicity of $\mathbf{q_2}$. $q_2(v_2,1)$, defined by the fixed price r_2, and $q_2(v_2,2)$, defined by the reduced form of (15), are non-decreasing. This follows from Assumption 1. Therefore, $q_2(v_2)$ is also non-decreasing and the optimal solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ together fulfill all constraints of $\mathcal{P}$. We have derived an optimal solution of $\mathcal{P}$.

The allocation rule is distorted in both periods. By Theorem 2, $\pi_1(\bar{U})$ is continuously differentiable. Therefore, $p_U > 0$ implies that the allocation is distorted in the first period. Hence, in all cases where the relaxed solution is not incentive compatible, the general solution involves a distortion in both periods. The relative magnitude of the distortion depends on ρ. If $d = 1$ is relatively unlikely (ρ small), then the distortion of the fixed price is bigger and the auction is closer to the Myerson solution. The reason is that distortions are more costly at the deadline that occurs more frequently.

Distortions. In the first period, the fixed price is $\max\{\bar{U} - \overline{v}, J_1^{-1}(V_2^{\text{opt}}|1)\}$. It is distorted upwards compared to the relaxed solution to make the fixed price less attractive.

To analyze the distortions in the auction in period two, note that

$$J_1^{p_U}(v_1) = J_1(v_1|2) + \frac{p_U}{f_1(v_1|2)} > J_1(v_1|2), \quad \forall v_1 \in [0, \overline{v}],$$

if the relaxed solution is not incentive compatible ($p_U > 0$). Therefore, the reserve price for buyer one is smaller than in the relaxed solution. Secondly, for all valuations above the reserve price, the winning probability is higher than in the relaxed solution because v_1 ties with a higher valuation v_2. Finally, in contrast to the relaxed solution, the winning probability of bidder two is strictly smaller than one for all $v_2 \in [0, \overline{v}]$. The reason is that for every $p_U > 0$, there is a non-empty interval $(c, \overline{v}]$ such that $J_1^{p_U}(v_1) > \overline{v}$ for all valuations $v_1 \in (c, \overline{v}]$. Buyer two cannot win against buyer one with valuation $v_1 > c$.

Bunching. We find that for the optimal allocation rule, there is bunching at the top of the type-space if the incentive compatibility constraint for the deadline is binding. The bunching region has full dimension. The optimal mechanism does not separate different types of buyer one if their valuations are very high ($v_1 > c$). These types win with probability one and make an expected payment equal to the fixed price in the first period. Therefore, we have pooling of valuations as well as deadlines. This finding is robust: a (small) bunch occurs even if the allocation is only slightly distorted.

Implementation. There are several ways to implement the optimal auction in period two. For example, it can be implemented by a generalized Vickrey auction. In this case, the winning bidder pays the valuation for which his (generalized) virtual valuation ties with the (generalized) virtual valuation of the losing bidder.

As in the standard auction model, there is also an open format that corresponds to this direct mechanism. Consider the following ascending clock auction. The auctioneer has a clock that runs from zero to $\overline{v}$. For each bidder i, the auctioneer's clock value c_a is translated into a bidder-specific clock value c_i. For bidder one, this is $c_1 = \left(J_1^{p_U}\right)^{-1}(c_a)$. For bidder two, this is $c_2 = J_2^{-1}(c_a)$. The auctioneer raises c_a continuously and bidders can drop out at any time. If bidder i drops out, the clock stops immediately. Bidder $j \neq i$ wins the object and has to make a payment equal to his bidder-specific clock-value. Given the informational assumptions made in this paper, this auction is strategically equivalent to the generalized Vickrey auction. It has the advantage that the winning bidder does not have to reveal his true valuation to the auctioneer.

4. Conclusion

To conclude, we discuss some assumptions, possible generalizations, extensions, and directions for future research.

Discounting. Whether discounting changes the analysis in this paper depends on the exact way in which it is incorporated in the utility function. If only payments are discounted and buyers and the seller use a common discount factor, the analysis is almost identical. On the other hand, if the whole payoff is discounted, Lemma 1 may not be valid. It may be optimal to allocate the object in the first period even if the deadline is two because the waiting cost due to discounting are too high. In this case it is more complicated to rule out upward deviations in the deadline.

Which modelling assumption is more pertinent depends on the application. In the example given in the introduction, the value of the object for the buyer is the present discounted value of the revenue stream from the contractual relationship with the third party. This could for example be a production contract. If production starts after the deadline and is independent of the time at which the firm obtains the object (as long as it gets it before the deadline), it seems reasonable that the firm only discounts payments. Similar arguments apply in any situation where the buyer plans to use the object at a fixed time after the deadline.

More bidders. Introducing more bidders who arrive in the second period is straight forward. The assumption that there is only one bidder in the first period is more important. It was used to show that in the first period the object is sold for a fixed price. We have shown that in this case, misreporting deadline one instead of deadline two is most profitable for the buyer with the highest valuation. Hence, we know exactly where the incentive compatibility constraint for the deadline binds. If more than one buyer arrives in the first period, a fixed price is no longer optimal and therefore, the incentive compatibility constraint for the deadline may bind for interior types. The exact points where it binds arise endogenously in the optimal solution.

Number of periods. Increasing the number of periods introduces several complications. Consider for example a model with three periods. Suppose that in each period a single bidder arrives, whose deadline can be at any period after his arrival. Now, from period two onwards, there is more than one bidder who participates in the mechanism. This introduces similar problems as the introduction of more bidders in the first period discussed in the preceding paragraph. Additional complications will arise because buyers from different periods will have to be treated asymmetrically. In the third period, the mechanism designer has to design an optimal auction with three different bidders, two of which have type-dependent participation constraints. In the case of two periods and two bidders, the feasibility constraint could be used to eliminate the winning probability of one bidder (see Appendix A). A generalization of this approach to three bidders is not obvious.

Stochastic exit. One could abandon the assumption that a bidder with deadline two is available in period two with certainty. In some situations, buyers may find opportunities to purchase a similar object if the seller does not sell in the first period. Therefore stochastic exits would be an interesting extension for future research.

Simple mechanisms. Another approach is to confine the analysis to a simpler class of mechanisms. The present paper shows that the optimal mechanism involves a non-linear bonus in the auction for a bidder that announces his presence in the first period. A simpler (although suboptimal) mechanism would be to use a standard auction with a fixed bonus for bidders who arrive earlier. Mares and Swinkels (2008) show that the equilibrium allocation rules that can be achieved with bonuses in first- and second-price auctions differ, and offer revenue comparisons. Combining their analysis with a dynamic framework is an interesting topic for future research.

Appendix A. Proof of Theorem 5

In this section, we will solve $\mathcal{P}_2$. It will be convenient to make the changes of variables $t_1 = F_1(v_1|2)$ and $t_2 = F_2(v_2)$. Defining $v_1(t_1) := F_1^{-1}(t_1|2)$ and $v_2(t_2) = F_2^{-1}(t_2)$, we have

$$\begin{aligned} t_i &\sim U[0,1], \\ v_1'(t_1) &= \frac{1}{f_1(v_1(t_1)|2)}, \\ \text{and} \quad v_2'(t_2) &= \frac{1}{f_2(v_2(t_2))}, \end{aligned}$$

Furthermore we introduce for $i = 1, 2$

$$q_i(t) = q_i(v_i(t), 2), \tag{16}$$

$$U(t) = U_1(v_1(t), 2), \tag{17}$$

$$M_1(t) = J_1(v_1(t)|2) = v_1(t) - (1-t)v_1'(t) \tag{18}$$

$$M_2(t) = J_2(v_2(t)) = v_2(t) - (1-t)v_2'(t) \tag{19}$$

$$t_1^0 = F_1(v_1^0|2\,|2). \tag{20}$$

$$\text{and} \quad t_2^0 = F_2(v_2^0). \tag{21}$$

The objective of the seller becomes

$$R[q_1, q_2] := \int_0^1 q_1(t)M_1(t) + q_2(t)M_2(t)dt. \tag{22}$$

In order to employ control theory, we have to formulate the feasibility constraint $(F2)$ in terms of q. Border (1991) provides a characterization of feasibility for symmetric reduced form allocation rules. The following Theorem generalizes the result to asymmetric allocation rules. Define the sets of types where the q_i are not constant joined with 0 as

$$X_i := \{t \in [0,1] \mid \forall \varepsilon > 0 : q_i([t-\varepsilon, t+\varepsilon] \cap [0,1]) \neq \{q_i(t)\}\} \cup \{0\}.$$

Theorem 6 (Mierendorff (2008))**.** *For $i = 1, 2$, let $q_i : [0,1] \to [0,1]$ be nondecreasing. (q_1, q_2) is the reduced form of a feasible allocation rule if and only if for all $t_1 \in X_1$, $t_2 \in X_2$,*

$$\int_{t_1}^{1} q_1(t)dt + \int_{t_2}^{1} q_2(t)dt \leq 1 - t_1 t_2.$$

Now we can restate $\mathcal{P}_2$ as $\mathcal{P}_2'$:

$$\pi_2(\bar{U}) = \sup_{(q_1,q_2)} R[q_1, q_2] \qquad (\mathcal{P}_2')$$

subject to

$$\forall t \in [0,1] : \qquad q_i(t) \in [0,1], \qquad (23)$$

$$\forall t > t', \qquad q_i(t) \geq q_i(t'), \qquad (24)$$

$$\forall t_1 \in X_1,\ t_2 \in X_2 : \qquad \int_{t_1}^{1} q_1(\theta)d\theta + \int_{t_2}^{1} q_2(\theta)d\theta \leq 1 - t_1 t_2, \qquad (25)$$

$$\forall t \in [0,1] : \qquad U(t) = \int_0^t q_1(\theta)v_1'(\theta)d\theta, \qquad (26)$$

$$U(1) \geq \bar{U}, \qquad (27)$$

Using $q_i(F_i(v_i|2)) = q_i(v_i, 2)$, a solution to $\mathcal{P}_2$ can be derived easily from a solution to $\mathcal{P}_2'$.

A direct solution of $\mathcal{P}_2'$ is difficult because (25) is not a standard constraint. Instead, we can use (25) to eliminate q_2 from the objective function. For $q_1 : [0,1] \to [0,1]$ non-decreasing, define the *inverse* as

$$q_1^{-1}(t) := \begin{cases} 1 & \text{if } q_1(1) < t, \\ \inf\{\theta \in [0,1] \mid q_1(\theta) \geq t\} & \text{otherwise.} \end{cases}$$

Lemma 2. *Let $q_1 : [0,1] \to [0,1]$ be non-decreasing. Then an optimal solution to*

$$\sup_{q_2} \int_0^1 q_2(t) M_2(t) dt,$$

subject to (23)–(25) is given by:

$$q_2^*(t) = \begin{cases} q_1^{-1}(t) & \text{if } t \geq t_2^0, \\ 0 & \text{otherwise.} \end{cases}$$

The solution is unique for almost every t.

Proof. (25) can be rewritten as

$$\forall t_2 \in X_2 : \quad \int_{t_2}^{1} q_2(\theta) d\theta \leq \min_{t_1 \in X_1} \left[1 - t_1 t_2 - \int_{t_1}^{1} q_1(\theta) d\theta \right]$$

If we minimize over $t_1 \in [0,1]$, the first-order condition yields $t_1 = 0$ if $t_2 < q_1(0)$, $t_2 = q_1(t_1)$ if $q_1(0) \leq t_2 \leq q_1(1)$, and $t_1 = 1$ if $q_1(1) < t_2$. This is sufficient since we minimize a convex function. $\inf\{t \,|\, q_1(t) = t_2\}$ is contained in X_1 whenever $q_1(0) \leq t_2 \leq q_1(1)$. This is implies that $t_1 = q_1^{-1}(t_2)$ is a minimizer in X_1. Substituting this into (25) yields

$$\forall t_2 \in X_2 : \quad \int_{t_2}^{1} q_2(\theta) d\theta \leq 1 - q_1^{-1}(t_2) t_2 - \int_{q_1^{-1}(t_2)}^{1} q_1(\theta) d\theta. \tag{28}$$

q_2^* fulfills this constraint with equality for every $t_2 \in [0,1]$.

Now consider an alternative solution $\tilde{q}_2$ that differs from q_2^* on a set of positive measure. Let $\tilde{X}_2$ be the set where $\tilde{q}_2$ is not constant joined with $\{0\}$. If $\tilde{q}_2(t) > 0$ for some $t < t_2^0$, than it is not a maximizer. So suppose $\tilde{q}_2(t) = 0$ for $t < t_2^0$.

For all $t \in [0,1]$, we must have $\int_t^1 \tilde{q}_2(\theta) d\theta \leq \int_t^1 q_2^*(\theta) d\theta$. Suppose to the contrary, that $\int_t^1 \tilde{q}_2(\theta) d\theta > \int_t^1 q_2^*(\theta) d\theta$ for some t. Then $t \notin \tilde{X}_2$ and as $\tilde{q}_2(t) > q_2^*(t)$ this implies that $\tilde{q}_2$ is constant on $[0,t]$ and hence $\int_0^1 \tilde{q}_2(\theta) d\theta > \int_0^1 q_2^*(\theta) d\theta$. This is a contradiction because $0 \in X_2$.

Next, observe that $\int_a^1 \tilde{q}_2(\theta) d\theta < \int_a^1 q_2^*(\theta) d\theta$ for some $a \in [t_2^0, 1]$. Otherwise, we would have $\tilde{q}_2(t) = q_2^*(t)$ for almost every t. Let $Q(t)$ be the concave hull of

$$t \longmapsto \begin{cases} \int_t^1 \tilde{q}_2(\theta) d\theta, & \text{if } t \neq a, \\ \int_a^1 q_2^*(\theta) d\theta, & \text{if } t = a, \end{cases}$$

and define $\hat{q}_2(t) = -\frac{dQ(t)}{dt}$ for almost every t. By definition, $Q(t) = \int_t^1 \hat{q}_2(\theta) d\theta \leq \int_t^1 q_2^*(\theta) d\theta$. Hence $\hat{q}_2$ is a solution. Furthermore, there are $\underline{a}, \bar{a}$ such that

$$\hat{q}_2(t) = \begin{cases} \tilde{q}_2(t), & \text{if } t \notin [\underline{a}, \bar{a}], \\ \frac{\int_{\underline{a}}^1 \tilde{q}_2(\theta) d\theta - \int_a^1 q_2^*(\theta) d\theta}{a - \underline{a}}, & \text{if } t \in [\underline{a}, a), \\ \frac{\int_a^1 q_2^*(\theta) d\theta - \int_{\bar{a}}^1 \tilde{q}_2(\theta) d\theta}{\bar{a} - a}, & \text{if } t \in (a, \bar{a}]. \end{cases}$$

Hence $\hat{q}_2(t) < \tilde{q}_2(t)$ for $t \in (\underline{a}, a)$, $\hat{q}_2(t) > \tilde{q}_2(t)$ for $t \in (a, \bar{a})$ and $\hat{q}_2(t) = \tilde{q}_2(t)$ otherwise. Furthermore,

$$\int_{\underline{a}}^{a} \hat{q}_2(\theta) - \tilde{q}_2(\theta) d\theta = \int_{a}^{\bar{a}} \tilde{q}_2(\theta) - \hat{q}_2(\theta) d\theta.$$

This implies that we have constructed $\hat{q}_2$ from $\tilde{q}_2$ by shifting winning probability from types in $[\underline{a}, a]$ to types in $[a, \overline{a}]$. By Assumption 1, this increases the objective function. Hence $\tilde{q}_2$ cannot be optimal. ☐

Using Lemma 2, (22) becomes

$$\int_0^1 q_1(t)M_1(t)dt + \int_{t_2^0}^1 q_1^{-1}(t)M_2(t)dt. \tag{29}$$

If we assume that q_1 is absolutely continuous, substituting $s = q_1(t)$ in the second integral yields

$$\int_0^1 q_1(t)M_1(t) + tq_1'(t)\tilde{M}_2(q_1(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt, \tag{30}$$

where we define $\tilde{M}_2(t) := \max\{0, M_2(t)\}$.[15]

Monotonicity implies some regularity of q_1. In particular $q_1 = q_1^C + q_1^J$ where q_1^C is a continuous function and q_1^J is a pure jump function. This leaves two problems unresolved. Firstly, we have to deal with jumps and secondly, absolute continuity of q_1^C is not guaranteed.[16]

These problems can be circumvented by solving the maximization problem under the restriction that q_1 be Lipschitz continuous with global Lipschitz constant K,

$$q_1 \in \mathcal{L}^K := \{q : [0,1] \to [0,1] \mid \forall t, t' \in [0,1] : \; |q(t) - q(t')| \le K|t - t'|\}.$$

We define the maximization problem $\mathcal{P}_2^K$ as $\mathcal{P}_2'$ subject to the additional constraint $q_1 \in \mathcal{L}^K$. It will be shown that optimal solutions of $\mathcal{P}_2^K$ converge to the optimal solution of $\mathcal{P}_2'$ as $K \to \infty$. Using Lipschitz functions is convenient to show existence because $\mathcal{L}^K$ is sequentially compact.

Theorem 7. *(a) An optimal solution of $\mathcal{P}_2'$ exists.*
(b) For every $K > 0$, an optimal solution of $\mathcal{P}_2^K$ exists.

Proof. (i) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2'$ such that $R[q_1^n, q_2^n] \to \pi_2(\bar{U})$. By Helly's Theorem, for $i = 1, 2$ there exists a subsequence $\left(q_i^{n_j}\right)_{j\in\mathbb{N}}$ and a non-decreasing function $q_i : [0,1] \to [0,1]$, such that $q_i^{n_j} \to q_i$ almost everywhere. If we consider q_i as elements of $L_2([0,1])$, the set of winning probabilities that satisfy (25) is weakly-compact (cf. Lemma 4 in Mierendorff (2008) and Lemma 5.4 in Border (1991)). Therefore, after taking subsequences again, $q_i^{n_j} \rightharpoonup q_i$ and q_i is feasible. Hence (q_1, q_2) satisfies (23)–(27). As $M_i \in L_2([0,1])$ and $v_1' \in L_2([0,1])$ the weak convergence of $q_i^{n_j}$ implies that $R[q_1, q_2] = \pi_2(\bar{U})$. Therefore (q_1, q_2) is an optimal solution.

[15]If q is not absolutely continuous, the substitution yields $\int_0^1 q_1(t)M_1(t)dt + \int_0^1 t\tilde{M}_2(t)dq_1(t) + \int_{q(1)}^1 \tilde{M}_2(t)dt$. In the second integral, q_1 is interpreted as a measure that does not admit a density. This is not useful if we want to apply optimal control theory.

[16]For example, the Cantor function is non-decreasing and continuous but it cannot be described as the integral of a function. Hence it is not absolutely continuous.

(ii) Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ be a sequence of solutions of $\mathcal{P}_2^K$ such that $R[q_1^n, q_2^n] \to \pi_2^K(\bar{U})$. After taking subsequences we can assume that this sequence converges to a solution satisfying (23)–(27) as in (i). Let q_1 be the limit of q_1^n. Since $q_i^n \in \mathcal{L}^K$, for all $s, t \in [0,1]$, $|q_1(t) - q_1(s)| = \lim_{n\to\infty} |q_1^n(t) - q_1^n(s)| \leq K|t-s|$. Hence $q_1 \in \mathcal{L}^K$.

□

The next step is to show that Lipschitz solutions converge to the general solution if K tends to infinity. The proof is based on Reid (1968).

Lemma 3. *Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $K_n \to \infty$ as $n \to \infty$. Then there exists a solution (q_1, q_2) of $\mathcal{P}_2'$ and a subsequence $(q_1^{n_j}, q_2^{n_j})_{j\in\mathbb{N}}$such that $q_i^{n_j}(t) \overset{j\to\infty}{\longrightarrow} q_i(t)$ for almost every t. This solution is optimal: $R[q_1, q_2] = \pi_2(\bar{U})$.*

Proof. After taking a subsequence, we can assume that (q_1^n, q_2^n) converges a.e. to a solution $(\hat{q}_1, \hat{q}_2)$ of $\mathcal{P}_2'$ (see proof of Theorem 7). To show optimality of $(\hat{q}_1, \hat{q}_2)$, let (q_1, q_2) be an optimal solution of $\mathcal{P}_2'$. We can extend q_1 to $\mathbb{R}$ by setting $q_1(t) = 0$ if $t < 0$ and $q_1(t) = 1$ if $t > 1$. Define $q_{d,1} : \mathbb{R} \to [0,1]$ as

$$q_{d,1}(t) := \frac{1}{2d} \int_{t-d}^{t+d} q_1(s) ds.$$

By the Lebesgue differentiation theorem $q_{d,1}(t) \to q_1(t)$ for almost every $t \in [0,1]$ as $d \to 0$. Since q_1 is non-decreasing and $q_1(t) \in [0,1]$, $q_{d,1}$ also has these properties. Furthermore $q_{d,1} \in \mathcal{L}^{\frac{1}{2d}}$:

$$\begin{aligned}
\forall t > t' : \qquad 0 \leq q_{d,1}(t) - q_{d,1}(t') &= \frac{1}{2d}\left(\int_{t-d}^{t+d} q_1(s)ds - \int_{t'-d}^{t'+d} q_1(s)ds\right) \\
&= \frac{1}{2d}\left(\int_{t'+d}^{t+d} q_1(s)ds - \int_{t'-d}^{t-d} q_1(s)ds\right) \\
&\leq \frac{1}{2d}\int_{t'+d}^{t+d} q_1(s)ds \\
&\leq \frac{1}{2d}(t-t')
\end{aligned}$$

Next, define $\tilde{q}_{d,1} = q_{d,1}$ if $\int_0^1 q_{d,1}(t)v_1'(t)dt \geq \bar{U}$ and otherwise $\tilde{q}_{d,1} = \lambda_d + (1-\lambda_d)q_{d,1}$, where $\lambda_d = \frac{\bar{U} - \int_0^1 q_{d,1}(t)v_1'(t)dt}{1 - \int_0^1 q_{d,1}(t)v_1'(t)dt}$. Furthermore, define

$$\tilde{q}_{d,2}(t) := \begin{cases} \tilde{q}_{d,1}^{-1}(t), & \text{if } M_2(t) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For every d, $(\tilde{q}_{d,1}, \tilde{q}_{d,2})$ is a solution of $\mathcal{P}_2^{\frac{1}{2d}}$. λ_d converges to zero as $d \to 0$. By Lemma 2, $q_2(t) = q_1^{-1}(t)$ for a.e. t such that $M_2(t) \geq 0$ and $q_2(t) = 0$ otherwise. Hence, for $i = 1, 2$, $\tilde{q}_{d,i} \to q_i$ almost everywhere as $d \to 0$. By the dominated convergence theorem, $R[\tilde{q}_{d,1}, \tilde{q}_{d,2}] \to R[q_1, q_2]$ and $R[q_1^n, q_2^n] \to R[\hat{q}_1, \hat{q}_2]$. Define $d_n = \frac{1}{2K_n}$. Then, $R[\tilde{q}_{d_n,1}, \tilde{q}_{d_n,2}] \leq R[q_1^n, q_2^n]$ and we have $R[q_1^n, q_2^n] \to R[q_1, q_2]$ and hence $R[\hat{q}_1, \hat{q}_2] = R[q_1, q_2]$. $\square$

A.1. Solution on the class $\mathcal{L}^K$

Using Lemma 2, we rewrite $\mathcal{P}_2^K$ as a control problem. The state variables are the expected utility of bidder one, denoted $U(t)$, and the winning probability, denoted $q(t)$, (In the control problem we write q instead of q_1). As q is absolutely continuous, we can use $u(t) = q'(t)$ as a control variable. The objective is defined as

$$R_c[U, q, u] := \int_0^1 q(t)M_1(t) + tu(t)\tilde{M}_2(q(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt.$$

where u is a measurable control

$$u : [0, 1] \to [0, K]. \tag{31}$$

The evolution of the state variables is governed by

$$U'(t) = q(t)v'(t), \tag{32}$$

$$q'(t) = u(t). \tag{33}$$

We impose the state constraint

$$\forall t \in [0, 1] : \qquad q(t) \leq 1. \tag{34}$$

Furthermore, we impose the following constraints on the start- and endpoints:

$$U(0) = 0, \tag{35}$$

$$q(0) \geq 0, \tag{36}$$

$$U(1) \geq \bar{U}, \tag{37}$$

$$q(1) \text{ free.} \tag{38}$$

To summarize, we have the following control problem:

$$\max_{(U,q,u)} R_c[U, q, u], \tag{$\mathcal{P}_C^K$}$$

subject to (31)–(38).

(32) is (26) in differential form. (33) and (31) ensure that $q \in \mathcal{L}^K$ and non-decreasing. (31), (36) and (33) imply $q(t) \geq 0$. Hence, we can dispense with a second state constraint.

To solve the control problem, we employ a version of the Pontryagin maximum principle that allows for an integrand in the objective function that is not differentiable with respect to the state variables at some points. This yields the following necessary conditions for an optimum.

Theorem 8 (Clarke (1983), pp. 210-212). *Let (U, q, u) be a solution of $\mathcal{P}_C^K$. If (U, q, u) is optimal, there exists $\omega \in \{0, 1\}$, an absolutely continuous function $p : [0, 1] \to \mathbb{R}^2$, the components of which we denote by (p_U, p_q), and a non-negative measure μ on $[0, 1]$, such that the following conditions hold:*

(i) For almost every $t \in [0, 1]$,

$$p_U'(t) = 0, \tag{39}$$

$$p_q'(t) \in -\omega \left[M_1(t) + tu(t)\partial \tilde{M}_2(q(t)) \right] - p_U v_1'(t), \tag{40}$$

where $\partial \tilde{M}_2$ is the generalized gradient of $\tilde{M}_2$.

(ii) For almost every $t \in [0, 1]$, $u(t)$ maximizes

$$\left[\omega t \tilde{M}_2(q(t)) + p_q(t) + \mu[0, t) \right] u.$$

(iii) μ is supported on $\{q(t) = 1\}$,

(iv) p satisfies the transversality conditions

$$p_q(0) \leq 0, \qquad (\textit{with equality if } q(0) > 0,)$$

$$p_U(1) \geq 0, \qquad (\textit{with equality if } U(1) > \underline{U},)$$

$$p_q(1) = \omega \tilde{M}_2(q(1)) - \mu[0, 1].$$

(v) $\omega + \|p\| + \|\mu\| > 0$.

Note that (39) implies that p_U is constant. First we show that trivial solutions do not occur.

Lemma 4 (Non-triviality). *If $\bar{U} < \overline{v}$, $\omega = 1$.*

Proof. Suppose that $\omega = 0$. By (40), $p_q'(t) = -p_U v_1'(t)$. By the transversality conditions, $p_U \geq 0$. $p_U = 0$ implies, $p_q'(t) = 0$ and $p_q(t) = p_q(0)$ for all t. $p_U > 0$ implies, $p_q'(t) < 0$ and $p_q(t) < 0$ for all $t > 0$.

Suppose $p_U > 0$. By, the transversality condition this implies $U(1) = \bar{U}$. By (ii), $u(t)$ maximizes $(p_q(t) + \mu[0, t))u$. If $q(0) < 1$, $\mu[0, t) = 0$ for t close to zero and hence $u(t) = 0$. As $\mu[0, t)$ cannot become positive we must have $q(t) = q(0) < 1$ for all t and consequently $\mu[0, 1] = 0$. The transversality condition therefore requires

$p_q(1) = 0$, a contradiction. If, however, $q(0) = 1$ we would have $U(1) = \overline{v} > \bar{U}$. Again a contradiction.

Now suppose that $p_U = 0$. If $q(1) < 1$, $\mu[0,1] = 0$ and by the transversality conditions, $p(t) = 0$ for all t. This implies $\omega + \|p\| + \|\mu\| = 0$, in contradiction to (v). Hence, $q(1) = 1$. Since $p_q(t) = p_q(1)$, we have $p_q(t) = -\mu[0,1]$. To fulfil (v) we must have $\mu[0,1] > 0$. $u(t)$ maximizes $(\mu[0,t) - \mu[0,1])u$. This implies that $u(t) = 0$ if $q(t) < 1$. Hence, we must have $q(t) = 1$ for all $t \in [0,1]$. This implies $U(1) = \overline{v}$ which cannot be optimal if $\bar{U} < \overline{v}$. □

In condition (i), $\partial\tilde{M}_2(q(t))$ is the generalized gradient of $\tilde{M}_2(q)$ at $q(t)$. If $q(t)$ is a point of differentiability of $\tilde{M}_2$, $\partial\tilde{M}_2(q(t)) = \{\tilde{M}_2'(q(t))\}$. At $q(t) = t_2^0$, where $\tilde{M}_2$ is not differentiable, $\partial\tilde{M}_2(q(t)) = [0, M_2'(t_2^0)]$. Since q is non-decreasing, either $q^{-1}(t_2^0)$ is a singleton or $u(t) = 0$ for almost every $t \in q^{-1}(t_2^0)$. Therefore (40) can be simplified to the condition that for almost every $t \in [0,1]$,

$$-p_q'(t) = M_1^{p_U}(t) + tu(t)\tilde{M}_2'(q(t)), \tag{41}$$

where

$$M_1^{p_U}(t) := M_1(t) + p_U v_1'(t).$$

Condition (ii) implies that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) > 0, \tag{42}$$

$$u(t) \in [0,K] \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) = 0, \tag{43}$$

$$u(t) = 0 \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) < 0. \tag{44}$$

In (42), $\mu[0,t)$ was omitted because $q(t) < 1$ if $u(t) = K$. Integrating (41) yields for $s,t \in [0,1]$:

$$\begin{aligned} p_q(t) &= p_q(s) - \int_s^t M_1^{p_U}(\theta) + \theta u(\theta)\tilde{M}_2'(q(\theta))d\theta \\ &= p_q(s) - \int_s^t M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + s\tilde{M}_2(q(s)). \end{aligned} \tag{45}$$

Defining $H^{p_U}(t) = \int_0^t M_1^{p_U}(\theta)d\theta$ and $m_q(t) = \int_0^t \tilde{M}_2(q(\theta))d\theta$ we have that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } p_q(0) + m_q(t) > H^{p_U}(t), \tag{46}$$

$$u(t) \in [0,K] \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) = H^{p_U}(t), \tag{47}$$

$$u(t) = 0 \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) < H^{p_U}(t). \tag{48}$$

Lemma 5 (Reid (1968))**.** *Suppose* $p_q(0) + m_q(t) = H^{p_U}(t)$ *for* $t \in \{\underline{t}, \bar{t}\}$*,* $\underline{t} < \bar{t}$*. Let* $\alpha, \beta \in \mathbb{R}$ *and* $l(t) = \alpha + \beta t$*. If* $l(t) \leq H^{p_U}(t)$ *for all* $t \in [\underline{t}, \bar{t}]$*, then* $p_q(0) + m_q(t) \geq l(t)$ *for all* $t \in [\underline{t}, \bar{t}]$*.*

Proof. Suppose that $m_q(s) + p_q(0) < l(s)$ for some $s \in (\underline{t}, \bar{t})$. Then there exists $\varepsilon > 0$ and $\underline{t} < t_1 < t_2 < \bar{t}$ such that $m_q(t) + p_q(0) < l(t) - \varepsilon$ for $t \in (t_1, t_2)$, $m_q(t_1) + p_q(0) = l(t_1) - \varepsilon$, and $p_q(0) + m_q(t_2) = l(t_2) - \varepsilon$. This implies that $m_q'(t) = \tilde{M}_2(q(t))$ cannot be constant on (t_1, t_2). On the other hand, $m_q(t) + p_q(0) < l(t) - \varepsilon < H(t)$ and hence $u(t) = 0$ for $t \in (t_1, t_2)$ which implies that $m_q'(t)$ is constant, a contradiction. □

An immediate implication of the Lemma is that $p_q(0) + m_q(t) \geq \bar{H}^{p_U}_{[\underline{t}, \bar{t}]}(t)$, where $\bar{H}^{p_U}_{[\underline{t}, \bar{t}]}(t)$ denotes the convex hull of H^{p_U} restricted to $[\underline{t}, \bar{t}]$, i.e. the greatest convex function $G : [\underline{t}, \bar{t}] \to \mathbb{R}$ such that $G(t) < H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$. Furthermore, $p_q(0) + m_q(t)$ is convex because q and $\tilde{M}_2$ are non-decreasing. This yields the following

Corollary 1. *If* $p_q(0) + m_q(t) \leq H^{p_U}(t)$ *for all* $t \in [\underline{t}, \bar{t}]$*, with equality at the endpoints of the interval, then* $p_q(0) + m_q(t) = H^{p_U}(t)$*, for all* $t \in [\underline{t}, \underline{t}]$*.*

If $M_1^{p_U}$ is non-decreasing on $[\underline{t}, \bar{t}]$, then $H^{p_U}(t) = \bar{H}^{p_U}_{[\underline{t}, \bar{t}]}(t)$. Differentiating $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t}, \bar{t}]}$ yields $M_1^{p_U} = \tilde{M}_2(q(t))$ for $t \in [\underline{t}, \bar{t}]$.

If, however, $M_1^{p_U}$ is not monotonic on $[\underline{t}, \bar{t}]$, differentiating yields $\bar{M}^{p_U}_{[\underline{t}, \bar{t}]}(t) = \tilde{M}_2(q(t))$, where $\bar{M}^{p_U}_{[\underline{t}, \bar{t}]} = \frac{d\bar{H}^{p_U}_{[\underline{t}, \bar{t}]}(t)}{dt}$ is non-decreasing. Hence, Reid's Lemma provides a control theoretic technique to show that Myerson's ironing procedure can be used to solve irregular instances of mechanism design problems.

Now we establish some properties of the optimal solution. Define $x_{p_U}(t)$ as the solution to

$$M_2(x) = M_1^{p_U}(t),$$

and $x_{p_U}^{[\underline{t}, \bar{t}]}(t)$ as the solution to

$$M_2(x) = \bar{M}^{p_U}_{[\underline{t}, \bar{t}]}(t).$$

The derivative of x_{p_U} is given by

$$x_{p_U}'(t) = \frac{M_1'(t) + p_U v''(t)}{M_2'(x_{p_U}(t))}.$$

The assumptions on f_i and F_i guarantee that $x_{p_U}'(t)$ is continuous on $[0, 1]$. Let $K^{p_U} := \max_{t \in [0,1]} x_{p_U}'(t)$. Then $x_{p_U} \in \mathcal{L}^{K^{p_U}}$. In what follows, we write $\bar{H}^{p_U}$ for $\bar{H}^{p_U}_{[0,1]}$ and $\bar{M}_1^{p_U}$ for $\bar{M}^{p_U}_{[0,1]}$.

Lemma 6 (interior solution)**.** *Suppose* $u(t) \in (0, K)$ *for* $t \in [\underline{t}, \bar{t}]$*,* $\underline{t} < \bar{t}$*.*

(i) If $q(\underline{t}) \geq t_2^0$*, then* $q(t) = x_{p_U}(t)$ *for almost every* $t \in [\underline{t}, \bar{t}]$*.*

(ii) If $q(\bar{t}) < t_2^0$*, then* $M_1^{p_U}(t) = 0$ *for almost every* $t \in [\underline{t}, \bar{t}]$*.*

Proof. If $u(t) > 0$, we must have $\mu[0,t) = 0$. (46) – (48) imply that $p_q(0) + m_q(t) = H^{p_U}(t)$ for a.e. $t \in (\underline{t}, \overline{t})$. Differentiating this w.r.t. t yields

$$\tilde{M}_2(q(t)) = M_1^{p_U}(t).$$

If $q(t) \geq t_2^0$, $\tilde{M}_2(q(t)) = M_2(q(t))$ and hence that $q(t) = x_{p_U}(t)$. If $q(t) < t_2^0$, $\tilde{M}_2(q(t)) = 0$ and hence $M_1^{p_U}(t) = 0$. By continuity, the results extend to $\underline{t}$ and $\overline{t}$. □

Note that setting $q(t) = x_0(t)$ for $t \geq t_1^0$ and $q(t) = 0$ otherwise, yields the optimal solution of Myerson (1981). This is not surprising because p_U would be zero if the incentive compatibility constraint for the deadline were ignored.

Next we derive a necessary condition for intervals where $u(t)$ is in $\{0, K\}$.

Lemma 7 (constant q)**.** *Suppose $q(t) = a \in [0,1]$ on $[\underline{t}, \overline{t}]$, $\underline{t} < \overline{t}$, and let $[\underline{t}, \overline{t}]$ be chosen maximally. Then*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$
$$p_q(0) + m_q(t) = H^{p_U}(t),$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \overline{t}$ if $\overline{t} < 1$, and furthermore

$$M_1^{p_U}(\underline{t}) \geq \tilde{M}_2(a), \qquad \text{if } \underline{t} > 0, \tag{49}$$
$$\text{and} \quad M_1^{p_U}(\overline{t}) \leq \tilde{M}_2(a), \qquad \text{if } \overline{t} < 1. \tag{50}$$

Proof. If $q(t)$ is constant, then for almost every $t \in (\underline{t}, \overline{t})$, $u(t) = 0$ and therefore $p_q(t) + t\tilde{M}_2(q(t)) + \mu[0,t) \leq 0$ and $p_q(0) + m_q(t) + \mu[0,t) \leq H^{p_U}(t)$. As $\mu \geq 0$ and by continuity, $p_q(t) + t\tilde{M}_2(q(t)) \leq 0$ and $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for $t \in \{\underline{t}, \overline{t}\}$.

Suppose $\underline{t} > 0$ and let $S_- := \{0 < t < \underline{t} | u(t) > 0\}$. Since $q(t) < a$ for $t < \underline{t}$, and q is absolutely continuous, $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$. Hence there exists a sequence $t_n \nearrow \underline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equalities in the Lemma follow for $\underline{t} > 0$. For $\overline{t} < 1$ set $S_+ := \{\overline{t} < t < 1 \mid u(t) > 0\}$. $S_+ \cap [\overline{t}, \overline{t} + \delta]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \searrow \overline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equations in the Lemma follow for $\overline{t} < 1$.

To show (49), note that for almost every $t \in S_-$, $p_q(t) \geq -t\tilde{M}_2(q(t))$. (45) yields

$$p_q(\underline{t}) = p_q(t) - \int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - \underline{t}\tilde{M}_2(q(\underline{t})) + t\tilde{M}_2(q(t)).$$

With $p_q(\underline{t}) = -\underline{t}\tilde{M}_2(q(\underline{t}))$ and $p_q(t) \geq -t\tilde{M}_2(q(t))$ this implies

$$\int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \geq 0,$$

for almost every $t \in S_-$. If this inequality is fulfilled, there must be a $t' \in [t, \underline{t}]$ with

$$M_1^{p_U}(t') - \tilde{M}_2(q(t')) \geq 0.$$

As $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$, t and hence t' can be chosen arbitrarily close to $\underline{t}$. By continuity this implies

$$M_1^{p_U}(\underline{t}) - \tilde{M}_2(q(\underline{t})) \geq 0.$$

To show (50), note that for almost every $t \in S_+$, $p_q(t) \geq -t\tilde{M}_2(q(t))$. (45) yields

$$p_q(t) = p_q(\bar{t}) - \int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + \bar{t}\tilde{M}_2(q(\bar{t})).$$

With $p_q(\bar{t}) = -\bar{t}\tilde{M}_2(q(\bar{t}))$ and $p_q(t) \geq -t\tilde{M}_2(q(t))$ this implies

$$\int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \leq 0,$$

for almost every $t \in S_+$. As above there exists $t' \in [\bar{t}, t]$ such that the integrand is non-positive at t'. t and t' can be chosen arbitrarily close to $\bar{t}$. Therefore, by continuity

$$M_1^{p_U}(\bar{t}) - \tilde{M}_2(q(\bar{t})) \leq 0.$$

□

Lemma 7 implies that there cannot be an interval where q is constant and $q \in (0, 1)$ if x_{p_U} is strictly increasing.

Lemma 8. *Suppose $u(t) = K$ for almost every $t \in (\underline{t}, \bar{t})$, $\underline{t} < \bar{t}$. Let $(\underline{t}, \bar{t})$ be chosen maximally. Then for $t = \underline{t}$ and for $t = \bar{t}$ if $\bar{t} < 1$,*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \bar{t}$ if $\bar{t} < 1$

$$p_q(0) + m_q(t) = H^{p_U}(t).$$

Furthermore,

$$M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t})), \qquad \text{if } \underline{t} > 0, \tag{51}$$

$$\text{and} \quad M_1^{p_U}(\bar{t}) \geq \tilde{M}_2(q(\bar{t})), \qquad \text{if } \bar{t} \in [0, 1]. \tag{52}$$

Proof. The proof is very similar to the proof of the preceding Lemma. To show the first equality for $\underline{t} = 0$, the transversality condition can be used to obtain $p_q(0) \leq 0$. For $\bar{t} = 1$, (52) follows from $M_1^{p_U}(1) \geq \bar{v}$ and $\tilde{M}_2(q(t)) \leq \bar{v}$. □

The following Lemma, which does not depend on the maximum principle, excludes solutions that have lower winning probabilities than the undistorted solution x_0.

Lemma 9. *For $K > K^0$, let $b \geq t_1^0$ be the unique solution to $(b - t_1^0)K = x_0(b)$. If $q(t) \leq x_0(t)$ for all $t \in [t_1^0, 1]$ and $q(t) < x_0(t)$ for some $t \in [b, 1]$, then q is not optimal.*

Proof. Suppose by contradiction that q is an optimal solution with the properties stated in the Lemma. Let $b' \in [0, b]$ be the unique solution to $q(t_1^0) + (b' - t_1^0)K = x_0(b')$. Increase q such that $q(t) = x_0(t)$ on $[b', 1]$. To do this while maintaining the Lipschitz property with constant K, it may be necessary to increase q on $[t_1^0, b']$ but not below t_1^0. This increases the objective function because x_0 is the optimal solution absent constraints. But an increase in q also increases $U(1)$. Therefore the new solution satisfies the constraints of the maximization problem. This contradicts the optimality of q. □

Lemma 10. *If $\bar{U} < \overline{v}$, then $p_U \leq \bar{p}_U := \max_{t \in [0,1]} \frac{\overline{v} - v_1(t)}{v_1'(t)} < \infty$.*

Proof. Suppose to the contrary that, $p_U > \bar{p}_U$. Then $M_1^{p_U}(t) > \tilde{M}_2(1) = \overline{v}$ for all $t \in [0, 1]$. By Lemma 6.ii, this implies $q(\underline{t}) > t_2^0$ if $u(t) \in (0, K)$ on a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 6.i, this implies $q(t) = x_{p_U}(t)$, for all $t \in [\underline{t}, \overline{t}]$, but this is impossible if $M_1^{p_U}(t) > \overline{v}$. Hence we have $u(t) \in \{0, K\}$ for all $t \in [0, 1]$.

Suppose $u(t) = 0$ on a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 7, this implies $\overline{t} = 1$. If $u(t) = K$ on a maximal interval $[\underline{t}, \overline{t}]$, Lemma 8 implies $\underline{t} = 0$. Therefore, there exists $a \in [0, 1]$ such that $u(t) = K$ for $t < a$ and $u(t) = 0$ for $t > a$. Suppose $a > 0$. Lemma 8 implies $p_q(0) = 0$ if $a > 0$. As $M_1^{p_U}(t) > \tilde{M}_2(q(t))$ for all t, we have $p_q(t) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. Hence, $u(t) = 0$ for all $t > 0$ and $a = 0$.

If $q(t) = q$ is constant, Lemma 9 implies that $q > t_2^0$. Therefore, $p_q(0) = 0$ by the transversality condition. Using (45), we get $p_q(1) = -\int_0^1 M_1^{p_U} dt < 0$. The transversality condition and $p_U > 0$ imply $U(1) = \bar{U}$. This yields $q = \frac{\bar{U}}{\overline{v}}$. If $q < 1$, then $\mu[0, 1] = 0$ and hence $p_q(1) = \tilde{M}_2(q(1)) > 0$ by the transversality condition. But this contradicts $p_q(1) < 0$, so we must have $q = 1$ and hence $\bar{U} = \overline{v}$ which is ruled out by assumption. □

Note that $|x_{p_U}'(t)| \leq \frac{M_1'(t) + p_U |v_1'(t)|}{\min_{x \in [0,1]} |M_2'(x)|}$. Defining $\overline{K} := \max_{t \in [0,1]} \frac{M_1'(t) + \bar{p}_U |v_1'(t)|}{\min_{x \in [0,1]} |M_2'(x)|}$ we have $x_{p_U} \in \mathcal{L}^{\overline{K}}$ for all $p_U \leq \bar{p}_U$.

Lemma 11. *Let $(\underline{t}, \overline{t})$ be a maximal interval such that $u(t) = K$ for all $t \in (\underline{t}, \overline{t})$ and $K > K^{p_U}$. Then $q(t) < \max\{t_2^0, x_{p_U}(t)\}$ for all $t \in [\underline{t}, \overline{t}]$. If $\underline{t} > 0$, then $q(\underline{t}) < t_2^0$. Furthermore $\overline{t} < 1$.*

Proof. If $q(t) \geq \max\{t_2^0, x_{p_U}(t)\}$, then $q(\overline{t}) > \max\{t_2^0, x_{p_U}(\overline{t})\}$ because $K > \overline{K}$. Hence $\tilde{M}_2(q(\overline{t})) > M_1^{p_U}(\overline{t})$, a contradiction by Lemma 8. If $\underline{t} > 0$, $q(\underline{t}) < t_2^0$ because otherwise (51) and $K > \overline{K}$ would imply $q(\bar{t}) > \max\{t_2^0, x_{p_U}(\bar{t})\}$, which is a contradiction. Finally, $\bar{t} = 1$ would imply $q(t) < x_0(t)$ for all $t \in [t_1^0, 1)$. This is also a contradiction by Lemma 9. □

Lemma 12. *For $K > K^0$, $q(1) = 1$.*

Proof. Suppose $q(1) < 1$. By Lemma 9, $q(1) > t_2^0$. As $\tilde{M}_2(q(1)) > 0$ for $q(1) > t_2^0$, by the transversality condition, $p_q(1) = \tilde{M}_2(q(1)) > 0$ and $p_q(t) + t\tilde{M}_2(q(t)) > 0$, for t close to one. Hence $u(t) = K$ on a maximal interval $[\underline{t}, 1]$. As $K > K^0$, $q(\underline{t}) < x_0(\underline{t}) \leq x_{p_U}(\underline{t})$. Then, either $\tilde{M}_2(q(\underline{t})) < M_1^{p_U}(\underline{t})$ or $q(\underline{t}) < t_2^0$. The former inequality contradicts optimality by Lemma 8 and the latter contradicts optimality by Lemma 9. □

Define $c := \min\{t \mid q(t) = 1\}$. By the preceding Lemma, this is well defined for $K > K^0$.

Lemma 13. *For $K > \overline{K}$,*

$$\begin{aligned} p_q(0) + m_q(c) &= H^{p_U}(c), \\ p_q(c) + c\tilde{M}_2(q(c)) &= 0, \\ M_1^{p_U}(c) &= \tilde{M}_2(1). \end{aligned}$$

Proof. If $c < 1$ the first two equations are implied by Lemma 7. If $c = 1$, $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to one. ($u(t) = 0$ is ruled out by $c = 1$, $u(t) \neq K$ follows from the same argument as in the proof of Lemma 12). Hence, the first two equalities hold for t close to c and by Lemma 6 also the third equality holds for t close to c. By continuity the equalities also hold for c. If $c < 1$, $M_1^{p_U}(c) \geq \tilde{M}_2(q(c))$ by Lemma 7. For $K > \overline{K}$, $u(t) = K$ for a maximal interval $[\underline{t}, c]$ is not possible as Lemma 8 requires $M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t}))$. Hence $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to c. By Lemma 6 and continuity, the third equality follows for c. □

Lemma 14. *Let (U, q, u) be an optimal solution to $\mathcal{P}_C^K$ for $K > \overline{K}$.*

(i) Let $\underline{b} = \min\{q(t) \geq t_2^0\}$. Then there exists $\overline{b} \in [\underline{b}, c]$ such that $u(t) = K$ for $t \in [\underline{b}, \overline{b}]$, and $\tilde{M}_2(q(t)) = \bar{M}_{[\overline{b},1]}^{p_U}(t)$ for $t \in [\overline{b}, c]$. Furthermore, $c = \min\{t | \bar{M}_{[\overline{b},1]}^{p_U}(t) = \overline{v}\}$.

(ii) Let $\hat{t}_1 = \min\{t \mid \bar{M}_1^{p_U}(t) \geq 0\}$. Then $\underline{b} \to \hat{t}_1$ and $\overline{b} \to \hat{t}_1$ as $K \to \infty$.

(iii) For almost every $t < \underline{b}$,

$$u(t) \begin{cases} = 0, & \text{if } p_q(0) < H^{p_U}(t), \\ \in [0, K], & \text{if } p_q(0) = H^{p_U}(t), \\ = K, & \text{if } p_q(0) > H^{p_U}(t). \end{cases}$$

Proof. (iii) follows directly from (46)–(48) as $q(t) \leq t_2^0$ for $t < \underline{b}$ and hence $m_q(t) = 0$.

If $p_q(0) < H^{p_U}(t)$ for all $t \in [0,1]$, then $p_q(0) < 0$ and therefore $q(0) = 0$ by the transversality condition. Hence $p_q(0) + m_q(t) < H^{p_U}(t)$, and $q(t) = 0$ for all t, contradicting Lemma 9. Therefore $p_q(0) \geq \min_t H^{p_U}(t)$.

To show (i), we first show that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$. Three cases have to be considered. To do this we need the following definitions:

$$\underline{p}_q := \begin{cases} \min\{p_q \mid \lambda\{H^{p_U}(t) \leq p_q\}K \geq t_2^0\}, & \text{if } \lambda\{H^{p_U}(t) \leq 0\}K \geq t_2^0, \\ 0, & \text{otherwise,} \end{cases}$$
$$b^{\max} := \max\{b \mid \underline{p}_q \geq H^{p_U}(b)\},$$

where λ denotes the Lebesgue measure on $[0,1]$.

Case 1: $H^{p_U}(t) > 0$ for all $t > 0$. ($\Rightarrow \underline{p}_q = 0$, $b^{\max} = 0$)
In this case, $q(0) \geq t_2^0$. Otherwise $p_q(0) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. This would imply $q(1) = q(0) < 1$, a contradiction. Suppose $u(t) = K$ for a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 11, $\underline{t} > 0$ would imply $q(\underline{t}) < t_2^0$. Hence $\underline{t} = 0$. Also by Lemma 11, $q(t) \leq x_{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$ and hence $q(0) < x_{p_U}(0)$. This implies $p_q(0) + m_q(t) < H^{p_U}(t)$ for t close to zero, contradicting $u(t) = K$. Hence $u(t) < K$ for all $t \in [0,1]$. This requires $p_q(0) + m_q(t) \leq H(t)$ for all t by (46)–(48), and by Reid's Lemma, we have $\tilde{M}_2(q(t)) = M^{p_U}_{[0,c]}(t)$ for all $t \in [0,c]$. With $\underline{b} = \bar{b} = 0$, this shows $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$ in case 1.

Case 2: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) = 0$.
In this case, $q(b^{\max}) = t_2^0$. Suppose to the contrary that $q(b^{\max}) < t_2^0$. This implies $p_q(0) \leq \underline{p}_q$. Hence $p_q(0) + m_q(t) \leq \underline{p}_q < H^{p_U}(t)$ for all $t > b^{\max}$. This is a contradiction to optimality. Next, suppose that $q(b^{\max}) > t_2^0$. This implies $p_q(0) \geq \underline{p}_q$ and therefore $p_q(0) + m_q(b^{\max}) > H^{p_U}(b^{\max})$. Therefore $b^{\max}$ is contained in an interval $[\underline{t}, \bar{t}]$ where $u(t) = K$ and $b^{\max} < \bar{t}$. By Lemma 11, this is a contraction. Therefore $q(b^{\max}) = t_2^0$. By (iii) we must have $p_q(0) = \underline{p}_q$ and hence $p_q(0) + m_q(b^{\max}) = \underline{p}_q = H^{p_U}(b^{\max})$. Set $\underline{b} = \bar{b} = b^{\max}$. Lemma 11 also implies that $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [b^{\max}, c]$. Reid's Lemma then implies that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\underline{b}, c]$ for case 2.

Case 3: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) > 0$.
In this case, $q(b^{\max}) > t_2^0$ because otherwise $q(1) = q(b^{\max}) < 1$, which is a contradiction. This implies $\underline{b} < b^{\max}$. Since $p_q(0) \geq \underline{p}_q$, $p_q(0) + m_q(b^{\max}) > H(b^{\max}) = \underline{p}_q$. Hence $b^{\max}$ is in the interior of a maximal interval $[\underline{t}, \bar{t}]$ such that $u(t) = K$ for all $t \in [\underline{t}, \bar{t}]$. By Lemma 11, $q(\underline{t}) < t_2^0$. This implies that $\underline{b} \in (\underline{t}, b^{\max})$. By Lemma 8, $p_q(0) + m_q(\bar{t}) = H(\bar{t})$ and by Lemma 11, $p_q(0) + m_q(t) \leq H(t)$, for $t \in [\bar{t}, c]$. Hence, we can set $\bar{b} = \bar{t}$ and have thus shown $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\bar{b}, c]$ for case 3.

Claim: $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\bar{b}, c]$.
Note that $\bar{M}^{p_U}_{[\bar{b},1]}(c) \leq \bar{M}^{p_U}_{[\bar{b},c]}(c)$. To show the converse, note that as q is constant on $[c, 1]$, $p_q(0) + m_q(t) + \mu[0, t) \leq H^{p_U}(t)$ for a.e. $t \geq c$. This implies

$$p_q(0) + m_q(c) + (t - c)\overline{v} + \mu[c, t) \leq H^{p_U}(c) + \int_c^t M_1^{p_U}(s)ds,$$

$$\Leftrightarrow \qquad \int_c^t M_1^{p_U}(s)ds \geq \overline{v}(t - c) + \mu[c, t). \tag{53}$$

If $\bar{M}^{p_U}_{[\bar{b},1]}(c) < \overline{v}$, then $\int_c^t M_1^{p_U}(s)ds = H^{p_U}(t) - H^{p_U}(c) \leq H^{p_U}(t) - \bar{H}^{p_U}(c) < (t - c)\overline{v}$ for some $t > c$. This would contradict (53) so we must have $\bar{M}^{p_U}_{[\bar{b},1]}(c) \geq \overline{v} = \bar{M}^{p_U}_{[\bar{b},c]}(c)$. If $\bar{M}^{p_U}_{[\bar{b},c]}(c) = \bar{M}^{p_U}_{[\bar{b},1]}(c)$ we must have $\bar{M}^{p_U}_{[\bar{b},c]}(t) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\bar{b}, c]$. This proves the claim and $c = \min\{t \mid \bar{M}^{p_U}_{[\bar{b},1]}(t) = \overline{v}\}$ follows immediately. Hence we have shown (i).

It remains to show (ii): $\underline{p}_q \to \min_{t\in[0,1]} H(t)$ as $K \to \infty$. This implies that $b^{\max} \to \hat{t}_1$. Furthermore $\overline{b} \geq b^{\max} \geq \underline{b}$ and $\underline{b} - \overline{b} < \frac{1}{K}$. Hence $\underline{b} \to \hat{t}_1$ and $\overline{b} \to \hat{t}_1$ as $K \to \infty$. □

Now we can turn to the limiting solution as $K \to \infty$.

Proof of Theorem 5. The reduced form of $\bar{x}_i$ as defined in (15) is

$$\bar{q}_1(v_1, 2) = \begin{cases} 0, & \text{if } \bar{J}_1^{p_U}(v_1) < 0 \\ \underline{x}_1^0 F_2(v_2^0), & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \\ F_2(J_2^{-1}(\bar{J}_1^{p_U}(v_1)), & \text{if } 0 < \bar{J}_1^{p_U}(v_1) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ F_1((\bar{J}_1^{p_U})^{-1}(J_2(v_2))), & \text{otherwise.} \end{cases}$$

Changing variables, we have

$$\bar{q}_1(t) = \begin{cases} 0, & \text{if } t < \underline{t}_1^0, \\ \underline{q}_1^0 t_2^0, & \text{if } t \in [\underline{t}_1^0, \overline{t}_1^0], \\ M_2^{-1}(\bar{M}_1^{p_U}(t)), & \text{if } 0 < \bar{M}_1^{p_U}(t) \leq \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(t) = \begin{cases} 0, & \text{if } M_2(t) < 0, \\ (\bar{M}_1^{p_U})^{-1}(M_2(t)), & \text{otherwise,} \end{cases}$$

where $[\underline{t}_1^0, \overline{t}_1^0] = (\bar{M}_1^{p_U})^{-1}(0)$ and $\overline{t}_1^0 = F_1^{-1}(\overline{v}_1^0)$.

Obviously $\bar{q}_2(t) = \bar{q}_1^{-1}(t)$ if $t \geq t_2^0$ and $\bar{q}_2(t) = 0$ otherwise. Therefore, by Lemma 2, we only have to show optimality of $\bar{q}_1$. Let (q_1^n, q_2^n) be a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $\overline{K} < K_n \to \infty$ as $n \to \infty$. Denote the adjoint variables in these solutions by p_U^n and p_q^n, respectively, and let (q_1, q_2) be the a.e.-limit of the sequence. By Theorem 3, (q_1, q_2) is an optimal solution. We will show that $(\bar{q}_1, \bar{q}_2)$ yields the same expected revenue as the limit of any such sequence. Since $\bar{M}^{p_U}_{[\hat{t}_1,1]}(t) = \bar{M}_1^{p_U}(t)$ for all $t \in [\hat{t}_1, 1]$, Lemma 14 implies that $q_1(t) = \bar{q}_1(t)$ for $t > \hat{t}_1$ where $p_U = \lim_{n\to\infty} p_U^n$. Next we consider the limiting solution for $t < \hat{t}_1$. We first show the following

Claim: If $q_1(t)$ is not constant at $t < \hat{t}_1$, then $H^{p_U}(t) = \min_\theta H^{p_U}(\theta)$.
Suppose to the contrary that $H^{p_U}(t) > \min_\theta H^{p_U}(\theta)$. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that $H^{p_U}(\tau) > \min_\theta H^{p_U}(\theta) + \delta$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$. Since $p_q^n(0) \to \min_\theta H^{p_U}(\theta)$ for $n \to \infty$, there exists $N > 0$ such that for all $n > N$, $p_q^n(0) < H^{p_U^n}(\tau)$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$. This implies that q_1^n is constant on $(t-\varepsilon, t+\varepsilon)$ for $n > N$ and hence q_1 is constant on $(t-\varepsilon, t+\varepsilon)$ which is a contradiction. This proves the claim.

Now set $\underline{x}_1^0 = \left[t_2^0(v_1(\bar{t}_1^0) - v_1(\underline{t}_1^0))\right]^{-1} \int_{\underline{t}_1^0}^{\bar{t}_1^0} q_1(s)v_1'(s)ds$ and let $[\underline{t}, \bar{t}]$ be the interval where $q_1(t) = \underline{x}_1^0 t_2^0$ (if $q_1(t) \neq \underline{x}_1^0 t_2^0$ for all t, set $\underline{t} = \bar{t}$ such that $q_1(t) < \underline{x}_1^0 t_2^0$ iff $t < \underline{t}$). With this definition, $q_1(t) < \underline{x}_1^0 t_2^0$ for $t < \underline{t}$ and $q_1(t) > \underline{x}_1^0 t_2^0$ for $t > \underline{t}$, and q_1 is not constant at $\underline{t}$ and $\bar{t}$. The claim implies that $[\underline{t}_1^0, \underline{t}]$ and $[\bar{t}, \bar{t}_1^0]$ are unions of intervals $[a, b]$ such that either $M_1^{p_U}(t) = 0$ for all $t \in [a, b]$, or q_1 is constant on $[a, b]$ and $\int_a^b M_1^{p_U}(t)dt = 0$. Hence, setting $q_1(t) = \underline{x}_1^0 t_2^0$ does not change the value of the objective and by definition of $\underline{q}_1^0$, $U_1(1)$ is left unchanged. Uniqueness of p_U and $\underline{x}_1^0$ are obvious.

For the proof of (ii) and (iii) note that π_2 can be written as

$$\pi_2(\bar{U}) = \int_0^1 \left[\bar{x}_{p_{\bar{U}}}(t) M_1(t) + \int_{\bar{x}_{p_{\bar{U}}}(t)}^1 \tilde{M}_2(q)dq \right] dt.$$

We first show that $\pi_2(\bar{U})$ is Lipschitz. For $\bar{U}' > \bar{U}$,

$$\begin{aligned}
\left|\pi_2(\bar{U}') - \pi_2(\bar{U})\right| &= \left| \int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} M_1(t) - \tilde{M}_2(q)dqdt \right|, \\
&\leq \int_0^1 \left| \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} \underbrace{M_1(t) - \tilde{M}_2(q)}_{|.|\leq M<\infty} dq \right| dt, \\
&\leq M \int_0^1 \bar{x}_{p_{\bar{U}'}}(t) - \bar{x}_{p_{\bar{U}}}(t)dt, \\
&\leq \frac{M}{\underline{v}_1'}(\bar{U}' - \bar{U}),
\end{aligned}$$

where $\underline{v}_1' = \min_{t\in[0,1]} v_1'(t) > 0$ by our assumptions on the type distributions. Next we show that $\pi_2'(\bar{U}^+) \leq -p_U \leq \pi_2'(\bar{U}^-)$.

$$\begin{aligned}\frac{1}{h}(\pi_2(\bar{U}+h)-\pi_2(\bar{U})) &= \frac{1}{h}\int_0^1\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t)-\tilde{M}_2(q)dqdt,\\ &= -p_{\bar{U}}\underbrace{\frac{1}{h}\int_0^1\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} v_1'(t)dqdt}_{=h} + \frac{1}{h}\int_0^1\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t)-\tilde{M}_2(q)dqdt,\\ &= -p_{\bar{U}} + \frac{1}{h}\int_\alpha^1\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t)-\tilde{M}_2(q)dqdt\\ &\quad + \frac{1}{h}\int_{\beta_h}^\alpha\int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t)-\tilde{M}_2(q)dqdt,\end{aligned}$$

where $\alpha = \inf\{t|\bar{x}_{p_{\bar{U}}}(t) \geq t_2^0\}$ and $\beta_h = \inf\{t|\bar{x}_{p_{\bar{U}+h}}(t) \geq 0\}$. The first integral converges to $\int_\alpha^1 (M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t)))\frac{\partial \bar{x}_{p_{\bar{U}}}(t)}{\partial p_{\bar{U}}}dt$. If $x_{p_U}'(t) \neq 0$ then $M_1^{p_{\bar{U}}}(t) = \bar{M}_1^{p_{\bar{U}}}(t)$ and hence the integrand vanishes. If, on the other hand, $x_{p_{\bar{U}}}'(t) = 0$ for all t in a maximal interval $[\underline{t}, \bar{t}]$, then $\frac{\partial}{\partial p_{\bar{U}}}\bar{x}_{p_{\bar{U}}}'(t)$ is constant on $(\underline{t}, \bar{t})$. This implies $\int_{\underline{t}}^{\bar{t}}(M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t)))\frac{\partial \bar{x}_{p_{\bar{U}}}(t)}{\partial p_{\bar{U}}}dt = \int_{\underline{t}}^{\bar{t}}(M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t)))dt\frac{\partial \bar{x}_{p_{\bar{U}}}(\frac{\bar{t}+\underline{t}}{2})}{\partial p_{\bar{U}}} = \left(\int_{\underline{t}}^{\bar{t}} M_1^{p_U}(t)dt - (\bar{t}-\underline{t})\tilde{M}_2(\bar{x}_{p_{\bar{U}}}(\underline{t}))\right)\frac{\partial \bar{x}_{p_{\bar{U}}}(\frac{\bar{t}+\underline{t}}{2})}{\partial p_{\bar{U}}} = 0$. Hence, the first integral converges to zero. The second integral is non-positive if $h > 0$. Hence $\pi_2'(\bar{U}^+) \leq -p_U$ wherever the limit exists. Similarly one can show that $-p_U \leq \pi_2'(\bar{U}^-)$ wherever the limit exists. As $\pi_2(\bar{U})$ is non-increasing, it is differentiable almost everywhere and we have $\pi_2'(\bar{U}) = -p_U$ for almost every $\bar{U}$.

Since $\pi_2(\bar{U})$ is Lipschitz continuous $\pi_2(\bar{U}) = \pi_2(0) - \int_0^{\bar{U}} p_U(s)ds$. Therefore, as $p_U(\bar{U})$ is non-decreasing, π_2 is weakly concave. If $\{\bar{M}_1^{p_U}(t) = 0\}$ is a singleton $p_U(\bar{U})$ is strictly increasing an hence π_2 strictly concave. □

Appendix B. Other Proofs

Proof of Lemma 1. Consider the following allocation rule:

$$\hat{x}_1^1(v_1, d) := \begin{cases} x_1^1(v_1, 1) & \text{if } d = 1,\\ 0 & \text{if } d = 2,\end{cases}$$

$$\hat{x}_1^2(v_1, d, v_2) := \begin{cases} 0 & \text{if } d = 1,\\ x_1^1(v_1, 2) + x_1^2(v_1, 2, v_2) & \text{if } d = 2,\end{cases}$$

$$\text{and}\quad \hat{x}_2 := x_2.$$

The payment rule is not changed. Obviously, $\hat{x}$ is a feasible allocation rule that allocates only at the deadline. Since $\hat{x}_2 = x_2$, we only have to consider incentive compatibility and individual rationality of buyer one. The definition of $\hat{x}$ implies that $\hat{q}_1^1(v,1) = q_1^1(v,1)$ and $\hat{q}_1^1(v,2) + \hat{q}_1^2(v,2) = q_1^1(v,2) + q_1^2(v,2)$. Hence $\hat{U}_1(v,d,v',d) = U_1(v,d,v',d)$: A bidder who reports his deadline truthfully gets the same expected payoff in both mechanisms. This implies (IR). Since $\hat{q}_1^1(v',2) = 0$, a bidder with $d = 1$ who reports $d' = 2$ gets less utility in $(\hat{x}, y)$ than in (x, y). Formally

$$\hat{U}_1(v,1,v',2) = \hat{q}_1^1(v',2)\, v - p_1(v',2) = -p_1(v',2) \leq U(v,1,v',2).$$

Similarly for $d' = 1$ and $d = 2$, $\hat{U}_1(v,2,v',1) \leq U_1(v,2,v',1)$. Therefore $(IC1)$ for U implies that for all $(v,d),(v',d') \in [0,1] \times \{1,2\}$,

$$\hat{U}_1(v,d) = U_1(v,d) \geq U_1(v,d,v',d') \geq \hat{U}_1(v,d,v',d')$$

which is $(IC1)$ for $(\hat{x}, y)$. □

Proof of Theorem 1.

(i) Monotonicity of $q_1(.,d)$ and $q_2(.)$ together with $(PE1)$–$(PE2)$ is the standard characterization of incentive compatibility w.r.t. the valuation (Myerson, 1981). The rest follows from the discussion preceding the Theorem, where we have used monotonicity of $U_1(v,d)$ in v. This follows from $(PE1)$.

(ii) Suppose that x_1 is deterministic and (ICD^d) holds for $v \in \{0, \overline{v}\}$. Then $q_1(v,1) \in \{0,1\}$. Monotonicity of $q_1(.,1)$ implies that there is a cutoff valuation $v_c \in [0, \overline{v}]$ such that $q_1(v,1) = 0$ for $v < v_c$, and $q_1(v,1) = 1$ for $v > v_c$. $(PE1)$ and (ICD^d) for $v = 0$ imply that for all $v \in [0, v_c]$, $U_1(v,1) = U_1(0,1) \leq U_1(0,2) \leq U_1(v,2)$. For $v > v_c$, $(PE1)$, and (ICD^d) for $\overline{v}$ imply

$$\begin{aligned} U_1(v,1) &= U_1(\overline{v},1) - \int_v^{\overline{v}} d\theta \\ &\leq U_1(\overline{v},2) - \int_v^{\overline{v}} q_1(\theta,2) d\theta \\ &= U_1(v,2). \end{aligned}$$

□

Proof of Theorem 2. Substituting V_2^{opt} into the objective function we get

$$\pi_1(U) = \max_{q_1(.,1)} \int_0^{\overline{v}} \left[q_1(v,1) J_1(v|1) + (1 - q_1(v,1)) V_2^{\text{opt}} \right] f_1(v|1) dv, \tag{54}$$

Subject to (5)–(7), this is a control problem with state $U_1(v) = U_1(v,1)$ and measurable control $q_1(.) = q_1(v,1)$. The law of motion for the state is $U_1'(v) = q_1(v)$. We account for (6)–(7) by imposing the state constraint $U_1(v) \leq U(v)$, requiring the state to start at zero, $U_1(0) = 0$, and the control to take values between zero and one, $q_1(v) \in [0,1]$. The monotonicity constraint (8) will be neglected for the moment.

The Hamiltonian of this problem is

$$\mathcal{H}(U_1, q_1, p, v) = pq_1 + \left[q_1 J_1(v|1) + (1-q_1)V_2^{\text{opt}}\right] f_1(v|1)$$

where p is the adjoint variable of the state U_1. Let (U_1, q_1) be an optimal solution. By the Pontryagin maximum principle (c.f. Clarke (1983, pp. 211-212)) we have that $p(v) = p$ is constant and $p + \mu[0,\overline{v}] = 0$, where μ is a non-negative measure supported on the set $\{v \mid U_1(v) = U(v)\}$. Furthermore, for almost every v, $q_1(v)$ maximizes $\mathcal{H}(U_1(t), q_1, p + \mu[0,v), v)$. This implies that for almost every v,

$$\begin{aligned} q_1(v) &= 1, && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) > 0,\\ q_1(v) &\in [0,1], && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) = 0,\\ \text{and}\quad q_1(v) &= 0, && \text{if } p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) < 0. \end{aligned}$$

Since $p + \mu[0,v) \leq 0$, $q_1(v) = 0$ if $J_1(v|1) < V_2^{\text{opt}}$. But if $J_1(v|1) \geq V_2^{\text{opt}}$, Assumption 2 implies that $(J_1(v|1) - V_2^{\text{opt}})f_1(v|1)$ is strictly increasing. Since $\mu[0,v)$ is non-decreasing, $p + \mu[0,v) + (J_1(v|1) - V_2^{\text{opt}})f_1(v|1) = 0$ implies $p + \mu[0,v') + (J_1(v'|1) - V_2^{\text{opt}})f_1(v'|1) > 0$ for all $v' > v$. Therefore there is a unique value r_1 such that

$$q_1(v_1) = \begin{cases} 0, & \text{if } v_1 < r_1\\ 1, & \text{if } v_1 > r_1. \end{cases}$$

Obviously, any such solution satisfies (8). r_1 can be determined without resorting to optimal control theory. As the mechanism is deterministic, it is the lowest value such that $J_1(r_1) \geq V_2^{\text{opt}}$ and $U_1(\overline{v}, 1) = \overline{v} - r_1 \leq U(\overline{v})$. This yields the solution stated in the Theorem.

Inserting the solution in the objective function, we obtain

$$\pi_1(\bar{U}) = \int_{\max\{J_1^{-1}(V_2^{\text{opt}}|1), \overline{v}-\bar{U}\}}^{\overline{v}} J_1(v|1)f_1(v|1)dv + V_2^{\text{opt}} F_1(\max\{J_1^{-1}(V_2^{\text{opt}}|1), \overline{v}-\bar{U}\}|1).$$

$$\pi_1'(\bar{U}) = \begin{cases} (J_1(\overline{v}-\bar{U}|1) - V_2^{\text{opt}})f_1(\overline{v}-\bar{U}|1), & \text{if } J_1(\overline{v}-\bar{U}|1) > V_2^{\text{opt}},\\ 0 & \text{otherwise.} \end{cases}$$

For $\bar{U} \to \overline{v} - J_1^{-1}(V_2^{\text{opt}}|1)$ we have $\pi_1'(\bar{U}) \to 0$ since f_1 is bounded. Hence, $\pi_1'(\bar{U})$ is continuous. Using Assumption 2, we conclude that $\pi_1'(\bar{U})$ is strictly decreasing if $J_1(\overline{v}-\bar{U}|1) > V_2^{\text{opt}}$ and hence π_1 is strictly concave. □

Proof of Theorem 3. Since the relaxed solution is deterministic and $U_1(0,d) = 0$ for $d = 1, 2$, it suffices to check $U_1(\overline{v}, 1) \leq U_1(\overline{v}, 2)$. Since buyer one with valuation $\overline{v}$ wins with probability one, regardless of the reported deadline, this is equivalent to

$p_1(\overline{v},1) \geq p_1(\overline{v},2)$. The expected payment in period one equals the reserve price $J_1^{-1}(V_2^{\text{opt}}|1)$.

Now suppose $J_1(.|1)$ is strictly concave on $[v_1^0|1,\overline{v}]$, and $J_1(v|1) \geq J_1(v|2)$ for all $v \in [v_1^0|1,\overline{v}]$. In this case, $J_1^{-1}(.|1)$ is strictly convex on $[0,\overline{v}]$, and $J_1^{-1}(v|1) \leq J_1^{-1}(v|2)$ for $t \in [0,\overline{v}]$. Using Jensen's inequality and monotonicity of $J_1^{-1}(v|.)$ we get

$$\begin{aligned}
p_1(\overline{v},1) &= J_1^{-1}(V_2^{\text{opt}}|1) \\
&= J_1^{-1}\left(\int_0^{\overline{v}} \max\{0,J_2(v_2)\}f_2(v_2)dv_2 \,\middle|\, 1\right) \\
&< \int_0^{\overline{v}} J_1^{-1}\left(\max\{0,J_2(v_2)\}f_2(v_2) \mid 1\right) dv_2 \\
&\leq \int_0^{\overline{v}} J_1^{-1}\left(\max\{0,J_2(v_2)\}f_2(v_2) \mid 2\right) dv_2 \\
&= \int_0^{\overline{v}} y_1(\overline{v},2,v_2)f_2(v_2)dv_2 \\
&= p_1(\overline{v},2).
\end{aligned}$$

Applying monotonicity first and then Jensen's inequality yields the same result for $J_1(.|2)$ strictly concave. The proof of (ii) is very similar. □

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