

Time Preferences and Bargaining*

Sebastian Schweighofer-Kodritsch[†]

May 12, 2016

Abstract

This paper presents an analysis of general time preferences in the canonical Rubinstein (1982) model of bilateral alternating-offers bargaining. I derive a simple sufficient structure for optimal punishments and thereby fully characterize (i) the set of equilibrium outcomes for any given preference profile, and (ii) the set of preference profiles for which equilibrium is unique. When both players have a present bias—empirically, a property of most time preferences regarding consumption, and implied, e.g., by any hyperbolic or quasi-hyperbolic discounting—equilibrium is unique, stationary and efficient. When, instead, one player finds a near-future delay more costly than delay from the present—empirically common for time preferences over money—non-stationary equilibria arise that explain inefficiently delayed agreement with gradually increasing offers.

Keywords: time preferences, alternating offers, bargaining, optimal punishments, dynamic inconsistency, delay, gradualism

JEL classification: C78, D03, D74

*This paper generalizes my earlier work “On Time-Inconsistency in Bargaining”, which was based on my dissertation submitted to the LSE. I am deeply grateful for the support, guidance and inspiration I have received there during this project from Erik Eyster and Balázs Szentes. I owe special thanks to Yves Breitmöser, for numerous insightful comments on earlier drafts. I also thank Can Çeliktemur, Leonardo Felli, Paul Heidhues, Alex Imas, Kristóf Madarász, Francesco Nava, Motty Perry, Michele Piccione, Matthew Rabin, Ariel Rubinstein, Justin Valasek, Roel van Veldhuizen and Georg Weizsäcker for illuminating discussions, and my audiences at the LSE and STICERD, the RES PhD Presentation Meeting 2013, DIW Berlin, University of Bonn, University of Cologne, Goethe-University Frankfurt, Florida State University, University of Oslo, NOeG 2013 Conference, University of Innsbruck, NAG 2014 Conference, 6th Israeli Game Theory Conference, CEA 2014 Conference, FUR 2014 Conference, 25th International Conference on Game Theory and 13th Viennese Workshop on Optimal Control and Dynamic Games for helpful remarks.

[†]Humboldt-University Berlin and WZB Berlin Social Science Center. Mail: Wirtschaftswissenschaftliche Fakultät, Humboldt-Universität zu Berlin, Spandauer Str. 1, D-10099 Berlin, Germany. Phone: +49 (0)30 2093 5998. Email: sebastian.kodritsch@gmail.com.

1 Introduction

As a mechanism for sharing economic surplus, bargaining is pervasive in decentralized exchange. Understanding this mechanism’s functioning, in particular its efficiency properties, is of fundamental importance for any applied work investigating search frictions; e.g., see [Browning and Chiappori \(1998\)](#) on household behavior, or [Hall and Milgrom \(2008\)](#) on wages and unemployment. The same is true for the optimal design of institutions, when a central authority might impose an allocation instead of leaving it up to decentralized bargaining; e.g., a manager allocating tasks to a team of employees, or a government regulating industry standards.

In the absence of irrevocable commitments, time is the prime variable of bargaining agreements: the parties may agree not only now or never, but also sooner or later. The question of how the parties’ attitudes to delay govern their bargaining lies at the heart of modern bargaining theory ([Ståhl, 1972](#); [Rubinstein, 1982](#)). This paper is the first to provide a general answer to this question, covering also various attitudes to delay other than exponential discounting (ED), while allowing the parties’ bargaining strategies to be arbitrarily history-dependent.

Specifically, I fully characterize bilateral alternating-offers bargaining (without a deadline) when each party i evaluates delayed agreements with a continuous utility function $U_i(x_i, t)$, assuming only that she always prefers a greater share x_i of the surplus, holding delay t constant, and a shorter delay t , holding her share $x_i > 0$ constant. With the sole exception of ED, all such preferences are dynamically inconsistent. Hence this paper demonstrates how various forms of dynamic inconsistency are analytically tractable, it sheds light on the robustness of the celebrated conclusions obtained under ED, and it brings the vast body of empirical research on time preferences to bear directly on the study of bargaining.

Dynamically inconsistent preferences require a new analytical approach to this game. As I show, the standard technique of characterizing equilibrium via recursions on the players’ equilibrium payoff/utility extrema (see [Shaked and Sutton, 1984](#)) generally fails in face of the possibility of multiple and delayed equilibrium agreements because a player may not rank these consistently across different points in time.

I circumvent this problem by directly analyzing the off-path punishments (continuation equilibria, upon rejection) that support all equilibrium play, i.e. *optimal penal codes* (cf. [Abreu, 1988](#)). I show that it is sufficient to consider *simple* penal codes described by four outcomes.¹ These define four punishments such that the exact same punishment is used to

¹[Mailath, Nocke, and White \(2015\)](#) present related examples of repeated *sequential* games where no simple penal code is optimal due to incentive trade-offs between within-round and continuation punishment. By contrast, here a single round’s play determines all payoffs.

deter *any* deviation by a given player in a given role (hence four), entirely independent of the deviation’s history; e.g., any deviation by player 1 as the proposer triggers the exact same continuation equilibrium (upon rejection), on as well as off the path. Loosely speaking, the strategic complexity doubles when we allow for dynamic inconsistency: while in general four punishment outcomes are also necessary to characterize optimal penal codes, under ED two of them are redundant. This fundamental insight renders the history-dependence of strategies analytically tractable, under minimal preference assumptions, and thus enables me to obtain the paper’s core results: a full characterization of both (i) equilibrium outcomes for any given preference profile, and (ii) those preference profiles that imply a unique equilibrium.

Viewed through the lens of the existing evidence on time preferences, this characterization yields the novel prediction that the bargaining mechanism’s functioning depends on whether parties share consumption (e.g., a literal cake, effort provision, social esteem) or money. The reason is that people’s time preferences differ systematically across these domains, in ways that imply different bargaining equilibria.²

Regarding consumption, a rather general *present bias*, where delay is most costly when it takes consumption away from the immediate present, is well-established (e.g., McClure, Ericson, Laibson, Loewenstein, and Cohen, 2007; Brown, Chua, and Camerer, 2009; Ned Augenblick and Sprenger, 2015). I show that when both parties’ preferences exhibit this type of bias, bargaining equilibrium is unique, stationary and efficient. This result proves the sharp conclusions under ED robust to various forms of present bias—in particular any hyperbolic (Chung and Herrnstein, 1967; Ainslie, 1975) or quasi-hyperbolic (Phelps and Pollak, 1968; Laibson, 1997) discounting—and finally opens the door to the use of non-cooperative bargaining in applied economic modeling that studies such preferences.

Regarding money, a careful examination of the existing experimental evidence (see appendix B.1) reveals that, across studies, at least around a third of individuals’ time preferences exhibit a (*near-*) *future bias*.³ This means that they are most impatient about delay beyond some time in the near future rather than the immediate present; near-future bias would for instance be implied by a discounting function that is initially concave (e.g., Ebert and Prelec, 2007). When (at least) one of the parties has a sufficiently strong bias of this type, bargaining has multiple non-stationary equilibria that necessarily involve inefficiently

²This introductory discussion focuses on preference profiles implying a unique *stationary* equilibrium, as is true under standard concavity assumptions concerning the surplus share.

³This finding has also been called “reverse time-inconsistency” (Sayman and Öncüler, 2009), “increasing impatience” (Attema, Bleichrodt, Rohde, and Wakker, 2010), “hypobolic discounting” (Eil, 2012) and “patient shifts” (Read, Frederick, and Airoldi, 2012). Due to the focus on (quasi-) hyperbolic discounting in the literature, it is often not made explicit: e.g., Andreoni and Sprenger (2012) estimate a median “beta” greater than one, i.e. the majority exhibits a near-future bias; however, they concentrate on the absence of present bias.

delayed, gradual agreement. Moreover, as the frequency of offers increases, an arbitrarily small bias becomes sufficient, rendering the conclusions from ED fragile overall.

Intuitively, a near-future biased individual does not mind bargaining for a few periods; further, future delay is, however, costly. In contrast to present bias, she therefore does not exert control over the delay she finds most painful. Indeed, after having bargained for a few periods, she will again be very willing to bargain for a few more, in return for only a slightly larger share. Given there would subsequently be further delay, her opponent is able to extract a premium for immediate agreement, because that avoids handing over control to her “excessively” patient future selves, and this premium in turn supports their delay. Thus delay is self-enforcing.⁴ Although the familiar proposer advantage exerts a countervailing force, it merely reduces the “inconsistency premium” that the opponent might credibly ask for and becomes negligible when offers are made frequently.

The non-stationary delay equilibria capture elementary strategic considerations. A party does not propose a Pareto-improving division early based on the belief that she would thus lead her opponent to expect an even superior—but not itself Pareto-improving—outcome and, consequently, reject it. To the extent that the set of Pareto-improving divisions shrinks as the parties approach the time of agreement, due to their impatience, those equilibria naturally permit gradualism (see [Compte and Jehiel, 2004](#)): the parties’ offers increase gradually towards those of the eventually agreed division. Moreover, when the two parties’ preferences are not too “asymmetric”, the set of non-stationary equilibrium outcomes also includes the focal point of an *immediate* equal split. Thus near-future bias offers a unified explanation of two prominent, yet seemingly incompatible tendencies in bargaining.⁵

More generally, the multiplicity of (non-stationary) equilibrium outcomes in this case yields an interval prediction. As a function of the agreement’s delay, the interval for a party’s share is monotonically shrinking (in terms of set inclusion). The basic multiplicity is roughly in line with experimental studies of bargaining (over money), which have observed that parties indeed reach different agreements—with and without delay—in objectively identical bargaining situations (see [Roth, 1995](#)). Given its explanation of two prominent behavioral tendencies, the interval prediction therefore opens new avenues for econometric analyses of bargaining data (see [Tamer, 2010](#); [De Paula, 2013](#)).

Multiplicity also arises, *a fortiori*, within the hitherto most successful explanation of inefficient delay through incomplete information (assuming ED), which creates a trade-off

⁴This nature of delay is novel. It means there exist “truly” non-stationary delay equilibria that are non-stationary in *every* subgame; prior constructions in related work have instead relied on stationary equilibrium off-path (see [Avery and Zemsky, 1994](#)).

⁵For evidence from experiments implementing an indefinite horizon see [Weg, Rapoport, and Felsenthal 1990](#); [Zwick, Rapoport, and Howard 1992](#).

between information and time (see, e.g., the survey by [Kennan and Wilson, 1993](#)). The model assumptions are certainly complementary, with both approaches being able to capture bargaining as we tend to observe it.⁶ A main difference is, however, that the explanation based on time preferences proposed here is more fundamental (there is no theory of delay without time preferences) and directly builds on independent empirical research. Moreover, the discipline on beliefs under perfect information makes it more readily applicable, in both theoretical and empirical work: equilibrium is fully characterized for a very general class of time preferences, and experimental researchers are likely to have better control over the participants’ induced preferences than their beliefs about others.⁷

Related Literature. There exists little prior work on bargaining that analyzes dynamically inconsistent time preferences: [Burgos, Grant, and Kajii \(2002a\)](#); [Akin \(2007\)](#); [Ok and Masatlioglu \(2007\)](#); [Noor \(2011\)](#).⁸ All of these papers restrict attention to stationary strategies, however, thus severely limiting the potential for dynamic inconsistency to matter.⁹ This paper studies a general class of preferences that covers all of those studied previously and at the same time generalizes the analysis to arbitrarily history-dependent strategies.

Other closely related work investigates non-stationary time preferences that are, however, dynamically consistent ([Binmore, 1987](#); [Rusinowska, 2004](#); [Pan, Webb, and Zank, 2015](#)); e.g., a player may apply different discount rates to June 30, 2016, and July 1, 2016, but independent of the delay to these dates.¹⁰ I abstract from such exogenous effects of time on the players’ preferences, which would also appear negligible under frequent offers; instead, the discount rate for any given period may depend only on the delay to this period, not on its absolute time. Moreover, I maintain that preferences are history-independent; i.e., unlike in [Fershtman and Seidmann \(1993\)](#), and [Li \(2007\)](#), where the best past offer acts as

⁶This comparison is valid only for bargaining without a strict deadline. Given a finite horizon and perfect information, any form of impatience results in a unique backwards induction solution, with immediate agreement in any round, that is fully determined by the parties’ attitudes to a single period of delay only.

⁷These arguments apply as well to recent related approaches in which players may hold incorrect beliefs about their opponent: under “optimism” (see the survey by [Yildiz, 2011](#)) they may incorrectly believe that they have better knowledge of their proposer advantage, and under “strategic uncertainty” ([Friedenberg, 2014](#)) they may incorrectly interpret opponent deviations as irrationality. These two approaches appear well-suited, however, to explain the effects introduced by deadlines (optimism for “loose” protocols, and strategic uncertainty for “strict” ones where backwards induction rationality is very sensibly at stake).

⁸[Burgos et al. \(2002a\)](#) study bargaining with breakdown risk for certain non-expected-utility preferences; [Akin \(2007\)](#) also investigates naïveté and learning by quasi-hyperbolic discounters.

⁹The sole exception is [Lu \(2015\)](#) who studies bargaining by sophisticated quasi-hyperbolic discounters. In his model bargaining is over an infinite stream of cakes rather than a single one, however, so agreements are infinite consumption commitments.

¹⁰This is similar to time-varying surplus as in [Coles and Muthoo \(2003\)](#); see also [Merlo and Wilson \(1995\)](#) and [Cripps \(1998\)](#), who investigate Markovian surplus processes. All of these models maintain dynamic consistency of preferences; indeed, delay typically occurs only when efficient.

a “reference point”, the parties are consequentialist, caring only about the eventual surplus division and its delay, not *how* agreement is reached.

Regarding the power of history-dependent strategies in generating delay, also the work that endogenizes the timing of offers, starting with [Perry and Reny \(1993\)](#) and [Sákovic \(1993\)](#), as well as that on “negotiation” by [Busch and Wen \(1995\)](#) where, as long as parties fail to agree, they repeatedly play a disagreement game, share similarities. The underlying reason for why history-dependent strategies are powerful here—namely, dynamic inconsistency—is fundamentally different, however.

Finally, this paper contributes to the wider literature that explores the bargaining implications of relaxing certain hitherto standard but “unrealistic” (or extreme) assumptions about the players. Whereas this model’s only non-standard feature is dynamically inconsistent *preferences*, relaxing ED, most of the recent literature has been concerned with non-standard *beliefs*, relaxing common knowledge of the bargaining protocol or of players’ rationality (e.g., [Yildiz, 2011](#); [Friedenberg, 2014](#)).

Outline. After introducing the formal model in section 2, section 3 already describes the main results of this paper for the special case where players maximize their discounted share of the surplus for arbitrary discounting; this generalizes the most widely used version of the [Rubinstein \(1982\)](#) model. Section 4 then contains the full-fledged equilibrium characterization, and I further investigate equilibrium uniqueness and multiplicity/delay in section 5. Finally, I offer some concluding remarks in section 6. All formal proofs (as well as additional notation) are found in appendix A; appendix B contains supplementary material.

2 Bargaining and Time Preferences

2.1 Bargaining Protocol, Histories and Strategies

I follow [Rubinstein \(1982\)](#) exactly with regards to the bargaining protocol of (possibly indefinitely) alternating offers. There are two players $\{1, 2\} \equiv I$, who bargain over a perfectly divisible surplus of (normalized) size one. Throughout the paper, whenever $i \in I$ denotes one player, $j \equiv 3 - i$ denotes the other. In round $n \in \mathbb{N}$, player $P(n)$ proposes a surplus division $x \in \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 + x_2 = 1\} \equiv X$ to her opponent $R(n)$ (equivalently, $P(n)$ offers $R(n)$ share $x_{R(n)}$), who then responds by either accepting or rejecting the proposal. If it is accepted, the game ends with agreement on x ; otherwise, one period of time elapses until the subsequent round $n + 1$, where the roles of proposer and respondent are reversed, so $P(n + 1) = R(n)$. This process of alternating offers begins with player 1’s proposal, i.e.

$P(1) = 1$, and continues until there is agreement, possibly without ever terminating.

A *history* of play to the beginning of round $n \in \mathbb{N}$ is a sequence of $n - 1$ rejected proposals $h^{n-1} \in X^{n-1}$, where $X^0 \equiv \{\emptyset\}$; throughout, “history” always refers to such a beginning-of-round history. A *strategy* σ_i of a player i assigns to every possible such history h^{n-1} an available action: if $i = P(n)$, then $\sigma_i(h^{n-1})$ specifies a proposal $x \in X$, and if $i = R(n)$, then it specifies for every possible proposal whether she accepts or rejects it; without loss of generality, I identify this response rule $\sigma_{R(n)}(h^{n-1})$ with the set of accepted proposals $Y \in \mathcal{P}(X)$. If i 's response rule Y has $x \in Y \Leftrightarrow x_i \geq q$, I say that i *accepts with threshold* q . A strategy σ_i is *stationary* if it specifies the same proposal x and response rule Y , irrespective of history. Finally, a *strategy profile* σ is a pair of strategies $(\sigma_{P(1)}, \sigma_{R(1)})$, and its prescribed *play* after history h^{n-1} is $\sigma(h^{n-1}) \equiv (\sigma_{P(n)}(h^{n-1}), \sigma_{R(n)}(h^{n-1}))$.

2.2 Outcomes and (Time) Preferences

If the players agree on division x with a delay of t periods, I call the outcome (x, t) , and if they perpetually fail to agree, I call it $((0, 0), \infty)$. Thus defined in terms of relative time (delay), the set of possible outcomes is the same after any history. A player i 's preferences are formulated over the set $A_i \equiv [0, 1] \times T$, for $T \equiv \mathbb{N}_0 \cup \{\infty\}$, of i 's *personal outcomes* that are her own share and the delay of agreement.

Assumption 1. *In any round n , a player i 's preferences over personal outcomes are represented by the same utility function $U_i : A_i \rightarrow \mathbb{R}$, satisfying the following properties:*

1. *Continuity:* $\{a \in A_i | U_i(a) \geq k\}$ and $\{a \in A_i | U_i(a) \leq k\}$ are closed for all $k \in \mathbb{R}$;¹¹
2. *Desirability:* $q < q'$ implies $U_i(q, t) < U_i(q', t)$ for all t ;¹²
3. *Impatience:*

(a) $t > t'$ implies $U_i(q, t) \leq U_i(q, t')$ for all q ,

(b) $q > 0$ implies $U_i(q, 0) > U_i(q, 1)$, and

(c) $\lim_{t \rightarrow \infty} U_i(1, t) \leq U_i(0, 0)$ or there exists a finite \hat{t} such that $U_i(q, t) = U_i(q, \hat{t})$ for all q and all $t \geq \hat{t}$.

¹¹Closedness refers to the product topology on A_i , where $[0, 1]$ and T are endowed with the relative standard and discrete topologies, respectively.

¹²Absent separability, desirability cannot be formulated independent of the time dimension; specifically, (2.) rules out that a player be entirely indifferent regarding her share once delay gets “too long”. A slight generalization can accommodate such preferences as well, however, without requiring a single change in the results or proofs presented: replace property (2.) with “for any $t \in T$, either U_i is constant on $[0, 1] \times \{t' \in T | t' \geq t\}$ or $q < q'$ implies $U_i(q, t) < U_i(q', t)$.”

Continuity (1.) is a standard technical assumption, and desirability (2.) defines the conflict of interest in the bargaining problem. Property (3.) corresponds to a general notion of impatience regarding agreement: for any given division of the surplus, players do not prefer later over sooner agreement (3.a), if a division yields them a positive share they prefer immediate agreement over delayed agreement (3.b), and whenever they do not become “overwhelmingly” impatient for delay approaching infinity (the standard case guaranteeing “continuity at infinity”), they must be impatient only regarding a finite horizon (3.c). In what follows, by “impatience” I refer only to the two properties (3.ab). The role of property (3.c) is technical: together with continuity, it guarantees existence of a “worst” equilibrium, and I point out explicitly where it is used.

Assumption 1 covers all models of time preferences with impatience put forward in the literature (see [Manzini and Mariotti, 2009](#)).¹³ It generalizes the most widely studied class of separable time preferences (i.e., discounted utility) axiomatized by [Fishburn and Rubinstein \(1982, thm. 1\)](#), where $U_i(q, t) = d(t) \cdot u(q)$ with $d(\cdot)$ a decreasing “discounting” function, to also cover various non-separable time preferences such as those proposed by [Benhabib, Bisin, and Schotter \(2010\)](#) or [Noor \(2011\)](#).¹⁴ An *instantaneous utility function* can nonetheless be defined by $u_i(q) \equiv U_i(q, 0)$, and it is continuous and increasing.

[Halevy \(2015, prop. 4\)](#) shows that a player’s preferences satisfying assumption 1 are dynamically consistent if and only if they satisfy the stationarity axiom. The latter requires that the preference over two outcomes (q, t) and (q', t') depend only on their relative delay: $U_i(q, t) \geq U_i(q', t')$ if and only if $U_i(q, t + \tau) \geq U_i(q', t' + \tau)$ for any $\tau \in T$; this would here yield ED, where $U_i(q, t) = \delta^t \cdot u(q)$ ([Fishburn and Rubinstein, 1982, thm. 2](#)). With the exception of ED, all time preferences studied here are therefore dynamically inconsistent.

2.3 Equilibrium Concept

I abstract from informational frictions by assuming that the players’ preferences are common knowledge. In the terminology coined by [O’Donoghue and Rabin \(1999\)](#), players are then fully “sophisticated” about their own as well as their opponent’s dynamic inconsistency.

¹³The focus of this paper is on time preferences in the usual broad sense of preferences over delayed rewards, which have been extensively researched empirically (see appendix [B.1](#)). However, assumption 1 can also (alternatively or additionally) accommodate costs that are proper to the bargaining activity; e.g., with $U_i(q, t) = q - c(t)$ for $c(\cdot)$ increasing, party i would rather quit bargaining altogether if she expected it to take some time but eventually result only in a very small payoff (e.g., consider $q = 0$).

¹⁴[Ok and Masatlioglu \(2007\)](#) propose a theory of *relative* discounting that relaxes transitivity for comparisons across three different delays, thus capturing also sub-additive discounting ([Read, 2001](#)) and similarity-based choice ([Rubinstein, 2003](#)). Within the simplified structure of equilibria established below, these failures of transitivity play no role, however. Hence, the characterization of equilibrium *outcomes* also covers these “preferences” (formally, in their notation, let $d(t) \equiv \eta(0, t)$).

The equilibrium concept has to incorporate how intertemporal conflict within a player’s own preferences is resolved. In single-person decision problems, the standard solution concept for such sophisticated decision makers is that of Strotz-Pollak equilibrium (Strotz, 1956; Pollak, 1968), also known as multiple-selves equilibrium (Piccione and Rubinstein, 1997); it is the subgame-perfect Nash equilibrium (SPNE) of an auxiliary game in which the decision-maker at any point in time is a distinct non-cooperative player. Technically, one then looks for strategy profiles that are robust to one-stage deviations, and this formalizes the presumption that a decision-maker cannot internally commit to future behavior.

The equilibrium notion employed here is the natural extension of this concept to strategic interaction by multiple decision-makers (cf. Chade, Prokopovych, and Smith, 2008). To facilitate its definition, let $z_i^{h^{n-1}}(x, Y|\sigma)$ denote the personal outcome of player i , as of round n , that obtains if, following history h^{n-1} , $P(n)$ proposes x , $R(n)$ uses response rule Y , and in case there is no agreement, i.e. $x \notin Y$, both players subsequently adhere to strategy profile σ ; e.g., if $\sigma_{P(n+1)}(h^{n-1}, x) = x' \in \sigma_{R(n+1)}(h^{n-1}, x)$, then $z_i^{h^{n-1}}(x, Y|\sigma)$ equals $(x_i, 0)$ whenever $x \in Y$, and $(x'_i, 1)$ otherwise; accordingly, $z_i^{h^{n-1}, x}(\sigma(h^{n-1}, x)|\sigma) = (x'_i, 0)$.

Definition 1. A strategy profile σ is a **multiple-selves equilibrium (“equilibrium”)** if, for any round n , history h^{n-1} , division x and response rule Y ,

$$\begin{aligned} U_{P(n)}\left(z_{P(n)}^{h^{n-1}}\left(\sigma\left(h^{n-1}\right)\middle|\sigma\right)\right) &\geq U_{P(n)}\left(z_{P(n)}^{h^{n-1}}\left(x, \sigma_{R(n)}\left(h^{n-1}\right)\middle|\sigma\right)\right); \\ U_{R(n)}\left(z_{R(n)}^{h^{n-1}}\left(x, \sigma_{R(n)}\left(h^{n-1}\right)\middle|\sigma\right)\right) &\geq U_{R(n)}\left(z_{R(n)}^{h^{n-1}}\left(x, Y\middle|\sigma\right)\right). \end{aligned}$$

Observe that this indeed defines the SPNE of the auxiliary game where the set of players is taken to be $I \times \mathbb{N}$. The well-known one-stage deviation principle (e.g., Fudenberg and Tirole, 1991, thm. 4.2) says that it coincides with SPNE of the actual game played by I whenever both players’ preferences satisfy ED; hence this paper’s model contains that of Rubinstein (1982) as a special case.¹⁵

2.4 Preliminaries

A central property for the analysis of this game is its stationarity: conditional on failure to agree, the game repeats itself every two rounds. Hence, ignoring history, all subgames beginning with the very same player i ’s proposal are identical and, in particular, have the same

¹⁵As in Rubinstein (1982), I consider only pure strategies—a common restriction in this literature, even in models with inherent risk (e.g., Merlo and Wilson, 1995; Cripps, 1998). Permitting randomization devices, while unlikely to enlarge the set of equilibrium outcomes (cf. Binmore, 1987), would come at the cost of augmenting the domain of preferences by risk, however, adding a layer of cardinality. Nonetheless, the model has a straightforward interpretation in terms of bargaining under the shadow of a constant risk of breakdown with non-expected-utility preferences; see appendix B.4.2.

equilibria; denote this game by G_i . The above defines G_1 ; the sole modification of specifying player 2 as the initial proposer, $P(1) = 2$, defines game G_2 . To distinguish absolute and relative time, throughout, I use n for rounds of a given bargaining game (absolute time) and t for delays to a given agreement (relative time).

Let then $A_i^* \subseteq A_i$ be the set of player i 's personal outcomes that are equilibrium outcomes in G_i . The equilibrium characterization will center on a player i 's *minimal proposer value* v_i^* and *minimal rejection value* w_i^* , as well as the *supremal delay* t_i^* in G_i , given by:

$$\begin{aligned} v_i^* &\equiv \min \{U_i(q, t) \mid (q, t) \in A_i^*\} \\ w_i^* &\equiv \min \{U_i(q, t + 1) \mid (q, t) \in A_i^*\} \\ t_i^* &\equiv \sup \{t \in T \mid \exists q \in [0, 1], (q, t) \in A_i^*\}. \end{aligned}$$

3 The Case of Discounted Shares

The most widely used version of the [Rubinstein \(1982\)](#) model has the bargainers maximize their exponentially discounted surplus share. To make the key results of this paper quickly accessible, this section illustrates them for the generalization of this case only in terms of *discounting*; in fact, under the following common strengthening of assumption [1](#) it summarizes all information necessary to apply the results in either theoretical or empirical work.

Assumption 2. *In any round n , a player i 's preferences over personal outcomes are represented by the same utility function $U_i : A_i \rightarrow \mathbb{R}$ such that*

$$U_i(q, t) = \left(\prod_{s=1}^t \delta_i(s) \right) \cdot q,$$

where (i) $0 < \delta_i(s) < 1$ for any positive s , and (ii) $\lim_{t \rightarrow \infty} \prod_{s=1}^t \delta_i(s) = 0$.¹⁶

A player i 's total discount factor for a delay of t periods, denoted $d_i(t)$, is the product $\prod_{s=1}^t \delta_i(s)$ of the intermittent per-period discount factors. Indifference between two outcomes $(q, t - 1)$ and (q', t) means that $q = \delta_i(t) \cdot q'$, so unless discounting is constant— $\delta_i(\cdot) = \delta_i$, i.e. ED—it is dynamically inconsistent.

The burden of deciding about delay in bargaining is ultimately on the player responding to an offer. Regardless of its exact form, the respondent's impatience bestows a strategic advantage upon the proposing player, guaranteeing the latter a minimal rent. In particular, perpetual disagreement can therefore not be an equilibrium outcome.

¹⁶I follow the convention that the empty product for $t = 0$ equals one.

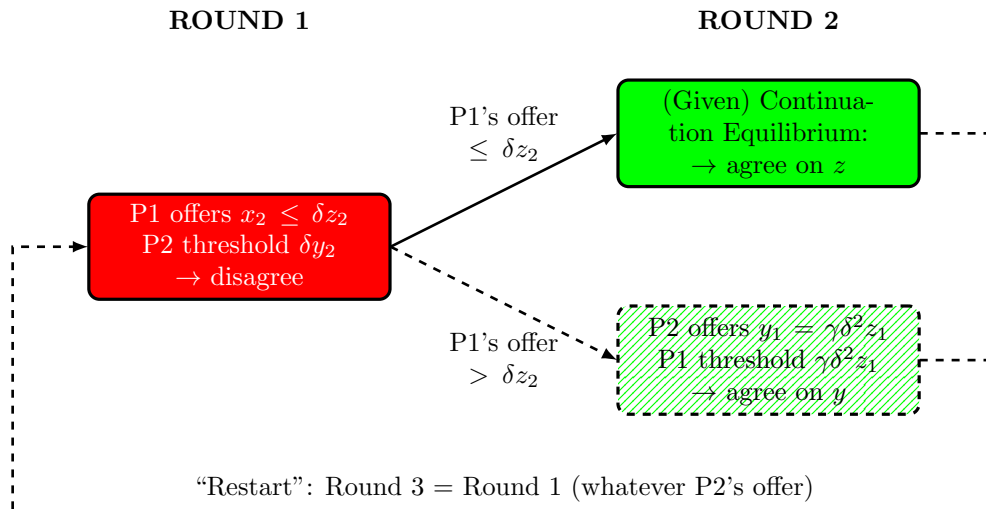


Figure 1: Delay equilibrium in example 1 (assuming $\delta y_2 \geq 1 - \delta z_1$). The equilibrium path uses solid lines/borders, and dashed ones indicate supporting off-path behavior.

Given the bargainers eventually agree, it is straightforward to characterize *stationary* equilibrium and establish equilibrium existence: starting from agreement on division x when player i makes an offer, two rounds of backwards induction must lead to the same agreement. Under assumption 2 there exists a unique such agreement, hence a unique stationary equilibrium: i always proposes the same division x and accepts with the same threshold y_i —equal to j 's offer—such that

$$x_i = 1 - \delta_j(1) \cdot (1 - y_i) \text{ and } y_i = \delta_i(1) \cdot x_i. \quad (1)$$

Stationary equilibrium assumes that bargainers are unresponsive to their opponent's past behavior (as well as their own) and predicts immediate agreement after any history; both is at odds with observed bargaining behavior. Consider then the following example.

Example 1. Od (player 1) and Eve (player 2) bargain over how to “split a dollar”. Their preferences satisfy assumption 2, where it is only specified that both discount a first period of delay with common factor $\delta_i(1) = \delta$, and that Od discounts a second period of delay with factor $\delta_1(2) = \gamma\delta$ for $\gamma < 1$. (It is instructive to think first of $\delta \approx 1$ and $\gamma \approx 0$.) Since $\delta_1(2) < \delta_1(1)$, Od is dynamically inconsistent with a “near-future bias”: e.g., facing the prospect of agreement on x in two periods, he would prefer agreeing instead next period for any share q with $\gamma\delta x_1 < q$, but in this next period reverse his preference if also $q < \delta x_1$. (ED would require $\gamma = 1$, hence $\delta_1(2) = \delta_1(1)$.)

Figure 1 describes equilibrium strategies for (once) delayed agreement on a given continuation equilibrium division z (filled green); for concreteness, take $z = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$, as under

continuation according to the unique stationary equilibrium. Delay requires a supporting off-path threat that prevents Od from exploiting his proposer advantage. (If the second-round had agreement on z regardless of first-round play, Od could simply offer Eve her (then unique) rejection value δz_2 —which she had no reason to reject—and thus appropriate the full efficiency gain from immediate rather than delayed agreement.) This threat is alternative second-round agreement y (shaded green), which is more favorable to Eve than z and played in case Od initially offered Eve a share in excess of δz_2 . Hence, Eve initially accepts with threshold δy_2 , and for $1 - \delta y_2 \leq \delta z_1$ initial proposer Od prefers the delayed z over any available immediate agreement; he therefore chooses his initial offer x_2 so low (e.g., zero) that Eve in turn prefers the delayed z over acceptance, $x_2 \leq \delta z_2$.

Of course, threat y such that $1 - \delta y_2 \leq \delta z_1$ (implying $y_2 > z_2$) must be credible. It is Od’s near-future bias that lends credibility to it: the strategies in figure 1 specify that *any* failure to agree when Eve makes her offer off-path (shaded green) leads to continuation play identical to that from round 1, with once delayed agreement on z . Od’s rejection would therefore always entail *two* periods of delay and have value $\gamma \delta^2 z_1$, enabling proposer Eve to appropriate the full efficiency gain from immediate agreement, with her share equal to $y_2 = 1 - \gamma \delta^2 z_1$. For a sufficiently strong bias of Od (sufficiently low γ), y satisfies the equilibrium condition $1 - \delta y_2 \leq \delta z_1$, and the delayed agreement on z produces its own supporting threat. Such values of γ exist for any z with $z_1 \geq \frac{1-\delta}{\delta}$; as $\delta \rightarrow 1$, this means *any* z (in particular the stationary continuation equilibrium). Moreover, regardless of how small Od’s bias is (γ close to one), the strategies then form an equilibrium for sufficiently frequent offers (δ large enough).

Two points are worth emphasizing about this example. (It is readily extended to exhibit also longer delays; see example 3 below.) First, for delay to occur it suffices that the proposer (Od) makes an unacceptably low offer. Though inefficient, he may well do so if he expects any attempt at compromise (Pareto-improvement) to be rejected as well. The intuitive difference between an “unambiguously” low and a compromise offer is, however, that the latter’s rejection would allow the respondent (Eve) to credibly adopt an uncompromising stance. It is this off-path belief that rationalizes the low offer that eschews the respondent’s such opportunity.

Second, near-future bias provides a foundation for this belief, hence delay. In contrast to prior explanations, which depended on multiple *stationary* equilibria (Avery and Zemsky, 1994), this dynamic inconsistency means delay can be “self-enforcing”: any delay at the proposer stage comes with the *threat* of one additional (future) delay at the respondent stage (see Od in round 2 off-path, shaded green); under near-future bias this additional delay can be so costly (γ low enough) as to rationalize an agreement that in turn supports unacceptable

offers—hence delay—at the proposer stage. To outweigh the proposer advantage, which ensures a minimal rent to the proposer over her worst threat, the bias needs to be sufficiently strong. As offers become frequent, this rent vanishes, however, and delay equilibria arise for arbitrarily small such biases.

Given the possibility of equilibrium delay, standard recursive arguments fail in characterizing equilibrium. When preferences are dynamically inconsistent, knowledge of a player’s *continuation value* is insufficient to determine her *rejection value*, which is the strategically relevant one. In particular, the relationship $w_i^* = \delta_i(1) \cdot v_i^*$ between i ’s minimal (continuation) value v_i^* as proposer and i ’s minimal (rejection) value w_i^* as respondent generally holds true only when no equilibrium of G_i has delay (cf. [Shaked and Sutton, 1984](#)).

To circumvent this problem, I directly investigate the structure of optimal punishments delivering the minimal values v_i^* and w_i^* . The main insight towards characterizing equilibrium is that, given any equilibrium delay t , a proposing player is indifferent between her least preferred immediate equilibrium agreement and her least preferred equilibrium agreement with that delay t ; both yield proposer i her minimal value v_i^* . This indifference property allows to solve for player i ’s minimal values (v_i^*, w_i^*) given the maximal delay t_i^* in game G_i : letting $\Delta_i(t) \equiv \inf \{\delta_i(s) \mid s \in T, 0 < s \leq t\}$ denote player i ’s minimal per-period discount factor over horizon t ,

$$v_i^* = 1 - \delta_j(1) \cdot (1 - w_i^*) \text{ and } w_i^* = \Delta_i(t_i^* + 1) \cdot v_i^*. \quad (2)$$

Proposer i cannot do worse than by making the smallest offer that respondent j would never refuse, j ’s maximal rejection value. This value obtains when j would subsequently receive her maximal share $1 - w_i^*$ with least delay—i.e., immediately following rejection—and equals $\delta_j(1) \cdot (1 - w_i^*)$. For the second equation in (2) suppose an equilibrium of game G_i with delay t . From the indifference property, initial proposer i ’s worst such equilibrium has her share equal to $\frac{1}{d_i(t)} \cdot v_i^*$, and this implies rejection value $\frac{d_i(t+1)}{d_i(t)} \cdot v_i^* \equiv \delta_i(t+1) \cdot v_i^*$ for i as the respondent prior to G_i . This rejection value is minimal whenever $\delta_i(t+1)$ is so, meaning that the one additional delay—the $t+1$ -th period—that i ’s rejection would entail is most costly (over $t \leq t_i^*$).

Conversely, the maximal delay t_i^* in game G_i is uniquely determined by the minimal proposer values v_i^* and v_j^* , as they capture the players’ incentives, as proposer, to make an unacceptable offer rather than settle for the worst immediate agreement:

$$t_i^* = \sup \left\{ t \in T \mid \kappa_i(t, v_i^*, v_j^*) \leq 1 \right\}, \text{ for } \kappa_i(t, v_i, v_j) \equiv \begin{cases} 0 & t = 0 \\ \frac{v_i}{d_i(t)} + \frac{v_j}{d_j(t-1)} & t > 0 \end{cases}. \quad (3)$$

The function $\kappa_i(t, v_i, v_j)$ measures the incentive cost of delay t in game G_i : if initial proposer i could obtain up to value v_i by making an accepted offer rather than incurring delay t , she requires at least the share $\frac{v_i}{d_i(t)}$ with this delay in order not to do so; similarly, player j 's share must be at least $\frac{v_j}{d_j(t-1)}$, since when she gets to propose the first time along the path, the delay would be $t-1$. As the delay shrinks, these shares become smaller, so the above two incentive constraints are not only necessary but sufficient. They can be satisfied under *some* feasible division if and only if $\kappa_i(t, v_i, v_j) \leq 1$. When the proposer values are minimal, so is the incentive cost, and an equilibrium with delay t exists as long as this minimal incentive cost does not exceed the total available surplus (3).

The values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ are jointly determined by the system of six equations in (2) and (3), to which they are the unique *extreme* solution: if $(v_i, w_i, t_i)_{i \in I}$ is any solution, then $v_i^* \leq v_i$, $w_i^* \leq w_i$ and $t_i^* \geq t_i$ for both i . They fully characterize equilibrium: agreement on division x with delay t is an equilibrium outcome of game G_i if and only if

$$t \leq t_i^* \text{ and } \frac{v_i^*}{d_i(t)} \leq x_i \leq \begin{cases} 1 - w_j^* & t = 0 \\ 1 - \frac{v_j^*}{d_j(t-1)} & t > 0 \end{cases}.$$

The set of divisions that players might agree upon is monotonically shrinking with the delay, where the bounds trace the players' time preferences according to the aforementioned indifference property.

The characterization yields several further insights. First, equilibrium is unique if and only if there is a unique solution to the system of equations. Indeed, the unique stationary equilibrium values in (1), together with $t_1^* = t_2^* = 0$, always form a solution. It is then immediate from (2) that a weak manifestation of present bias, namely $\delta_i(1) \leq \delta_i(s)$ for all $s \geq 1$, is sufficient for uniqueness (then $\Delta_i(\infty) = \delta_i(1)$, and hence $w_i^* = \delta_i(1) \cdot v_i^*$): if both parties find the first period of delay that rejection always entails most costly, then the proposer advantage is only reinforced and delay cannot be self-enforcing. Thus the uniqueness under ED extends to any form of present bias, in particular any quasi-hyperbolic or hyperbolic discounting.

A future bias of at least one of the bargainers is therefore necessary for equilibrium delay. When this bias concerns the relatively *near future*—relative referring to the players' overall impatience that drives the incentive cost in (3)—then it is sufficient (e.g., under frequent offers). The resulting set of equilibria then has two structural features that capture prominent tendencies in real bargaining behavior: i) gradual agreement, and ii) immediate equal division under symmetry (e.g., [Zwick et al., 1992](#); [Roth, 1995](#)).

First, any delayed agreement is reached through gradual agreement, where as bargaining

unfolds, each party’s “concessions” (offers as proposer, and maximum accepted/conceded opponent shares as respondent) increase towards that of the eventual agreement (see section 5.3.1 for a formal definition). The closer in time is the agreement, the smaller is the set of Pareto-improvements, hence ever higher concessions are consistent with delay. For instance, in example 1’s delay equilibrium, Od’s concessions are x_2 and z_2 , and Eve’s are $1 - \delta y_2$ and z_1 ; both sequences are increasing.

Second, if both players’ preferences are symmetric, existence of a delay equilibrium always implies a credible threat such that the minimal proposer value/share is less than one half; $\kappa_i(1, v^*, v^*) \leq 1$ implies $v^* < \frac{1}{2}$. At the same time, due to the proposer advantage, there is then also an equilibrium in which the proposer obtains a value/share greater than one half (e.g., the symmetric stationary equilibrium), hence this threat supports an immediate equal split.¹⁷ In example 1 (which permits symmetry) the equilibrium condition for delayed agreement when z is the stationary equilibrium division implies $1 - \delta y_2 < \frac{1}{2}$, and immediate agreement on an equal division can be supported by only slightly modified threats: if round 2 is reached following an offer of less than *one half*, they agree on y , otherwise on z .

The rest of this paper formally establishes analogous results for the general time preferences of assumption 1, paying special attention to the simplification of equilibria using optimal punishments. Though hardly discussed here, all results build on this insight.

4 Equilibrium for General Time Preferences

This paper studies a highly stylized model, with the objective of capturing the fundamental strategic considerations and explaining the main behavioral tendencies of parties engaged in bargaining. To achieve this, I allow for arbitrary history-dependence of their strategies. The common assumption of stationary strategies would conflict with this objective, because it strongly restricts the parties’ beliefs *a priori*: however systematically player i has deviated from a given stationary strategy in the past, it restricts the other to still believing that i will comply with it (see Rubinstein, 1991, p. 912). This point is of special importance here due to the additional presence of *intra*-personal conflict (dynamic inconsistency). First, a player’s beliefs about her own future behavior are as central as those regarding the opponent, as she may have reason to “doubt herself”. Second, the potential of stationary strategies for creating/exploiting dynamic preference reversals is severely limited.

The combination of dynamically inconsistent preferences with the possibility of multiple equilibria and delay (through history-dependent strategies) poses an analytical challenge,

¹⁷More generally, for any $t < t^*$, an equal division with t periods of delay is an equilibrium outcome; also, whenever an equal split is an equilibrium agreement for some delay t , so is an *immediate* equal split.

however.¹⁸ Standard recursive techniques (see [Shaked and Sutton, 1984](#)) fail to be applicable, because a player’s *continuation value* alone provides insufficient information to pin down a unique *rejection value*; yet, this is the strategically relevant value one round earlier, hence required for recursion.

To illustrate, consider a (β, δ) -discounter with linear instantaneous utility, say player 1. Immediate agreement on x and once delayed agreement on y with the same (continuation) value $U_1 = x_1 = \beta\delta y_1$ imply the different rejection values $\beta\delta x_1 = \beta\delta U_1$ and $\beta\delta^2 y_1 = \delta U_1$, respectively. Without further knowledge regarding the underlying equilibrium outcomes, a player i ’s minimal proposer value v_i^* (which is i ’s minimal continuation value when responding) is hence insufficient to determine her minimal rejection value w_i^* .

The approach proposed in this paper directly analyzes the off-path “punishments” (continuation equilibria) that support all equilibrium play and underlie the minimal values (v_i^*, w_i^*) . Its basic idea is that the game’s stationarity property will nonetheless entail a tractable structure for such punishments, since only two types of round need to be distinguished in terms of deviations: any round in which the same party $i \in \{1, 2\}$ gets to make an offer has the same sets of both equilibrium plays and continuation equilibria. If a particular “optimal” assignment of the latter as punishments deters deviations from *any* equilibrium play, it achieves this at any such stage, also off-path, independent of history. How much tractability is thus gained then depends on how “simple” this optimal assignment can be made. In the next section I show what optimality of punishment means, and how four appropriately chosen equilibrium outcomes suffice to describe all off-path play.

The following two reservation shares of a player i (subject to feasibility) will feature prominently in the analysis. (Under the stronger assumption 2 this extra notation could easily be dispensed with.) First, her (immediate) *reservation share* for a given rejection value $U \in U_i(A_i)$ is

$$\pi_i(U) \equiv \min \{q \in [0, 1] \mid u_i(q) \geq U\};$$

player i then accepts any offer above $\pi_i(U)$ whose rejection would yield value U . Second, her *delayed reservation share* for delay t and immediate value (instantaneous utility) $u \in u_i([0, 1])$ is

$$\phi_i(u, t) \equiv \max \{q \in [0, 1] \mid u \geq U_i(q, t)\};$$

player i then rejects offer q with value $u = u(q)$ for any promised agreement with delay t that has her share greater than $\phi_i(u, t)$.¹⁹

¹⁸It is straightforward to show that stationary equilibrium implies immediate agreement after any history.

¹⁹Since T contains infinity, for completeness, set $\phi_i(u, \infty) = 1$ for any $u \in u_i([0, 1])$.

4.1 Optimal Simple Penal Codes and Simple Play

Due to the conceptual similarity, I adopt the terminology introduced by [Abreu \(1988\)](#) for infinitely repeated games.²⁰ The major difference as well as innovation is that, due to the sequential nature of moves (see below), I base the analysis on sequences of play—for short “plays”—rather than paths; such a play extends paths to include the entire response *rules* used along the path rather than just the on-path responses.²¹ I then call an assignment of punishments supporting all equilibrium play (of both G_1 and G_2) an *optimal penal code* (OPC), and I call it an *optimal simple penal code* (OSPC) if punishment is history-independent, with a single punishment per player per role (proposer or respondent) in which this player may deviate.

The sequential nature of moves within a round complicates the analysis relative to repeated games because an OPC cannot simply assign a deviant player’s worst continuation equilibrium. The proposer’s punishment for a deviant offer is constrained by the respondent’s incentives after such a deviation, which affords the proposer a strategic advantage; e.g., a worse continuation equilibrium for the proposer may at the same time weaken the respondent’s current bargaining position and thus make deviant offers more attractive. Indeed, [Mailath et al. \(2015\)](#) present related examples of infinitely repeated *sequential-move* games in which the second mover’s “incentive constraint” forces any OPC to fine-tune punishment to the first mover’s particular deviation, so that no OSPC exists.

Optimal Simple Punishment. The trade-off between providing incentives within-round and under continuation is, however, less complicated here: the respondent’s acceptance ends the game, and the agreement round’s actions determine all payoffs. Punishment therefore takes place only after deviations that result in a rejection, and for a given punishment the offer that led to it is inconsequential. Call then (i) any deviant rejection of an offer a *respondent deviation*, and (ii) any deviant offer that the respondent may reject without deviating herself a *proposer deviation*. These two types exhaust all (one-stage) deviations that lead to punishment: e.g., given a strategy profile prescribes proposal x and response rule Y , if a proposal $x' \in Y$ is rejected, this constitutes a respondent deviation, and if a proposal $x' \notin Y \setminus \{x\}$ is rejected, this constitutes a proposer deviation. The following result shows that optimality of punishments is a property of their rejection values and optimal punishments can always be made simple. (Existence of an OPC will be established constructively, as part of the equilibrium characterization in theorem 1.)

²⁰I am deeply grateful to my former colleague Can Çeliktemur for pointing out this similarity to me at an early stage of this project.

²¹Against the background of [Abreu’s](#) influential work, I define various concepts of this section only verbally; the full-fledged formalism can be found in appendix A.

Lemma 1. *Any OPC's punishments (i) minimize the respondent's rejection value after respondent deviations, and (ii) maximize the respondent's rejection value after proposer deviations. Whenever an OPC exists, there exists an OSPC.*

The first property, regarding a responding player's *deviant rejection*, is straightforward: if rejection of some offer cannot be deterred by her least preferred continuation equilibrium (i.e., one with minimal rejection value) then there cannot be an equilibrium in which she accepts this offer; conversely, if it can be deterred by some continuation equilibrium then *a fortiori* by her least preferred one. Hence any outcome $(x^{R,i}, t^{R,i})$ of a player i 's optimal respondent punishment—an equilibrium outcome of game G_i —satisfies $w_i^* = U_i(x_i^{R,i}, t^{R,i} + 1)$.

The second property is driven by the proposer advantage. A proposer can always deviate to an offer that the respondent will accept and thus evade punishment. In particular, a responding player accepts any offer whose value exceeds her maximal rejection value, in any equilibrium. This guarantees a minimal rent to the proposer, equal to the full efficiency gain from immediate agreement over the respondent's most preferred rejection outcome (which is inefficient due to the delay). Given (ii), any deviant offer that the respondent compliantly rejects would dissipate this rent, as the respondent obtains the same value—her maximal rejection value—but in this case inefficiently. Hence, a proposer can never do better by deviating than by making the lowest accepted offer. However, a play where at some stage the proposing player would gain by deviating to an accepted offer could not be supported by *any* specification of punishments.²²

Note the following immediate consequence: letting $(x^{P,i}, t^{P,i})$ be any outcome of player i 's optimal proposer punishment—i.e., an equilibrium outcome of game G_j such that respondent j 's rejection value $U_j(x_j^{P,i}, t^{P,i} + 1)$ is maximal—it must be that i 's minimal proposer value satisfies $v_i^* = u_i(1 - \pi_j(U_j(x_j^{P,i}, t^{P,i} + 1)))$. Not only could proposer i always obtain at least this value by making an accepted offer, but immediate agreement on the division x with $x_j = \pi_j(U_j(x_j^{P,i}, t^{P,i} + 1))$ is itself clearly also an equilibrium outcome of game G_i (take $(x^{P,i}, t^{P,i})$ as “unconditional” continuation outcome). Because she may always make an offer that the respondent would never refuse, there cannot be a delay equilibrium that is worse for the proposer than her least preferred immediate-agreement equilibrium.

The first part of lemma 1 shows that it is without loss of generality to restrict OPCs to four optimal punishments, one per player per type of deviation, with the respective properties (i) and (ii); these then support any equilibrium play, of both G_1 and G_2 . Given how it identifies the perpetrator, an OPC is then simple in the sense that punishment need not

²²Recall that we are concerned with one-stage deviations only; hence, whether such a deviation exists can be determined from play alone. Allowing for any punishments, there may also be a deviation to a rejected offer that is even more attractive, but it would be a profitable deviation from prescribed play in any case.

fit the crime. However, so far this simplicity concerns only first deviations from prescribed play; the punishments themselves may still be rather complex.

The second part of lemma 1 extends the simplicity of an OPC to its own punishments, thus creating an OSPC. It is based on the observation that any OPC supports, in particular, the play of its own constituent punishments. Intuitively, we can therefore iteratively apply the same optimal punishments also to deviations from first punishment play (second deviations), and then also to deviations from second punishment play (third deviations) etc. Thus we create an OPC in which player i 's proposer and respondent deviations are followed by the same respective punishment, entirely independent of their history, i.e. an OSPC; e.g., a proposer deviation by player 1 from its own punishment's play then simply "restarts" this very punishment play. It is therefore without loss of generality to restrict OPCs to OSPCs, and these are fully described by four optimal punishment *plays*.

Simple Play. Consequentialist parties care only about outcomes of play, not play itself; making an offer that is commonly known to be rejected is therefore tantamount to not offering anything at all. The final simplification result removes such redundancy regarding equivalent types of equilibrium play (in particular, optimal punishment play).

Call a play that ends in agreement on division x in round n (perpetual disagreement means $x = (0, 0)$ and $n = \infty$) a *simple play* if (i) all rejected offers are *minimal offers* (i.e., zero offers), and (ii) all response rules specify *maximal acceptance thresholds*, equal to the respective respondent's reservation share for her maximal rejection value in a disagreement round $m < n$, and to $x_{R(n)}$ in the terminal agreement round n . Note that, given the players' maximal rejection values, simple play is fully determined by its ultimate outcome, here (x, t) for $t = n - 1$.²³ For the purpose of characterizing equilibrium outcomes, with optimal punishments, this is indeed without loss of generality.

Lemma 2. *Whenever an OPC exists and (x, t) is an equilibrium outcome of game G_i , the simple play of this outcome is an equilibrium play of G_i .*

In conclusion, all strategic complexity off the equilibrium path can be summarized by merely four optimal punishment *outcomes* $\left((x^{P,i}, t^{P,i}), (x^{R,i}, t^{R,i}) \right)_{i \in I}$; these define four simple plays that form an OSPC supporting all equilibrium play, of both (sub-) games G_1 and G_2 . Moreover, to check whether an outcome is an equilibrium outcome it suffices to check only for one-stage deviations from its simple play, which is straightforward. These insights afford a greatly simplified structure for equilibrium analysis.

²³As defined here, simple play exists for every equilibrium outcome, but not necessarily for every *possible* outcome; e.g., if player 2's maximal rejection value implies a zero reservation share, then there is no simple play of G_1 with delayed agreement.

4.2 Equilibrium Characterization

The equilibrium characterization exploits a “fixed-point property” of any quadruple of optimal punishment outcomes: by means of their implied OSPC they support themselves as the most extreme outcomes—in terms of their rejection values (lemma 1)—among all the outcomes that they support. Since there may be multiple OSPCs, I first map this fixed-point property into a system of equations that the unique associated punishment values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ necessarily solve. These equations, in general, have multiple solutions, and the values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ are found as their unique *extreme* solution, whose existence follows from the continuity assumptions on preferences. These values then characterize the set of OSPCs, and thus also the set of equilibrium outcomes. This is the central result of this paper.

Define first the function $\kappa_i : T \times U_i(A_i) \times U_j(A_j) \rightarrow \mathbb{R}_+$ such that

$$\kappa_i(t, v_i, v_j) \equiv \begin{cases} 0 & t = 0 \\ \phi_i(v_i, t) + \max\{\phi_j(v_j, t-1), \phi_j(u_j(0), t)\} & t > 0 \end{cases},$$

which measures the *surplus-cost of delay* t in G_i given proposer values v_i and v_j , and which is non-decreasing in each of its arguments. Its significance derives from the fact that, given the minimal proposer values v_i^* and v_j^* from optimal punishment, game G_i has an equilibrium outcome with (positive) delay t if and only if $\kappa_i(t, v_i^*, v_j^*) \leq 1$. The restriction to simple play allows to reduce the necessary and sufficient incentive constraints for agreement on x with this delay to $x_i \geq \phi_i(v_i^*, t)$ and $1 - x_i \equiv x_j \geq \max\{\phi_j(v_j^*, t-1), \phi_j(u_j(0), t)\}$; κ_i therefore measures the incentive cost of delay t as the minimal amount of surplus, so that both players can be promised a large enough share with this delay.

Let then $E \subseteq \prod_{i \in I} (u_i([0, 1]) \times U_i(A_i) \times T)$ be the set of sextuples $(v_i, w_i, t_i)_{i \in I}$ such that, for each $i \in I$,

$$v_i = u_i(1 - \pi_j(U_j(1 - \pi_i(w_i), 1))) \quad (4)$$

$$w_i = \inf\{U_i(\phi_i(v_i, t), t+1) \mid t \in T, t \leq t_i\} \quad (5)$$

$$t_i = \sup\{t \in T \mid \kappa_i(t, v_i, v_j) \leq 1\} \quad (6)$$

Lemma 6 in appendix A.3 shows how each element $(v_i, w_i, t_i)_{i \in I}$ of E corresponds to a quadruple of punishment outcomes that are “constrained” optimal in the following sense: by means of a construction similar to an OSPC, they support a *subset* of equilibrium outcomes that includes themselves (so they are indeed equilibrium outcomes), and on which they are optimal; i.e., constrained to this subset, they yield the minimal punishment values $(v_i, w_i)_{i \in I}$ and supremal delays $(t_i)_{i \in I}$.

If optimal punishments, and thus an OSPC, exist, the associated values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ are necessarily in E . However, in general, due to the interdependency of punishments—harsher punishments permit longer delays, and longer delays permit harsher punishments—there may be (other) constrained OSPCs. In fact, the set E always contains an element $(v_i, w_i, t_i)_{i \in I}$ with $t_1 = t_2 = 0$ that corresponds to a “trivial” constrained OSPC: irrespective of who deviated in a given round, it specifies the same punishment; thus this OSPC reduces to a single stationary equilibrium in which player i always offers $1 - \phi_i(v_i, 0) = \pi_j(w_j)$ and always accepts with threshold $\pi_i(w_i) = \pi_i(U_i(\phi_i(v_i, 0), 1))$, so there is immediate agreement after any history.

In view of potential multiplicity in E , the actual values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ must then be its unique *extreme element*; i.e., any other element $(v_i, w_i, t_i)_{i \in I}$ satisfies $v_i^* \leq v_i$, $w_i^* \leq w_i$ and $t_i^* \geq t_i$ for both i .

Theorem 1. *The values $(v_i^*, w_i^*, t_i^*)_{i \in I}$ exist, and they are equal to the unique extreme element of the set E . For each $i \in I$, $(x^{P,i}, t^{P,i})$ and $(x^{R,i}, t^{R,i})$ are outcomes of player i 's optimal proposer and respondent punishment, respectively, if and only if*

$$\left\{ \begin{array}{l} t^{P,i} = 0 \\ x_i^{P,i} = \pi_i(w_i^*) \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} t^{R,i} \in \arg \min \{U_i(\phi_i(v_i^*, t), t+1) \mid t \in T, t \leq t_i^*\} \\ x_i^{R,i} = \phi_i(v_i^*, t^{R,i}) \end{array} \right\},$$

and the set A_i^* of player i 's personal equilibrium outcomes in game G_i equals

$$\left\{ (q, t) \in A_i \mid \phi_i(v_i^*, t) \leq q \leq \begin{cases} 1 - \pi_j(w_j^*) & t = 0 \\ 1 - \max\{\phi_j(v_j^*, t-1), \phi_j(u_j(0), t)\} & t > 0 \end{cases} \right\}.$$

A few features of optimal punishments are noteworthy in view of the strategic advantage enjoyed by a proposing player. First, a player's optimal proposer punishment is unique and involves no delay: given her impatience, the respondent's rejection value is maximized by the maximal credible share with least delay following rejection (4). Second, an initially proposing player i 's least preferred equilibrium outcomes for various delays are necessarily indifferent, all yielding her the same minimal value v_i^* , and this allows to pin down optimal respondent punishment (5). Finally, whether and how long agreement may be delayed is fully determined by the players' incentives as proposer (6); this drives the aforementioned indifference property (see also the characterization of A_i^* in theorem 1).

Example 1 shows that the equilibrium characterization neither reduces to uniqueness nor to stationarity of equilibrium, nor to stationarity of optimal punishments. It is never an “anything goes”-type result, however, as the players' impatience imposes a certain structure on equilibrium through the proposer advantage: as a function of delay, the set of equilibrium

divisions monotonically shrinks (since $\phi_i(u, \cdot)$ is increasing, the upper and lower bounds on each player’s share converge), and perpetual disagreement is never an equilibrium outcome (note that $v_i^* > u_i(0) \geq U_i(0, \infty)$). In section 5, I present further detail, examples and discussion regarding the structure of equilibria for various preferences.

Theorem 1 is partly reminiscent of Merlo and Wilson (1995, thms 7 and 8), who assume ED and analyze bargaining by multiple players under a Markovian process governing the protocol as well as the size of the cake. They also characterize the set of equilibrium values by means of an extremal fixed point, but its nature differs significantly. ED implies that there is a stationary equilibrium outcome that maximizes one player’s value at the same time as it minimizes all other players’ values. In the two-player case this simple relationship between punishment and reward implies that optimal punishments are efficient and, without loss of generality, also stationary. Only in the case of more than two players, one player’s optimal punishment might necessitate some punishment of another player and some inefficiency, thus complicating the incentive structure (cf. Burgos et al., 2002b).

By contrast, here such a complication arises already with two players, and from a very different source: the dynamic inconsistency of a player’s time preferences. Optimal punishment might *require* delay, in which case it is both inefficient and non-stationary. The extreme equilibria are then “truly” non-stationary in the sense that their continuation is non-stationary after any history. Equilibrium delay does not necessitate multiple stationary equilibria; indeed, it does not even depend on the existence of a stationary equilibrium.

This distinguishes the delay obtained here from that obtained in other extensions of the original Rubinstein (1982) model that maintain a stationary game structure and ED, all of which rely on multiple stationary equilibria to support delay (Haller and Holden, 1990; Muthoo, 1990; van Damme, Selten, and Winter, 1990; Fernandez and Glazer, 1991; Myerson, 1991; Avery and Zemsky, 1994). The sole exception I am aware of is that of Busch and Wen (1995).²⁴ Their model of negotiation enriches bargaining by a disagreement game, which is a fixed simultaneous-move game played after any rejected offer and determines a stream of payoffs before agreement. The truly non-stationary equilibria they construct exploit the resultingly richer preference domain through non-stationary play of the disagreement game similar to folk theorems for repeated games, but constrained by the parties’ incentives to reach agreement.

Existence of an OSPC is equivalent to the existence of minimum values v_i^* and w_i^* (as argued, a “constrained” OSPC and hence an equilibrium always exist, however). This is non-trivial here, as the set of equilibrium outcomes need not be closed.²⁵ The generality of

²⁴I am indebted to Paola Manzini for drawing my attention to these authors’ work.

²⁵Although the equilibrium concept introduced in definition 1 is equivalent to a version of subgame-perfect

assumption 1 means that the length of equilibrium delay might have no upper bound, despite the fact that perpetual disagreement is never an equilibrium outcome due to the proposer advantage (see appendix B.3 for an example). While existence of a minimal value v_i^* follows from standard continuity even with unbounded delay, the (only) role played by impatience property (3.c) is to ensure that the minimal value w_i^* also exists in this case, because the delay of agreement that is required for *optimal punishment* is then bounded.

5 Uniqueness v. Multiplicity, and Delay

For economic applications, where bargaining arises naturally in various contexts (household decision-making, wage setting, international trade agreements etc.), uniqueness of the bargaining prediction is an important concern. Any uncertainty about this one aspect of a model feeds through all of the conclusions drawn from it. The following characterization of those preference profiles (within the general class defined by assumption 1) for which equilibrium is indeed unique is immediate from theorem 1. It sets the stage for the remainder of this section, which relates the paper’s theoretical results to the empirical evidence on time preferences, as well as bargaining.

Corollary 1. *Equilibrium is unique if and only if the set E is a singleton. Whenever unique, equilibrium is stationary and has immediate agreement after any history: player i always offers the share $1 - \phi_i(v_i^*, 0) = \pi_j(w_j^*)$ and always accepts with the threshold $\pi_i(w_i^*) = \pi_i(U_i(\phi_i(v_i^*, 0), 1))$, $i \in I$.*

These necessary and sufficient conditions for uniqueness do not isolate preference properties at the individual level, however: fixing one party’s preferences, whether equilibrium is unique or displays multiplicity and delay generally depends on those of the opponent. To relate the theoretical results on bargaining to the empirical evidence on time preferences I therefore investigate what broad qualitative properties of preferences at the individual level imply uniqueness on the one hand, and multiplicity and delay on the other hand. For the latter case, I provide additional guidance to empirical bargaining research by showing how the structure of equilibrium “naturally” captures what seem to be the most important observed tendencies in real bargaining.

Nash equilibrium, existing results based on the upper hemi-continuity of its equilibrium correspondence (e.g., Börgers, 1991) cannot be applied here, because they assume finitely many players.

5.1 Preliminary Remarks on Stationary Equilibrium

In general, already *stationary* equilibrium need not be unique, and this is so even under ED (see Rubinstein, 1982). Multiplicity of stationary equilibrium is, however, hardly of practical interest, as it requires empirically implausible degrees of convexity of the players' utility functions in their surplus share. These curvature properties of time preferences in the reward are essentially orthogonal to whether or how these preferences are dynamically inconsistent. Indeed, the same axioms that have been postulated in order to guarantee uniqueness of stationary equilibrium under ED (e.g., Binmore, Rubinstein, and Wolinsky, 1986; Hoel, 1986; Osborne and Rubinstein, 1990) also do so within the much more general class of preferences analyzed here. For instance, consider the following property.²⁶

Definition 2. Player i 's preferences exhibit **immediacy** if, for any two shares q and q' , and any positive ϵ ,

$$u_i(q) = U_i(q', 1) \Rightarrow u_i(q + \epsilon) > U_i(q' + \epsilon, 1).$$

Starting from indifference between an immediate and a once delayed agreement, immediacy says that an increase in one's surplus share is more valuable when immediate. With impatience, indifference requires that the delayed share exceed the immediate one, so immediacy extends a basic property of any discounted concave utility to non-separable preferences. Because it is concerned with comparisons of only immediate and once delayed agreements, it does not restrict whether or how preferences are dynamically inconsistent.

Lemma 3. *If both players' preferences exhibit immediacy, stationary equilibrium is unique.*

For the purposes of applied work—the focus of this section—stationary equilibrium will be unique.²⁷ Immediacy ensures that the proposer's surplus rent in immediate rather than once delayed (history-independent) agreement is monotonically increasing in the share that the respondent would obtain by rejecting; e.g., if any offer's rejection would subsequently result in immediate agreement on division x , then the proposing player i 's such surplus rent equals $(1 - \pi_j(U_j(x_j, 1))) - (1 - x_j) = x_j - \pi_j(U_j(x_j, 1))$. Its increasingness ensures that the backwards-induction dynamics are well behaved: starting from any (history-independent) agreement, backwards induction produces a unique limit, i.e. a unique stationary point.²⁸

²⁶This is essentially a reformulation in utility terms of the “increasing loss to delay” axiom of Osborne and Rubinstein, 1990, pp. 35-36.

²⁷Appendix A.5 provides a full characterization of stationary equilibrium, for the general case.

²⁸If the rent were non-monotonic, the limit may depend on the starting division, yielding multiple stationary points.

5.2 Present Bias and Uniqueness

A unique stationary equilibrium is the only equilibrium with immediate agreement after any history. This equilibrium is unique overall whenever delay is not self-enforcing in the sense that it enlarges the scope for punishment so much that it effectively supports itself. Consider then the following preference property.

Definition 3. Player i 's preferences exhibit a **weak present bias** if, for any two shares q and q' , and any delay t ,

$$u_i(q) = U_i(q', t) \Rightarrow U_i(q, 1) \leq U_i(q', t + 1). \quad (7)$$

Present bias means that a party becomes more patient when an immediate and an indifferent delayed reward are pushed into the future. Hence, if a present-biased individual, in a period's time, would be indifferent between receiving a reward q immediately and receiving a reward q' with t periods of delay, she currently prefers the larger later reward.

Recall now that, due to the proposer advantage, delay cannot hurt a proposing party beyond her least preferred immediate agreement. Under weak present bias, delay cannot hurt this party as the respondent either: rejection necessarily entails a minimal delay of one period, but beyond this "critical" period she is more patient. Hence, subject to indifference as the proposer, she cannot be made worse off as the respondent; delay cannot be self-enforcing.

Proposition 1. *If, in addition to immediacy, both players' preferences exhibit a weak present bias, then equilibrium is unique.*

Together with immediacy, weak present bias provides a simple set of sufficient conditions for uniqueness. Both properties are readily checked for any given preferences, and both are readily testable empirically.

The interpretation of property (7) as weak present bias is most straightforward for discounted utility, where $U(q, t) = d(t) \cdot u(q)$. Letting $d(t) \equiv \prod_{s=1}^t \delta(s)$, weak present bias then reduces to $\delta(1) \leq \delta(t)$, saying that no future period of delay is discounted more heavily than the first one from the immediate present.²⁹ Any hyperbolic or quasi-hyperbolic discounting exhibits this property, with an actual bias: the (β, δ) -model of quasi-hyperbolic discounting has $\delta(1) = \beta\delta < \delta = \delta(t)$ for any $t > 1$, and hyperbolic discounting has $\delta(\cdot)$ increasing.³⁰

²⁹Halevy (2008) introduces a strict version of this discounting property, which he calls "diminishing impatience", and relates it to non-linear probability weighting of consumption risk (see also appendix B.4.2). The weak formulation of property (7) means it also covers ED as the limiting case where $\delta(\cdot)$ is constant.

³⁰The non-separable models of Benhabib et al. (2010) and Noor (2011) were both designed to capture the very same pattern of preference reversals that hyperbolic and quasi-hyperbolic discounting explain, and it can easily be verified that they, too, exhibit a weak present bias.

Proposition 1 establishes the robustness of the bargaining wisdom received from the study of ED to various forms of present bias: equilibrium is unique as well as efficient, it is easily computed on the basis of only the players’ attitudes to a single (the first) period of delay and has familiar comparative statics. If one believes in the essence of present bias but finds the evidence inconclusive as to what exact functional form it assumes, it is comforting to learn that equilibrium is robust to any mis-specification of higher-order delay attitudes. Moreover, the finding that the historically main mode of surplus sharing is efficient under present bias is good news for its evolutionary explanations (e.g., Dasgupta and Maskin, 2005; Netzer, 2009): otherwise, communities without a present bias would have had an evolutionary advantage, making its survival hard to understand.

Most importantly, proposition 1 expands the scope of applied work, which shows strong interest in the study of present-biased time preferences—in particular (β, δ) -discounting—but has hitherto lacked a strategically founded bargaining solution. Its application requires some caution, however, as the following example indicates—even after putting aside the empirical issue of whether present bias is prevalent on the most commonly considered bargaining domain of monetary rewards (see appendix B.1).

Example 2. Let the two parties’ preferences be given by $U_i(q, t) = d_i(t) \cdot q$ with $d_i(0) = 1 > d_i(t) = \beta_i \delta_i^t$ for all $t > 0$, $(\beta_i, \delta_i) \in (0, 1)^2$. The unique equilibrium of the game in which player 1 makes the initial offer has immediate agreement on division x such that

$$x_1 = \frac{1 - \beta_2 \delta_2}{1 - \beta_1 \delta_1 \beta_2 \delta_2}.$$

For a given positive period-length, this prediction is indistinguishable from that under ED where each player i has preferences $U_i(q, t) = \tilde{\delta}_i^t q$ with $\tilde{\delta}_i \equiv \beta_i \delta_i$ (cf. Bernheim and Rangel, 2009, pp. 69-71).

Whichever continuous-time version of (β, δ) -discounting is adopted (cf. Harris and Laibson, 2013; Pan et al., 2015), the limiting case of very frequent offers that is commonly focused on in applications becomes problematic. Either a player’s bias is taken to discontinuously differentiate instantaneous from delayed gratification (let $t \in \mathbb{R}_+$ above), in which case $x_1 \rightarrow \frac{1-\beta_2}{1-\beta_1\beta_2}$ as $\delta_i \rightarrow 1$ (regardless of relative speeds of convergence), and the bargaining outcome is fully determined by the players’ very short-run impatience; the initial proposer’s advantage then prevails for arbitrarily frequent offers, and—failing to generate an equal split—the model is at odds with the Nash bargaining solution.³¹

Or an extended notion of the “present” of length $\tau_i > 0$ is adopted, such as $d_i(t)$ equal to

³¹Notice that any bias $\beta_i < 1$, however small, means that in the limit this player obtains none of the surplus in bargaining against an exponential discounter.

δ_i^t whenever $t \leq \tau_i$ and $\beta_i \delta_i^t$ otherwise. Then, however, as the length of a bargaining period falls below some player’s τ_i , the model exhibits multiple equilibria and delay, of the type presented in example 1 (there $1 \leq \tau_1 < 2$).

A related conceptual issue arises concerning the possibly distinct times of agreement and consumption feasibility. If there is an exogenous lag $\hat{\tau}$ between agreement and consumption exceeding the length of time for which there is a “present bias”, the unique equilibrium has immediate agreement with player 1’s share equal to $x_1 = \frac{1-\delta_2^{\hat{\tau}+1}}{1-\delta_1^{\hat{\tau}+1}\delta_2^{\hat{\tau}+1}}$; only the “long-run” discounting matters, because each player i discounts even immediate *agreements* with extra factor β_i .

Taking a broad perspective on what is being consumed, it could also be a bargainer’s relevant others’ esteem, proportional to the surplus she fetches (e.g., when a union leader negotiates on behalf of her union). The agreement reached might then differ drastically, depending on whether the bargaining is done behind closed doors (there is a lag between agreement and consumption, and only long-run discounting matters) or in the presence of such relevant others (when the timing of agreement and consumption coincide, and the degrees of present bias are the main determinant of the division).³²

5.3 Near-Future Bias, Multiplicity and Delay

A major contribution of this paper is to establish (non-stationary) delay equilibria for empirically plausible time preferences, satisfying the usual curvature assumptions. Before discussing the qualitative features of these preferences, I first argue for the relevance of such equilibria by showing how they capture two prominent tendencies in real bargaining: gradual agreement and equal surplus division.

5.3.1 Gradual Agreement and Equal Split

All delay equilibria share the same fundamental strategic reasoning. Although Pareto-improvements are available, none of them gets proposed, because a proposing player believes that, by doing so, she would induce the opponent to expect an even superior (non-Pareto-improving) agreement and, accordingly, reject the proposal. This belief leads to offers that are, in turn, unfavorable vis-à-vis the delayed outcome for the respondent. As the time of agreement draws closer, the set of Pareto-improvements shrinks at the rate of the parties’ impatience, so they may reason and behave in this way while making ever greater concessions,

³²I thank Erik Eyster and David Cooper for independently pointing out the following: any (common) lag between time of agreement and time of consumption does not affect the unique bargaining outcome under ED (this can be seen from the functions π_i), but under (β, δ) -discounting would shift bargaining power toward the player who is more patient in the long-run.

thus agreeing gradually.

Formally, for any equilibrium play $(x^n, Y^n)_{n=1}^{t+1}$, define party i 's *concession* in round n , denoted b_i^n , as her offer if i is the proposer, i.e. $b_i^n = x_j^n$ if $i = P(n)$, and as the supremal opponent share that she would accept if i is the respondent, i.e. $b_i^n = \sup \{x_j \in [0, 1] \mid x \in Y^n\}$ if $i = R(n)$. Call an equilibrium with outcome (x, t) a *gradual-agreement equilibrium* if its play has both players' concessions b_i^n increasing in n , i.e. $b_i^{n+1} > b_i^n$ for both i and all $n \leq t$. Gradual agreement meaningfully applies only to equilibria with delay, of course; then, however, its requirement is rather strong, as it treats a player's offers and response rules symmetrically in terms of concessions (it clearly implies increasing offers by each player).

Proposition 2. *If both parties $i \in I$ are uniformly impatient, so that for any positive share q , $t < t'$ implies $U_i(q, t) > U_i(q, t')$, then every equilibrium outcome is the outcome of a gradual-agreement equilibrium.³³*

Under gradual agreement, a player's concession has the interpretation of the credible promise that she will subsequently always be willing to give up at least this share, as long as the other player keeps to her promise. The fact that this promise has no material counterpart makes it distinct from the commitment mechanisms in related work explaining such "gradualism" (Admati and Perry, 1991; Compte and Jehiel, 2004).³⁴ These authors also provide anecdotal evidence for such behavior; it is, however, evident in the laboratory as well (e.g., Weg et al., 1990; Kahn and Murnighan, 1993).

Another prominent empirical finding is that, in payoff-symmetric bargaining problems, the parties tend to share the surplus equally, typically without delay (e.g., Roth, 1995). This coincides with the Nash bargaining solution, and it is also the limiting outcome of the unique symmetric equilibrium under ED as offers become arbitrarily frequent (Binmore et al., 1986, prop. 4). For the more general time preferences considered here, the following holds true.

Proposition 3. *If the two bargaining parties' preferences are symmetric, then an immediate equal split is an equilibrium outcome whenever there exists an equilibrium with delayed agreement. More generally, an equal split with delay $t - 1$ is an equilibrium outcome whenever there exists an equilibrium in which agreement is delayed by t periods.*

In reasonably symmetric bargaining situations, the possibility of delay implies that the parties may instead quickly agree on an equal split. This holds true here without recourse

³³If the requirement for gradual agreement were weakened to *non-decreasing* concessions, this proposition would hold true for any preference profile.

³⁴In these papers the value of a player's outside option increases in the opponent's past concessions. Li (2007) obtains a similar effect with history-dependent preferences.

to a limiting argument, hence even for non-negligible costs of disagreement; as offers become more frequent, the required delay equilibria are, however, more likely to exist (see below).

The unified explanation for these seemingly incompatible tendencies via non-stationary delay equilibria comes with equilibrium multiplicity. Yet, upon taking the perspective of a bargainer facing such indeterminacy, one may reasonably expect behavior to become prone to influence by commonly shared norms of behavior (promise-keeping) or focal points (symmetry) that coordinate beliefs. Such auxiliary assumptions can therefore serve to sharpen predictions.

Importantly, this explanation does not require time preferences that would be peculiar to the bargaining problem. Rather, the property of time preferences that implies delay equilibria is commonly found in experimental studies of how people generally evaluate delayed monetary rewards: *near-future bias*.

5.3.2 Near-Future Bias and Delay

In contrast to immediacy, there is significant evidence for violations of weak present bias, for the domain of monetary rewards (see appendix B.1). Notably, these violations occur almost exclusively within short horizons of less than a few weeks, where many people display *less* patience for a given delay when it is placed into the slightly more distant future (holding payoffs constant); hence, in contradiction to (7), an indifference $u_i(q) = U_i(q', t)$ is then broken in favor of the sooner agreement, $U_i(q, 1) > U_i(q', t + 1)$, for t not too large. Given immediacy, a violation of weak present bias is necessary for the emergence of delay equilibria; when it concerns the relatively near future, it is sufficient.³⁵

Under discounting, using decomposition $d_i(t) \equiv \prod_{s=1}^t \delta_i(s)$, a near-future bias means that $\delta_i(s) < \delta_i(1)$ for $s > 1$ not too large; i.e., a near-future period of delay is discounted more heavily than the first one. Whereas under weak present bias the minimal per-period discount factor $\Delta_i(t) \equiv \inf \{\delta_i(s) | s \in T, 0 < s \leq t\}$ is independent of the horizon t and constant at $\Delta_i(\infty) = \delta_i(1)$, under near-future bias it initially decreases as the horizon is extended: $\Delta_i(s) < \Delta_i(1)$ for $s > 1$ not too large. Ebert and Prelec (2007), Bleichrodt, Rohde, and Wakker (2009), Takeuchi (2011) and Pan et al. (2015) have advanced functional forms for near-future-biased discounting; in graphical terms, all of these discounting functions are initially concave, so their decline is steepest at some positive delay rather than at zero.

For a near-future biased bargainer a further period of delay in the near future is more critical than the first, initial period of delay. To avoid a costly future delay, she has to rely

³⁵Of course, offers must not take too much time for the “bias horizon” to be relevant; e.g., if a counter-offer would take forever, the first offer is an ultimatum, and there is a unique equilibrium in which the initial proposer obtains the entire surplus without delay.

on her future self. However, to her future self the same delay, in absolute time, will not be as critical any more, in relative time. Put succinctly, a given future delay is more painful now than it will be later—she will subsequently become more patient and, accordingly, tougher in bargaining than she would initially want herself to be.³⁶

This type of dynamic inconsistency makes delay self-enforcing, because any delay on path automatically implies the threat of an additional delay off-path, in the event of a rejection: assuming the additional delay would be particularly costly to her, such a bargainer may accept so bad a deal as the respondent now, that—in terms of a threat—this supports her unacceptable offers as the proposer later, when she will be more patient. Although her proposer advantage limits the power of this threat, as offers become frequent and this advantage vanishes, delay equilibria emerge for an arbitrarily small such bias.

The following simple parametric example of a near-future bias extends example 1 to conclusively illustrate various points made in this section.

Example 3. Let the two parties’ preferences be symmetrically given by $U_i(q, t) = d(t) \cdot q$ with

$$d(t) = \begin{cases} \delta^t & t \leq \tau \\ \gamma\delta^t & t > \tau \end{cases}, \text{ for } (\delta, \gamma) \in (0, 1)^2 \text{ and } \tau > 0.$$

First, note that the $\tau + 1$ -th period of delay is discounted most heavily: whereas the per-period discount factors are $\delta(t) = \delta$ for all $t \neq \tau + 1$, for that period it is $\delta(\tau + 1) = \gamma\delta$. Since $\tau > 0$, weak present bias is violated, and there is instead a bias toward not experiencing more than τ periods of delay. (Immediacy is clearly satisfied.) Hence $\Delta(t)$ equals δ for all $t \leq \tau$ and $\gamma\delta$ for all $t > \tau$; given Δ determines whether non-stationary delay equilibria emerge, this minimal deviation from ED is made only for convenience, to keep the number of parameters down to a mere three, $\{\delta, \gamma, \tau\}$. Due to preference symmetry, the player subscript is omitted throughout this example.

Suppose there is an equilibrium in which agreement is delayed by τ periods: then $v^* = \frac{1-\delta}{1-\gamma\delta^2}$ and $w^* = \gamma\delta v^*$ (see (2) in section 3); delay $\tau > 0$ is then “self-enforcing” if and only if $1 \geq \kappa(\tau, v^*, v^*) = \frac{v^*}{\delta^\tau} + \frac{v^*}{\delta^{\tau-1}}$, which reduces to

$$\delta^\tau \geq (1 + \delta) \cdot \frac{1 - \delta}{1 - \gamma\delta^2} \quad (8)$$

after substituting for v^* . The left-hand side is the present value of the surplus, and the right-

³⁶As an extreme but instructive example imagine someone who—at any point in time—does not mind bargaining for, say, 5 rounds, but is extremely averse to bargaining any longer; such a shifting personal “deadline” (in relative time) is dynamically inconsistent, since as soon as the first round is over this player will already not mind delaying agreement until round 6.

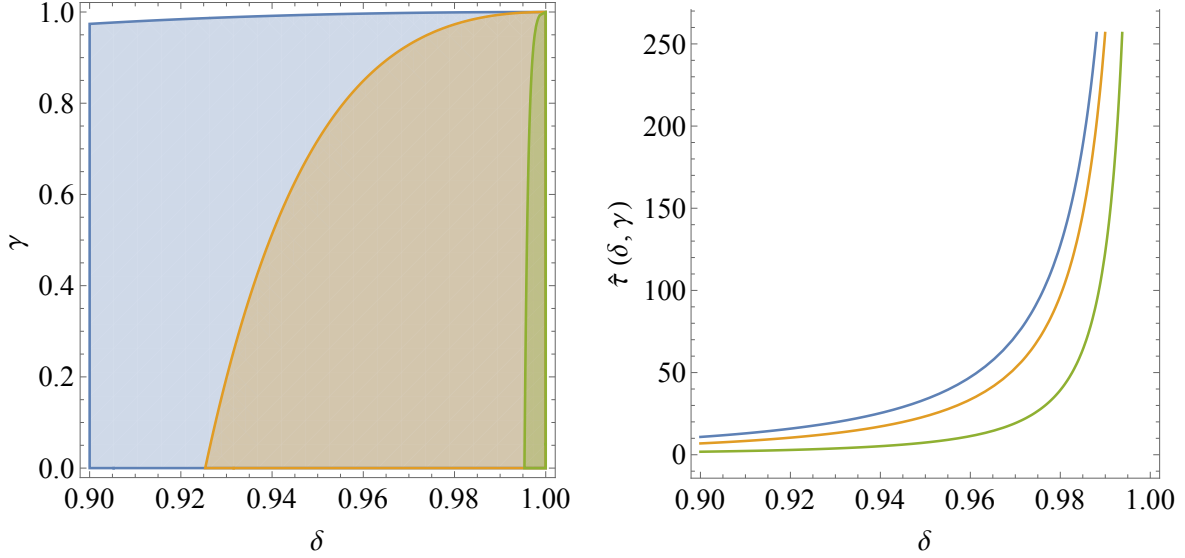


Figure 2: Graphs regarding equilibrium delay in example 3. The panel on the left shows the parametric regions (δ, γ) such that delay equilibria exist for three given values of τ , which are 1 (blue, orange and green), 25 (brown and green) and 1000 (green). The panel on the right plots $\hat{\tau}(\delta, \gamma)$ as a function of δ for three given values of γ , which are 0.5 (blue), 0.75 (orange) and 0.99 (green).

hand side is the present value of the incentive cost of a delay of τ periods: each proposer requires $v^* = \frac{1-\delta}{1-\gamma\delta^2}$, and the factor $(1 + \delta)$ is due to the fact that the initial proposer does so immediately whereas the other player does so only next round. Observe that, for any given $\tau > 0$ and $\gamma < 1$, there exist large enough values of δ such that inequality (8) is satisfied (the left-hand side limits to one whereas the right-hand side limits to zero as $\delta \rightarrow 1$); generally, as δ increases, the set of parameters γ and τ for which delay equilibria exist expands, as the left-hand-side panel of figure 2 illustrates. Whenever such delay equilibria exist, the minimal proposer and rejection values are obtained only by means of a “truly” non-stationary delay equilibrium, using optimal punishments.

Notice also that inequality (8) implies $w^* < v^* < \frac{v^*}{\delta^{\tau-1}} \leq \frac{1}{2}$, and an equal split with any delay up to $\tau - 1$ periods is then an equilibrium outcome (in particular under immediate agreement). It may also be reached gradually, say with delay \hat{t} , $0 < \hat{t} < \tau$: define a sequence $(b^n)_{n=1}^{\hat{t}+1}$ of concessions such that $b^1 \equiv 0$ and $b^n \equiv \frac{1}{2} \left(b^{n-1} + \delta^{\hat{t}+1-n} \cdot \frac{1}{2} \right)$, noting that the sequence is increasing, and that b^n falls short of a player’s present value of agreeing on an equal split with the delay $\hat{t} + 1 - n$ that remains as of the n -th round, which is $\delta^{\hat{t}+1-n} \cdot \frac{1}{2}$. It is straightforward to verify that the following describes equilibrium play with gradual agreement: in any (disagreement) round $n < \hat{t} + 1$ the proposing player $P(n)$ offers the share b^n , and the responding player $R(n)$ accepts with threshold $1 - b^{n+1}$ ($b^n < 1 - b^{n+1}$

follows from $b^n < b^{n+1} < \frac{1}{2}$); in the (agreement) round $n = \hat{t} + 1$ the proposing player $P(\hat{t} + 1)$ offers the share $\frac{1}{2}$, and the responding player $R(\hat{t} + 1)$ accepts with threshold $\frac{1}{2}$.

Solving for τ , inequality (8) becomes

$$\tau \leq \frac{\ln(1 - \delta^2) - \ln(1 - \gamma\delta^2)}{\ln(\delta)} \equiv \hat{\tau}(\delta, \gamma),$$

and if it is satisfied, the maximal delay t^* equals $\lfloor \hat{\tau}(\delta, \gamma) \rfloor$, i.e. the greatest integer not exceeding $\hat{\tau}(\delta, \gamma)$. For any $\gamma < 1$, this maximal delay approaches infinity as $\delta \rightarrow 1$, showing how small deviations from ED result in the emergence of delay equilibria as offers become very frequent; e.g., $\lfloor \hat{\tau}(\delta, \gamma) \rfloor = 404$ in case $\delta = \gamma = 0.999$. The right-hand-side panel of figure 2 illustrates this numerically.

The resulting delays can be very costly. The present value of the surplus in an equilibrium where agreement is maximally delayed equals $\gamma\delta^{t^*}$ whenever $\tau \leq \hat{\tau}(\delta, \gamma)$. As $\delta \rightarrow 1$, for any given $\gamma < 1$, not only is $\tau \leq \hat{\tau}(\delta, \gamma)$ going to be satisfied, but the entire surplus vanishes. For instance, while in the case of $\delta = \gamma = 0.99$ the maximal surplus loss amounts to roughly one third of the total, for values of γ that fall short of δ , the loss can be dramatic: up to 99.8% of the surplus can be lost through delay when $\delta = 0.99999$ and $\gamma = 0.99$.

When players discount the future only up to a finite number of delays, equilibrium delay can even be unbounded. Example 5 in appendix B.3 demonstrates this point, by only slightly modifying the example given here.

6 Concluding Remarks

This paper has revisited a central question in bargaining theory: how do bargaining outcomes, in particular the incidence of inefficient delay, depend on the parties' attitudes to delay? Previous research either assumed that these attitudes conform to ED or that the parties' bargaining behavior is unresponsive to how and why they fail to reach agreement; both assumptions are at odds with the empirical evidence. Here, I have provided the first general answer to this question, fully characterizing equilibrium in the canonical Rubinstein (1982) model of alternating-offers bargaining under minimal assumptions on the parties' attitudes to delay, and for arbitrarily history-dependent behavior.

This generality of both preferences and strategies required a novel analytical approach. I derived a simple, yet sufficiently general structure for off-path punishments that makes non-stationary equilibria analytically tractable even under dynamic inconsistency. To the best of my knowledge, this is the first stationary sequential-move game for which such optimal simple penal codes are generally established. Future research should clarify how useful this

approach is in other, related games, especially regarding analyses of dynamically inconsistent preferences.

When at least one of the parties finds a delay beyond some time in the near future more costly than initial delay from the present, this basic and otherwise disciplined model provides a rich descriptive theory of bargaining. Although there is no material reason for why consequentialist bargainers should ever care about how they have failed to agree in the past—they can therefore always ignore history and stick to a stationary equilibrium—such dynamic inconsistency provides a motive for treating observed past behavior as indicative of future behavior. Thus the model captures important tendencies in real bargaining.

Preferences of this type could be peculiar to bargaining situations, but the recent empirical research on time preferences shows that this is not the case. It suggests that near-future bias may be a more general phenomenon in how many people evaluate delayed *monetary* rewards (appendix B.1). It is a recent “discovery” mainly because this empirical research is only beginning to move on from disproving the universal nature of present bias, especially of the quasi-hyperbolic type, towards investigating what forms time preferences over money actually take.

Adopting a hedonic notion of utility, a present bias (as a bias toward *instantaneous* gratification) is compelling intuitively, and the quasi-hyperbolic model of discounted utility has proven enormously useful in explaining many important behaviors (e.g., Laibson, 1997; O’Donoghue and Rabin, 1999; Bénabou and Tirole, 2002). Yet, money would seem to be only a means to pleasure, in which case only people’s borrowing and lending rates could be elicited using monetary rewards. What drives the observation that the vast majority of participants nonetheless reveal significant biases—either present or near-future bias—is currently not understood and awaits further investigation. Whatever the conclusions of this research, it seems fair to assume that any useful model of time preferences will be covered by this paper’s minimal assumptions, hence its basic bargaining implications will have been laid out here.

In fact, for many settings—certainly those in which intertemporal trade-offs involve sufficiently long delays (beyond a week, say; cf. Sayman and Öncüler, 2009, p. 470)—a genuine present bias is indistinguishable from a near-future bias. Bargaining is somewhat special in that the parties may interact at very high frequency, so this distinction becomes behaviorally relevant. Since present bias implies immediate agreement, this observation suggests an intervention to restore bargaining efficiency under near-future bias: limiting the frequency of offers. Conditional on failing to agree, parties would thus be committed to not agreeing for a sufficiently long period of time so that the first period of delay would be most costly; thus a present bias is induced, which leads to immediate agreement.

I have conducted the analysis under the assumption that each party has perfect knowledge of her own preferences as well as those of her opponent (more precisely, preferences are common knowledge). A near-future biased bargainer suffers from her dynamic inconsistency because it is understood she will take a tougher stance in the future than she currently would like to; present bias is here a force toward immediate agreement because it is also correctly anticipated. The question arises how much sophistication is required for these results, especially in view of potential pitfalls of the intervention just suggested. For the case of quasi-hyperbolic discounting, [Akin \(2007\)](#) has already shown that when the parties are persistently naïve about their own present bias but sophisticated about that of the opponent, there can be severe delays.

Although this paper has emphasized time preferences in a strict sense, there certainly exist sources other than pure time that induce various dynamically inconsistent time preferences (of both the present- and future-biased type) in bargaining. In [appendix B.4](#) I outline bargaining models for two such alternative (or additional) sources, where the results of this paper directly apply: imperfect altruism regarding future generations' bargaining outcomes (cf. [Phelps and Pollak, 1968](#)), and nonlinear probability weighting when bargaining takes place under the shadow of breakdown risk (cf. [Barberis, 2012](#)). In conclusion, both the analytical approach and the mechanisms discovered in this paper for reduced-form time preferences should therefore serve as a useful guide for further analyses of psychologically enriched preferences in strategic bargaining, and—plausibly—even beyond.

References

- Abreu, D. (1988). On the theory of infinitely repeated games with discounting. *Econometrica* 56(2), 383–396.
- Admati, A. R. and M. Perry (1991). Joint projects without commitment. *The Review of Economic Studies* 58(2), 259–276.
- Ainslie, G. (1975). Specious reward: A behavioral theory of impulsiveness and impulse control. *Psychological Bulletin* 82(4), 463–496.
- Akin, Z. (2007). Time inconsistency and learning in bargaining games. *International Journal of Game Theory* 36(2), 275 – 299.
- Andreoni, J. and C. Sprenger (2012). Estimating time preferences from convex budgets. *The American Economic Review* 102(7), 3333–3356.
- Attema, A. E., H. Bleichrodt, K. I. M. Rohde, and P. P. Wakker (2010). Time-tradeoff sequences for analyzing discounting and time inconsistency. *Management Science* 56(11), 2015–2030.
- Avery, C. and P. B. Zemsky (1994). Money burning and multiple equilibria in bargaining. *Games and Economic Behavior* 7(2), 154–168.

- Barberis, N. (2012). A model of casino gambling. *Management Science* 58(1), 35–51.
- Bénabou, R. and J. Tirole (2002). Self-confidence and personal motivation. *The Quarterly Journal of Economics* 117(3), 871–915.
- Benhabib, J., A. Bisin, and A. Schotter (2010). Present-bias, quasi-hyperbolic discounting, and fixed costs. *Games and Economic Behavior* 69(2), 205–223.
- Bernheim, B. D. and A. Rangel (2009). Beyond revealed preference: Choice-theoretic foundations for behavioral welfare economics. *The Quarterly Journal of Economics* 124(1), 51–104.
- Binmore, K. G. (1987). Perfect equilibria in bargaining models. In K. G. Binmore and P. Dasgupta (Eds.), *The Economics of Bargaining*, pp. 77–105. Oxford: Basil Blackwell.
- Binmore, K. G., A. Rubinstein, and A. Wolinsky (1986). The Nash bargaining solution in economic modelling. *The RAND Journal of Economics* 17(2), 176–188.
- Bleichrodt, H., K. I. M. Rohde, and P. P. Wakker (2009). Non-hyperbolic time inconsistency. *Games and Economic Behavior* 66(1), 27–38.
- Börger, T. (1991). Upper hemicontinuity of the correspondence of subgame-perfect equilibrium outcomes. *Journal of Mathematical Economics* 20(1), 89–106.
- Brown, A. L., Z. E. Chua, and C. F. Camerer (2009). Learning and visceral temptation in dynamic saving experiments. *The Quarterly Journal of Economics* 124(1), 197–231.
- Browning, M. and P.-A. Chiappori (1998). Efficient intra-household allocations: A general characterization and empirical tests. *Econometrica* 66(6), 1241–1278.
- Burgos, A., S. Grant, and A. Kajii (2002a). Bargaining and boldness. *Games and Economic Behavior* 38(1), 28–51.
- Burgos, A., S. Grant, and A. Kajii (2002b). Corrigendum to “Bargaining and boldness”. *Games and Economic Behavior* 41(1), 165–168.
- Busch, L.-A. and Q. Wen (1995). Perfect equilibria in a negotiation model. *Econometrica* 63(3), 545–565.
- Chade, H., P. Prokopovych, and L. Smith (2008). Repeated games with present-biased preferences. *Journal of Economic Theory* 139(1), 157–175.
- Chung, S.-H. and R. J. Herrnstein (1967). Choice and delay of reinforcement. *Journal of the Experimental Analysis of Behavior* 10(1), 67–74.
- Coles, M. G. and A. Muthoo (2003). Bargaining in a non-stationary environment. *Journal of Economic Theory* 109(1), 70–89.
- Compte, O. and P. Jehiel (2004). Gradualism in bargaining and contribution games. *The Review of Economic Studies* 71(4), 975–1000.
- Cripps, M. W. (1998). Markov bargaining games. *Journal of Economic Dynamics and Control* 22(3), 341–355.

- Dasgupta, P. and E. Maskin (2005). Uncertainty and hyperbolic discounting. *The American Economic Review* 95(4), 1290–1299.
- De Paula, A. (2013). Econometric analysis of games with multiple equilibria. *Annual Review of Economics* 5, 107–131.
- Ebert, J. E. J. and D. Prelec (2007). The fragility of time: Time-insensitivity and valuation of the near and far future. *Management Science* 53(9), 1423–1438.
- Eil, D. (2012, June). Hypobolic discounting and willingness-to-wait. GMU Working Paper in Economics No. 12-28.
- Fernandez, R. and J. Glazer (1991). Striking for a bargain between two completely informed agents. *The American Economic Review* 81(1), 240–252.
- Fershtman, C. and D. J. Seidmann (1993). Deadline effects and inefficient delay in bargaining with endogenous commitment. *Journal of Economic Theory* 60(2), 306–321.
- Fishburn, P. C. and A. Rubinstein (1982). Time preference. *International Economic Review* 23(3), 677–694.
- Friedenberg, A. (2014, May). Bargaining under strategic uncertainty.
- Fudenberg, D. and J. Tirole (1991). *Game Theory*. The MIT Press.
- Halevy, Y. (2008). Strotz meets Allais: Diminishing impatience and the certainty effect. *The American Economic Review* 98(3), 1145–1162.
- Halevy, Y. (2015). Time consistency: Stationarity and time invariance. *Econometrica* 83(1), 335–352.
- Hall, R. E. and P. R. Milgrom (2008). The limited influence of unemployment on the wage bargain. *American Economic Review* 98(4), 1653–1674.
- Haller, H. and S. Holden (1990). A letter to the editor on wage bargaining. *Journal of Economic Theory* 52(1), 232–236.
- Harris, C. and D. Laibson (2013). Instantaneous gratification. *The Quarterly Journal of Economics* 128(1), 205–248.
- Hoel, M. (1986). Perfect equilibria in sequential bargaining games with nonlinear utility functions. *The Scandinavian Journal of Economics* 88(2), 383–400.
- Kahn, L. M. and J. K. Murnighan (1993). A general experiment on bargaining in demand games with outside options. *The American Economic Review* 83(5), 1260–1280.
- Kennan, J. and R. Wilson (1993). Bargaining with private information. *Journal of Economic Literature* 31(1), 45–104.
- Laibson, D. (1997). Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics* 112(2), 443–478.
- Li, D. (2007). Bargaining with history-dependent preferences. *Journal of Economic Theory* 136(1), 695–708.

- Lu, S. E. (2015, April). Self-control and bargaining.
- Mailath, G. J., V. Nocke, and L. White (2015, February). When and how the punishment must fit the crime. University of Mannheim Working Paper 15-04.
- Manzini, P. and M. Mariotti (2009). Choice over time. In P. Anand, P. Pattanaik, and C. Puppe (Eds.), *The Handbook of Rational and Social Choice*, pp. 239–270. Oxford University Press.
- McClure, S. M., K. M. Ericson, D. I. Laibson, G. Loewenstein, and J. D. Cohen (2007). Time discounting for primary rewards. *The Journal of Neuroscience* 27(21), 5796–5804.
- Merlo, A. and C. Wilson (1995). A stochastic model of sequential bargaining with complete information. *Econometrica* 63(2), 371–399.
- Muthoo, A. (1990). Bargaining without commitment. *Games and Economic Behavior* 2(3), 291–297.
- Myerson, R. B. (1991). *Game Theory: Analysis of Conflict*. Harvard University Press.
- Ned Augenblick, M. N. and C. Sprenger (2015). Working over time: Dynamic inconsistency in real effort tasks. *The Quarterly Journal of Economics* 130(3), 1067–1115.
- Netzer, N. (2009). Evolution of time preferences and attitudes toward risk. *The American Economic Review* 99(3), 937–955.
- Noor, J. (2011). Intertemporal choice and the magnitude effect. *Games and Economic Behavior* 72(1), 255–270.
- O’Donoghue, T. and M. Rabin (1999). Doing it now or later. *The American Economic Review* 89(1), 103–124.
- Ok, E. A. and Y. Masatlioglu (2007). A theory of (relative) discounting. *Journal of Economic Theory* 137(1), 214–245.
- Osborne, M. J. and A. Rubinstein (1990). *Bargaining and Markets*. Academic Press, Inc.
- Pan, J., C. S. Webb, and H. Zank (2015). An extension of qasi-hyperbolic discounting to continuous time. *Games and Economic Behavior* 89, 43–55.
- Perry, M. and P. J. Reny (1993). A non-cooperative bargaining model with strategically timed offers. *Journal of Economic Theory* 59(1), 50–77.
- Phelps, E. S. and R. A. Pollak (1968). On second-best national saving and game-equilibrium growth. *The Review of Economic Studies* 35(2), 185–199.
- Piccione, M. and A. Rubinstein (1997). On the interpretation of decision problems with imperfect recall. *Games and Economic Behavior* 20, 3–24.
- Pollak, R. A. (1968). Consistent planning. *The Review of Economic Studies* 35(2), 201–208.
- Read, D. (2001). Is time-discounting hyperbolic or subadditive? *Journal of Risk and Uncertainty* 23(1), 5–32.
- Read, D., S. Frederick, and M. Airoldi (2012). Four days later in Cincinnati: Longitudinal tests of hyperbolic discounting. *Acta Psychologica* 140(2), 177–185.

- Roth, A. E. (1995). Bargaining experiments. In J. H. Kagel and A. E. Roth (Eds.), *The Handbook of Experimental Economics*, Chapter 4. Princeton University Press.
- Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica* 50(1), 97–109.
- Rubinstein, A. (1991). Comments on the interpretation of game theory. *Econometrica* 59(4), 909–924.
- Rubinstein, A. (2003). “Economics and psychology”? The case of hyperbolic discounting. *International Economic Review* 44(4), 1207–1216.
- Rusinowska, A. (2004). Bargaining model with sequences of discount rates and bargaining costs. *International Game Theory Review* 6(2), 268–280.
- Sákovics, J. (1993). Delay in bargaining games with complete information. *Journal of Economic Theory* 59(1), 78–95.
- Sayman, S. and A. Öncüler (2009). An investigation of time inconsistency. *Management Science* 55(3), 470–482.
- Shaked, A. and J. Sutton (1984). Involuntary unemployment as a perfect equilibrium in a bargaining model. *Econometrica* 52(6), 1351–1364.
- Ståhl, I. (1972). *Bargaining Theory*. EFI The Economics Research Institute, Stockholm.
- Strotz, R. H. (1955-1956). Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies* 23(3), 165–180.
- Takeuchi, K. (2011). Non-parametric test of time consistency: Present bias and future bias. *Games and Economic Behavior* 71(2), 456–478.
- Tamer, E. (2010). Partial identification in econometrics. *Annual Review of Economics* 2, 167–195.
- van Damme, E., R. Selten, and E. Winter (1990). Alternating bid bargaining with a smallest money unit. *Games and Economic Behavior* 2(2), 188–201.
- Weg, E., A. Rapoport, and D. S. Felsenthal (1990). Two-person bargaining behavior in fixed discounting factors games with infinite horizon. *Games and Economic Behavior* 2(1), 76–95.
- Yildiz, M. (2011). Bargaining with optimism. *Annual Review of Economics* 3(4), 451–478.
- Zwick, R., A. Rapoport, and J. C. Howard (1992). Two-person sequential bargaining behavior with exogenous breakdown. *Theory and Decision* 32(3), 241–268.

Appendix

A Proofs

A.1 Additional Notation

The set $\mathcal{A} \equiv X \times \mathcal{P}(X)$ defines the possible pairs of proposals and response rules. The stationary strategy σ_i that specifies “always propose x ” and “always respond using rule Y ” is identified with the pair $(x, Y) \in \mathcal{A}$. The particular division that has player i ’s share equal to one (player j ’s share is zero) is denoted by $e^{(i)}$, and a player i ’s response rule “accept if and only if your share is at least q ” is denoted by $X_{i,q}$.

Take any strategy profile σ , and suppose that if both players act according to σ the outcome is division x in round m (hence with delay $m - 1$), where $x = (0, 0)$ and $m = \infty$ in case of perpetual disagreement. For any $n \leq m$, let then $h^{n-1}(\sigma) \in X^{n-1}$ be the round- n history σ induces, and let $(h^{n-1}(\sigma), x) \in X^m$ be its induced (terminal) path. I formally define σ ’s *play* to be the sequence $\langle \sigma \rangle \equiv (\langle \sigma \rangle_n)_{n=1}^m \in \mathcal{A}^m$ of offers and response *rules* it prescribes along its induced path, i.e. $\langle \sigma \rangle_n \equiv \sigma(h^{n-1}(\sigma))$ for any $n \leq m$.

To isolate plays from strategy profiles, call any sequence $(x^n, Y^n)_{n=1}^m \in \mathcal{A}^m$, for $m \in \mathbb{N}$, a play of game G_i if there exists a strategy profile σ in this game such that $\langle \sigma \rangle = (x^n, Y^n)_{n=1}^m$; this holds true if and only if $x^n \in Y^n \Leftrightarrow n = m$ (the condition is identical for both games G_1 and G_2), and for a given game, a play defines an equivalence class of strategy profiles.

Next, consider the following mapping that produces “simple” strategy profiles. Given any quadruple of plays $S \equiv (\langle \sigma^{P,i} \rangle, \langle \sigma^{R,i} \rangle)_{i \in I}$, define, for each $i \in I$, a mapping $\sigma^{S,i}(\cdot)$ that assigns to any play $\langle \hat{\sigma} \rangle$ a strategy profile in game G_i as follows: interpreting any play $\langle \sigma \rangle \in \{\langle \hat{\sigma} \rangle\} \cup \{\langle \sigma^{P,i} \rangle, \langle \sigma^{R,i} \rangle\}_{i \in I}$ as a sequence of “states”, say a strategy profile is in “state” $\langle \sigma \rangle_n$ if it prescribes play $\langle \sigma \rangle_n$ after a given history, and then define $\sigma^{S,i}(\langle \hat{\sigma} \rangle)$ by the rule that

(1) in round 1 $\sigma^{S,i}(\langle \hat{\sigma} \rangle)$ is in state $\langle \hat{\sigma} \rangle_1$, and

(2) if in round m it is in state $\langle \sigma \rangle_m = (x, Y)$, and proposal x' is rejected, then in round $m + 1$ it is in state

$$\tau(\langle \sigma \rangle_n, x') = \begin{cases} \langle \sigma \rangle_{n+1} & x' = x \notin Y \\ \langle \sigma^{P,P(n)} \rangle_1 & x' \neq x \notin Y \\ \langle \sigma^{R,R(n)} \rangle_1 & x' \neq x \in Y \end{cases}.$$

This is a well-defined strategy profile with the property that it distinguishes only four types of deviations from a given prescribed play—one per player per role—and always specifies the same continuation play after the same type of deviation. It is thus simple in the sense of minimal history-dependence.

Finally, given any pair of reservation shares $Q \equiv (q_1, q_2)$, define, for each $i \in I$, the mapping $\alpha^{Q,i}(\cdot)$ that assigns to any outcome (\hat{x}, \hat{t}) the sequence $(x^n, Y^n)_{n=1}^{\hat{t}+1} \in \mathcal{A}^{\hat{t}+1}$ such that

$$(x^n, Y^n) = \begin{cases} (e^{(P(n))}, X_{R(n), q_{R(n)}}) & n < \hat{t} + 1 \\ (\hat{x}, X_{R(n), \hat{x}_{R(n)}}) & n = \hat{t} + 1 \end{cases} \text{ for } (P(n), R(n)) \equiv \begin{cases} (i, j) & n \text{ odd} \\ (j, i) & n \text{ even} \end{cases}.$$

Note that $\alpha^{Q,i}(\hat{x}, \hat{t})$ is a play of game G_i if and only if

$$\left\{ \begin{array}{l} \hat{t} = 1 \Rightarrow q_j > 0 \\ \hat{t} > 1 \Rightarrow q_1 \cdot q_2 > 0 \end{array} \right\}.$$

A.2 Lemmas 1 and 2

Take any strategy profile σ and any round- n history h^{n-1} : first, let $\sigma|_{h^{n-1}}$ denote the restriction of σ to continuation histories of h^{n-1} , i.e. histories of the form (h^{n-1}, h^{m-1}) where $h^{m-1} \in X^{m-1}$ for $m \in \mathbb{N}$, and second, let $\sigma|^{h^{n-1}}$ denote the strategy profile in game $G_{P(n)}$ that is obtained from $\sigma|_{h^{n-1}}$ upon replacing h^{n-1} by the initial history h^0 . (Observe that, given h^{n-1} , $\sigma|^{h^{n-1}}$ completely characterizes $\sigma|_{h^{n-1}}$.) Fixing any quadruple of strategy profiles $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ such that, for each $i \in I$, $\sigma^{P,i}$ is a strategy profile in game G_j and $\sigma^{R,i}$ is a strategy profile in game G_i , define, for each $i \in I$, the mapping $\sigma^{*,i}(\cdot | (\sigma^{P,i}, \sigma^{R,i})_{i \in I})$ as follows: for any strategy profile σ in game G_i , it is the unique strategy profile $\sigma^{*,i}$ in this game such that $\langle \sigma^{*,i} \rangle = \langle \sigma \rangle$ and

$$\sigma^{*,i}|_{(h^{n-1}(\sigma), x)} = \begin{cases} \sigma^{P,P(n)} & x \notin \sigma_{R(n)}(h^{n-1}(\sigma)) \setminus \{\sigma_{P(n)}(h^{n-1}(\sigma))\} \\ \sigma^{R,R(n)} & x \in \sigma_{R(n)}(h^{n-1}(\sigma)) \end{cases}.$$

Using this definition, lemmas 1 and 2 are formally summarized in the proposition below; part (i) establishes the defining property of optimal punishment, part (ii) shows that it is without loss of generality for optimality to restrict attention to simple punishment, and part (iii) shows it is without loss of generality for equilibrium to restrict attention to simple play.

Proposition 4. *Let the quadruple of outcomes $((x^{P,i}, t^{P,i}), (x^{R,i}, t^{R,i}))_{i \in I}$ be such that, for each $i \in I$,*

$$(x_j^{P,i}, t^{P,i}) \in \arg \max_{(q,t) \in A_j^*} U_j(q, t+1) \quad \text{and} \quad (x_i^{R,i}, t^{R,i}) \in \arg \min_{(q,t) \in A_i^*} U_i(q, t+1). \quad (9)$$

(i) Fix a quadruple of equilibria $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ such that, for each $i \in I$, $\sigma^{P,i}$ is an equilibrium of game G_j supporting outcome $(x^{P,i}, t^{P,i})$ and $\sigma^{R,i}$ is an equilibrium of game G_i supporting outcome $(x^{R,i}, t^{R,i})$. Then, for any $k \in I$ and strategy profile $\hat{\sigma}$ in game G_k , $\langle \hat{\sigma} \rangle$ is an equilibrium play of G_k if and only if $\sigma^{*,k} \left(\hat{\sigma} \left| (\sigma^{P,i}, \sigma^{R,i})_{i \in I} \right. \right)$ is an equilibrium of G_k .

(ii) The quadruple of equilibria $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ in (i) can be chosen such that

$$\sigma^{P,i} = \sigma^{*,j} \left(\sigma^{P,i} \left| (\sigma^{P,i}, \sigma^{R,i})_{i \in I} \right. \right) \quad \text{and} \quad \sigma^{R,i} = \sigma^{*,i} \left(\sigma^{R,i} \left| (\sigma^{P,i}, \sigma^{R,i})_{i \in I} \right. \right). \quad (10)$$

(iii) For any $k \in I$, (\hat{x}, \hat{t}) is an equilibrium outcome of game G_k if and only if $\alpha^{Q^*,k}(\hat{x}, \hat{t})$, with $Q^* = (\pi_1(U_1(x_1^{P,2}, t^{P,2} + 1)), \pi_2(U_2(x_2^{P,1}, t^{P,1} + 1)))$, is an equilibrium play of G_k .

Proof. Part (i). Sufficiency is immediate, since $\langle \sigma^{*,k} \rangle = \langle \hat{\sigma} \rangle$.

For necessity, let $\langle \hat{\sigma} \rangle$ be an equilibrium play of G_k with outcome (\hat{x}, \hat{t}) , where it is without loss of generality to assume $\hat{\sigma}$ is itself an equilibrium of G_k , and also let $\sigma^* = \sigma^* \left(\hat{\sigma} \left| (\sigma^{P,i}, \sigma^{R,i})_{i \in I} \right. \right)$. By construction, $\langle \sigma^* \rangle = \langle \hat{\sigma} \rangle$, and continuation play under σ^* following any deviation from its path is an equilibrium of the resulting subgame. In order to verify that σ^* is an equilibrium it therefore suffices to verify that there are no profitable one-stage deviations at the histories $h^{n-1}(\sigma^*)$ along its path.

Take then any such history $h = h^{n-1}(\sigma^*)$, where player P makes an offer to player R , and $\sigma^*(h) = \hat{\sigma}(h) = (\tilde{x}, \tilde{Y})$. Consider any proposal $x' \in \tilde{Y}$; $\hat{\sigma}$'s being an equilibrium and the construction of σ^* imply that

$$u_R(x'_R) \geq U_R(z_R^h(x', \emptyset | \hat{\sigma})) \geq \min \{U_R(x_R, t + 1) | (x_R, t) \in A_R^*\} = U_R(z_R^h(x', \emptyset | \sigma^*)),$$

whereby acceptance is optimal for R under σ^* .

Next, consider any proposal $x' \notin \tilde{Y} \setminus \{\tilde{x}\}$; $\hat{\sigma}$'s being an equilibrium and the construction of σ^* imply that

$$u_R(x'_R) \leq U_R(z_R^h(x', \emptyset | \hat{\sigma})) \leq \max \{U_R(x_R, t + 1) | (x_R, t) \in A_R^*\} = U_R(z_R^h(x', \emptyset | \sigma^*)),$$

whereby rejection is optimal for R under σ^* .

The only remaining case at the responding stage is that of proposal \tilde{x} such that $\tilde{x} \notin \tilde{Y}$; this implies that $n < \hat{t} + 1$, and then $\hat{\sigma}$'s being an equilibrium play and the construction of σ^* imply that

$$u_R(\tilde{x}_R) \leq U_R(z_R^h(\tilde{x}, \emptyset | \hat{\sigma})) = U_R(\hat{x}_R, \hat{t} + 1 - n) = U_R(z_R^h(\tilde{x}, \emptyset | \sigma^*)),$$

whereby rejection is optimal for R under σ^* .

Finally, consider the proposing player P 's incentive to propose $x' \neq \tilde{x}$: if $x' \in \tilde{Y}$, then $u_P(x'_P) \leq U_P(z_P^h(\tilde{x}, \tilde{Y} | \hat{\sigma}))$ by $\hat{\sigma}$'s being an equilibrium, and because of $z_P^h(\tilde{x}, \tilde{Y} | \sigma^*) = z_P^h(\tilde{x}, \tilde{Y} | \hat{\sigma}) = (\hat{x}_P, \hat{t} + 1 - n)$ such deviations are not profitable to P under σ^* .

Letting $q_R^* = \pi_R(U_R(x_R^{P,P}, t^{P,P} + 1))$, it follows from $\hat{\sigma}$'s being an equilibrium that $\{x \in X | x_R > q_R^*\} \subseteq \tilde{Y}$ and $u_P(1 - q_R^*) \leq U_P(\hat{x}_P, \hat{t} + 1 - n)$: R must accept any offer which exceeds her maximal credible reservation share, and if $u_P(1 - q_R^*) > U_P(\hat{x}_P, \hat{t} + 1 - n)$ were true, then, because $u_P(\cdot)$ is continuously increasing and $q_R^* < 1$ due to R 's impatience, there would exist $\epsilon > 0$ such that P 's offering the accepted share $q_R^* + \epsilon$ would be a profitable deviation under $\hat{\sigma}$. Under σ^* any deviant proposal $x' \notin \tilde{Y}$ yields utility $U_P(x_P^{P,P}, t^{P,P} + 1)$; using the fact that $\pi_P(U_P(x_P^{P,P}, t^{P,P} + 1)) + \pi_R(U_R(x_R^{P,P}, t^{P,P} + 1)) < 1$ by impatience,

$$U_P(x_P^{P,P}, t^{P,P} + 1) \leq u_P(\pi_P(U_P(x_P^{P,P}, t^{P,P} + 1))) < u_P(1 - \pi_R(U_R(x_R^{P,P}, t^{P,P} + 1)));$$

hence no such deviation is profitable for P , concluding the proof.

Part (ii). If $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ is a quadruple of equilibria as in part (i), then $S = (\langle \sigma^{P,i} \rangle, \langle \sigma^{R,i} \rangle)_{i \in I}$ is a quadruple of plays, so the quadruple of strategy profiles $(\sigma^{S,j}(\langle \sigma^{P,i} \rangle), \sigma^{S,i}(\langle \sigma^{R,i} \rangle))_{i \in I}$ is well-defined. When used as punishments in mapping $\sigma^{*,i}$ this quadruple supports the same set of plays in game G_i , $i \in I$, as does $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$, since the punishments for various deviations from initial play are outcome equivalent. In particular, $(\sigma^{S,j}(\langle \sigma^{P,i} \rangle), \sigma^{S,i}(\langle \sigma^{R,i} \rangle))_{i \in I}$ therefore supports its own constituent (equilibrium) plays in S , so at no point is there a profitable deviation from any of these strategy profiles; it is therefore a quadruple of equilibria as in part (i).

Finally, by construction, any of them specifies the same punishment after any deviation by the same player in the same role, irrespective of history: if proposing player i makes a deviant offer that is compliantly rejected, this is $\sigma^{S,j}(\langle \sigma^{P,i} \rangle)$, and if responding player i deviantly rejects an offer, this is $\sigma^{S,i}(\langle \sigma^{R,i} \rangle)$. Hence it satisfies (10).

Part (iii). Sufficiency is immediate. Suppose then that agreement on \hat{x} with delay \hat{t} is an equilibrium outcome of G_k , and let $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ be a quadruple of equilibria as in part (i). Define also each player i 's shares $q_i^* \equiv \pi_i(U_i(x_i^{P,j}, t^{P,j} + 1))$ and $q_i^{**} \equiv \pi_i(U_i(x_i^{R,i}, t^{R,i} + 1))$.

The first step is to show that $\alpha^{Q^*,k}(\hat{x}, \hat{t})$ is a play. This is immediate only for $\hat{t} = 0$; for $\hat{t} = 1$, it is necessary and sufficient that $q_{3-k}^* > 0$, and for $\hat{t} > 1$, it is necessary and sufficient that both $q_2^* > 0$ and $q_1^* > 0$. Suppose then that $q_i^* = 0$ and note that any equilibrium must then have respondent i accept any offer. While immediate for any positive offer, there

cannot be an equilibrium in which respondent i rejects a zero offer by proposer j , because $u_j(1 - \epsilon) > U_j(1, 1)$ for small enough positive and hence accepted offers ϵ ; i 's rejecting a zero offer would therefore imply that such offers constitute profitable deviations by proposer j . Hence, $\hat{t} = 1$ implies $q_{3-k}^* > 0$, and $\hat{t} > 1$ implies both $q_2^* > 0$ and $q_1^* > 0$.

The second step is to show that, whenever σ' is a strategy profile in game G_k whose play equals $\alpha^{Q^*,k}(\hat{x}, \hat{t})$, then $\sigma \equiv \sigma^{*,k} \left(\sigma' \left| \left(\sigma^{P,i}, \sigma^{R,i} \right)_{i \in I} \right. \right)$ is an equilibrium of G_k . It suffices to verify that there are no profitable one-stage deviations at the histories $h^{n-1}(\sigma)$ for $n \leq \hat{t} + 1$, since the continuation strategy profiles $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ are all equilibria of their respective subgames. Consider then any such history $h = h^{n-1}(\sigma)$, where player P makes an offer to player R and $\sigma(h) = (\tilde{x}, X_{R,\tilde{q}})$. Observe the following inequalities:

$$q_R^{**} \leq \tilde{q} \leq q_R^*. \quad (11)$$

While (11) holds by construction if $n < \hat{t} + 1$, in the case of $n = \hat{t} + 1$ it means that $q_R^{**} \leq \hat{x}_R \leq q_R^*$; however, $\hat{x}_R < q_R^{**}$ would imply that there could not be an equilibrium in which R accepts an offer as low as \hat{x}_R , and $\hat{x}_R > q_R^*$ would imply that there could not be an equilibrium in which P offers as much as \hat{x}_R .

R 's rejection of any *deviant* offer $q \neq \tilde{x}_R$ such that $q < \tilde{q}$ is optimal: by (11), such offers exist only if $q_R^* > 0$, in which case their rejection value $U_R(x_R^{P,P}, t^{P,P} + 1)$ equals $u_R(q_R^*)$, and this exceeds that of acceptance, $u(q)$, since $q_R^* \geq \tilde{q} > q$. Moreover, R 's impatience implies that $x_R^{P,P} > q_R^*$, and combined with (11) this yields $U_P(x_P^{P,P}, t^{P,P} + 1) < u_P(1 - q_R^*) \leq u_P(1 - \tilde{q})$, showing that P has no profitable deviation to rejected offers $q < \tilde{q}$.

Also, R 's acceptance of any offer $q \geq \tilde{q}$ is optimal, because it yields a value of at least $u_R(\tilde{q})$, whereas rejection yields no more than $u_R(q_R^{**})$, where $u_R(\tilde{q}) \geq u_R(q_R^{**})$ by (11). Among these offers, \tilde{q} is clearly the best accepted offer for P .

For $n = \hat{t} + 1$, we can already conclude that there is no profitable deviation for either player, since all offers $q < \tilde{q}$ are deviant. Consider then the remaining case of deviations in a round $n < \hat{t} + 1$: if R 's rejection of the minimal possible, i.e. the zero offer failed to be optimal, then $u_R(0) > U_R(\hat{x}_R, \hat{t} + 1 - n)$, so there is no offer that R could optimally reject in favor of agreement on \hat{x} after $\hat{t} + 1 - n$ more rounds—in contradiction to this outcome's equilibrium property; to a similar effect, if P 's compliant zero offer were worse than the lowest accepted offer $\tilde{q} = q_R^*$, then $u_P(1 - q_R^*) > U_P(\hat{x}_P, \hat{t} - n + 1)$, so there is no rejected offer that P could optimally make in return for agreement on \hat{x} after $\hat{t} + 1 - n$ more rounds. \square

A.3 Theorem 1

In what follows, let

$$\begin{aligned}\tilde{v}_i &\equiv \inf \{U_i(q, t) \mid (q, t) \in A_i^*\} \\ \tilde{w}_i &\equiv \inf \{U_i(q, t+1) \mid (q, t) \in A_i^*\}\end{aligned}$$

denote each player i 's infimal punishment values. The theorem is proven via a series of lemmas. The first one, lemma 4, shows that the set E is non-empty. Lemma 5 then shows that for every element $(v_i, w_i, t_i)_{i \in I}$ of E there exists a quadruple of outcomes that deliver the values $(v_i, w_i)_{i \in I}$ when used as punishment outcomes. (This is the only result that uses impatience property (3.c), and it will imply that optimal punishments exist.) Lemma 6 goes on to establish that any such quadruple of outcomes in fact defines a “constrained” OSPC: as punishment outcomes they support a subset of equilibrium outcomes that includes them, and constrained to which they are optimal (see equation (9)). This means, in particular, that for any element $(v_i, w_i, t_i)_{i \in I}$ of E , $\tilde{v}_i \leq v_i$ and $\tilde{w}_i \leq w_i$ for each i . The final two lemmas show that E also contains an element $(v_i, w_i, t_i)_{i \in I}$ such that $v_i \leq \tilde{v}_i$ and $w_i \leq \tilde{w}_i$ for each i . Thus we conclude that E has an extreme element, which is $(v_i^*, w_i^*, t_i^*)_{i \in I}$. (Lemma 6 then implies the characterization of equilibrium outcomes based on the associated OSPC from lemma 5.)

Lemma 4. *The set E is non-empty.*

Proof. Consider the following functions $f_i : [0, 1] \rightarrow [0, 1]$ for each i :

$$f_i(q) \equiv 1 - \pi_j(U_j(1 - \pi_i(U_i(q, 1)), 1)). \quad (12)$$

f_i is continuous, and it is non-decreasing, with $0 < f_i(0) \leq f_i(1) \leq 1$. Hence it possesses a fixed point that is positive. Take any $\hat{q}_1 = f_1(\hat{q}_1)$ and define $\hat{q}_2 \equiv 1 - \pi_1(U_1(\hat{q}_1, 1))$; note that then also $\hat{q}_1 = 1 - \pi_2(U_2(\hat{q}_2, 1))$ and

$$\begin{aligned}\hat{q}_2 &= 1 - \pi_1(U_1(1 - \pi_2(U_2(\hat{q}_2, 1)), 1)) \\ &\equiv f_2(\hat{q}_2).\end{aligned}$$

I will prove that E contains the values $(v_i, w_i, t_i)_{i \in I} = (u_i(\hat{q}_i), U_i(\hat{q}_i, 1), 0)_{i \in I}$.

Given $t_i = 0$, the identity $\phi_i(u_i(\hat{q}_i), 0) \equiv \hat{q}_i$ immediately yields that the chosen values satisfy equations (4) and (5), for each i . At the same time, again for each i , whenever t is

positive,

$$\begin{aligned}
\kappa_i(t, u_i(\hat{q}_i), u_j(\hat{q}_j)) &\geq \kappa_i(1, u_i(\hat{q}_i), u_j(\hat{q}_j)) \\
&\geq \hat{q}_i + \hat{q}_j \\
&= \hat{q}_i + 1 - \pi_i(U_i(\hat{q}_i, 1)) \\
&> 1,
\end{aligned}$$

where the last inequality uses that $\hat{q}_i > 0$ implies $\hat{q}_i > \pi_i(U_i(\hat{q}_i, 1))$. This shows that the chosen values also satisfy equation (6), for each i . \square

Lemma 5. *For every element $(v_i, w_i, t_i)_{i \in I}$ of the set E , there exists a quadruple of outcomes $((y^{(i)}, 0), (x^{(i)}, t^{(i)}))_{i \in I}$ such that, for each $i \in I$,*

$$v_i = u_i\left(1 - \pi_j\left(U_j\left(1 - y_i^{(i)}, 1\right)\right)\right) \quad (13)$$

$$w_i = U_i\left(x_i^{(i)}, t^{(i)} + 1\right). \quad (14)$$

Proof. Let $(v_i, w_i, t_i)_{i \in I} \in E$ and define a quadruple of outcomes $((y^{(i)}, 0), (x^{(i)}, t^{(i)}))_{i \in I}$ such that, for each $i \in I$,

$$y_i^{(i)} = \pi_i(w_i) \text{ and } \left\{ \begin{array}{l} t^{(i)} \in \arg \min \{U_i(\phi_i(v_i, t), t + 1) \mid t \in T, t \leq t_i\} \\ x_i^{(i)} = \phi_i(v_i, t^{(i)}) \end{array} \right\}. \quad (15)$$

Recalling equations (4) and (5), it only remains to show that such values $t^{(i)}$ exist, so that the quadruple is well-defined. This is clearly true when each t_i is finite, and the following three steps prove it also for the case that $t_i = \infty$ (for some i).

Step 1: For any t , $\phi_i(v_i, t) > 0$. From equation (4) it follows that $v_i \geq u_i(1 - \pi_j(U_j(1, 1))) > u_i(0)$, since $\pi_j(U_j(q, t + 1)) \leq \pi_j(U_j(1, 1)) < 1$ for all $(q, t) \in A_j$ due to j 's impatience. Using identity $v_i \equiv u_i(\phi_i(v_i, 0))$, $v_i > u_i(0)$ is equivalent to $\phi_i(v_i, 0) > 0$, and the claim follows from the non-decreasingness of $\phi_i(u, \cdot)$ for any $u \in u_i([0, 1])$.

Step 2: For any $t \leq t_i$, $U_i(\phi_i(v_i, t), t) = v_i$. Since this holds true for $t = 0$ by definition, consider it for $0 < t \leq t_i$ and note that it suffices to show that $\phi_i(v_i, t) < 1$ (recall the definition of ϕ_i): from equation (6), $\kappa_i(t, v_i, v_j) \leq 1$, and using that $\phi_j(v_j, t - 1) > 0$ from step 1, this implies $\phi_i(v_i, t) < 1$.

Step 3: There exists a finite \bar{t}_i such that $w_i = \min \{U_i(\phi_i(v_i, t), t + 1) \mid t \in T, t \leq \bar{t}_i\}$. Since we can simply set $\bar{t}_i = t_i$ if t_i is finite, consider the case of $t_i = \infty$ and distinguish the two possible cases according to impatience property (3.c). Suppose first that player i 's preferences satisfy $\lim_{t \rightarrow \infty} U_i(1, t) \leq u_i(0)$. Since $v_i > u_i(0)$ from step 1, there then exists

a finite delay \hat{t} such that $t \geq \hat{t}$ implies $U_i(1, t) < v_i$, and hence $U_i(\phi_i(v_i, t), t) < v_i$, which contradicts step 2. The alternative case is that there exists a finite delay \hat{t} such that $t \geq \hat{t}$ implies $U_i(q, t) = U_i(q, \hat{t})$ for all q ; hence $U_1(\phi_1(v_1, t), t + 1) = U_1(\phi_1(v_1, \hat{t}), \hat{t} + 1)$ for all such t , which proves the claim upon setting $\bar{t}_1 = \hat{t}$. \square

Statement and proof of the next lemma use the following definition: for any values $(v_k, w_k)_{k \in I} \in \times_{k \in I} (u_k([0, 1]) \times U_k(A_k))$ and any player i , $A_i(v_1, w_1, v_2, w_2)$ is the set

$$\left\{ (q, t) \in A_i \left| \phi_i(v_i, t) \leq q \leq \begin{cases} 1 - \pi_j(w_j) & t = 0 \\ 1 - \max\{\phi_j(v_j, t - 1), \phi_j(u_j(0), t)\} & t > 0 \end{cases} \right. \right\}.$$

Lemma 6. *Take any element $(v_i, w_i, t_i)_{i \in I}$ of the set E and associated quadruple of outcomes $((y^{(i)}, 0), (x^{(i)}, t^{(i)}))_{i \in I}$ satisfying (15). Then, for each $i \in I$,*

$$\left\{ (1 - y_j^{(j)}, 0), (x_i^{(i)}, t^{(i)}) \right\} \subseteq A_i(v_1, w_1, v_2, w_2) \subseteq A_i^*,$$

and the following equalities hold true:

$$\begin{aligned} v_i &= \min \{U_i(q, t) \mid (q, t) \in A_i(v_1, w_1, v_2, w_2)\} \\ w_i &= \min \{U_i(q, t + 1) \mid (q, t) \in A_i(v_1, w_1, v_2, w_2)\} \\ t_i &= \sup \{t \in T \mid \exists q \in [0, 1], (q, t) \in A_i(v_1, w_1, v_2, w_2)\}. \end{aligned}$$

Proof. The following observation, for each i , will be helpful:

$$v_i > \max \{u_i(0), w_i\}. \quad (16)$$

Since $v_i > u_i(0)$ was established in step 1 of lemma 5, it only remains to prove that $v_i > w_i$: this follows from equation (5), implying $w_i \leq U_i(\phi_i(v_i, 0), 1)$, because $\phi_i(v_i, 0) > 0$ and i is impatient.

Let then, for each i , $\hat{q}_i \equiv \pi_i(U_i(1 - y_j^{(j)}, 1))$ and note that equation (13) implies that

$$\hat{q}_i = 1 - \phi_j(v_j, 0). \quad (17)$$

First, I will show that, given $Q = (\hat{q}_1, \hat{q}_2)$, $\alpha^{Q, i}(x^{(i)}, t^{(i)})$ is a play of game G_i , for each i . To simplify notation, let $i = 1$, which is without loss of generality. There is nothing to check if $t^{(1)} = 0$, so consider the case of $t^{(1)} > 0$. This implies that $t_1 > 0$ and hence $\kappa_1(1, v_1, v_2) \leq 1$; using that $\phi_2(v_2, 0) > 0$ by (16), we obtain $\phi_1(v_1, 1) < 1$, which implies $\phi_1(v_1, 0) < 1$, and hence, via equation (4) (for $i = 1$), $\hat{q}_2 > 0$. While necessary for any $t^{(1)} > 0$, this is

sufficient to prove the claim for $t^{(1)} = 1$. Suppose then $t^{(1)} > 1$; this implies $t_1 > 1$ and hence $\kappa_1(2, v_1, v_2) \leq 1$. Using $\phi_1(v_1, 2) > 0$ from combining (16) with the non-decreasingness of $\phi_1(u, \cdot)$, this in turn implies that $\phi_2(v_2, 0) < 1$, from which $\hat{q}_1 > 0$ follows via equation (4) (for $i = 2$).

Since any immediate-agreement outcome defines a play, it immediately follows from the previous argument that $S \equiv \left(\alpha^{Q,j} \left(y^{(j)}, 0 \right), \alpha^{Q,i} \left(x^{(i)}, t^{(i)} \right) \right)_{i \in I}$ is a quadruple of plays. I will now show that, for any outcome (\hat{x}, \hat{t}) and each i , $\alpha^{Q,i}(\hat{x}, \hat{t})$ is a play of G_i such that $\sigma^{S,i}(\alpha^{Q,i}(\hat{x}, \hat{t}))$ is an equilibrium of G_i if and only if $(\hat{x}_i, \hat{t}) \in A_i(v_1, w_1, v_2, w_2)$. Since $\left\{ (1 - y_j^{(j)}, 0), (x_i^{(i)}, t^{(i)}) \right\} \subseteq A_i(v_1, w_1, v_2, w_2)$, it is sufficient to prove that $\alpha^{Q,i}(\hat{x}, \hat{t})$ is a play of G_i such that there are no profitable deviations from this play under the strategy profile $\sigma^{S,i}(\alpha^{Q,i}(\hat{x}, \hat{t}))$ if and only if $(\hat{x}_i, \hat{t}) \in A_i(v_1, w_1, v_2, w_2)$. Again, only to simplify notation, I prove this claim for $i = 1$; also, I let $\hat{\sigma} \equiv \sigma^{S,1}(\alpha^{Q,1}(\hat{x}, \hat{t}))$ and $\hat{A}_1 \equiv A_1(v_1, w_1, v_2, w_2)$.

First, consider **immediate-agreement outcomes** $(\hat{x}, 0)$; $\alpha^{Q,1}(\hat{x}, 0)$ is a play for any division \hat{x} , and it remains to show that there is no profitable deviation from this play under $\hat{\sigma}$ if and only if $(\hat{x}_1, 0) \in \hat{A}_1$. Player 2's accepting all offers $q \geq \hat{x}_2$ is optimal if and only if $\hat{x}_2 \geq \pi_2(w_2)$, because deviantly rejecting such an offer would trigger her respondent punishment, which has continuation outcome $(x^{(2)}, t^{(2)})$ and associated rejection value w_2 ; her rejecting all other offers is optimal if and only if $\hat{x}_2 \leq \hat{q}_2$ because non-deviantly rejecting such a deviant offer would trigger player 1's proposer punishment, which has continuation outcome $(y^{(1)}, 0)$ and associated rejection value $U_2(1 - \pi_1(w_1), 1)$; using equation (17), $\hat{x}_2 \leq \hat{q}_2$ is equivalent to $\phi_1(v_1, 0) \leq \hat{x}_1$. To summarize, in terms of player 1's share in \hat{x} , player 2's response rule is optimal if and only if $\phi_1(v_1, 0) \leq \hat{x}_1 \leq 1 - \pi_2(w_2)$; this is equivalent to $(\hat{x}_1, 0) \in \hat{A}_1$.

Given player 2 optimally accepts with threshold \hat{x}_2 , this is the lowest immediately accepted offer, and there is no profitable deviation for player 1 if and only if $u_1(\hat{x}_1) \geq U_1(\pi_1(w_1), 1)$, because any deviation to a rejected offer triggers her proposer punishment which has continuation outcome $(y^{(1)}, 0)$ and associated rejection value $U_1(\pi_1(w_1), 1)$; inequality (16) implies $\phi_1(v_1, 0) > \pi_1(w_1)$, whereby $v_1 \geq U_1(\pi_1(w_1), 1)$ from player 1's impatience, and there is no profitable deviation for proposing player 1 whenever there is none for responding player 2. Hence, there is no profitable deviation from $\alpha^{Q,1}(\hat{x}, 0)$ if and only if $(\hat{x}_1, 0) \in \hat{A}_1$.

Next, consider **once delayed agreement outcomes** $(\hat{x}, 1)$; $\alpha^{Q,1}(\hat{x}, 1)$ is a play if and only if $\hat{q}_2 > 0$. Observe that $\hat{q}_2 = 0$ is equivalent to $\phi_1(v_1, 0) = 1$, by equation (17), and jointly with inequality (16) (for $i = 2$), this would indeed mean that \hat{A}_1 contains no delayed agreements at all. Hence it remains to establish the claim for this case under the assumption that $\hat{q}_2 > 0$.

Regarding the second round on the path, the above finding for the case of immediate-agreement outcomes—by mere relabeling—shows that there are then no profitable one-stage deviations if and only if $\phi_2(v_2, 0) \leq \hat{x}_2 \leq 1 - \pi_1(w_1)$. In terms of player 1's share this is equivalent to

$$\pi_1(w_1) \leq \hat{x}_1 \leq 1 - \phi_2(v_2, 0).$$

In the first round $\hat{\sigma}$ specifies that player 2 respond to offers by accepting with threshold \hat{q}_2 . Accepting offers $q \geq \hat{q}_2$ is optimal if and only if accepting offer \hat{q}_2 is optimal, i.e. if $u_2(\hat{q}_2) \geq U_2(x_2^{(2)}, t^{(2)} + 1)$, since the (deviant) rejection of any such offer is followed by continuation outcome $(x^{(2)}, t^{(2)})$. Note that $U_2(x_2^{(2)}, t^{(2)} + 1) = w_2$ from equation (14), and $w_2 \leq U_2(\phi_2(v_2, 0), 1)$ from equation (5); recalling equation (17), if acceptance were not optimal, then $u_2(1 - \phi_1(v_1, 0)) < U_2(\phi_2(v_2, 0), 1)$, which would imply that $\phi_2(v_2, 0) + \phi_1(v_1, 0) > 1$ and there would be no delayed agreement in \hat{A}_1 .

Rejection of all (deviant) offers q such that $0 < q < \hat{q}_2$ is followed by continuation outcome $(y^{(1)}, 0)$ and is optimal by construction, since $\hat{q}_2 > 0$ implies that $u_2(\hat{q}_2) = U_2(1 - \pi_1(w_1), 1)$ is the associated rejection value. Rejecting the zero offer specified for the proposer in this round is optimal if and only if $u_2(0) \leq U_2(\hat{x}_2, 1)$; either $u_2(0) \leq U_2(1, 1)$, in which case $u_2(0) \leq U_2(\hat{x}_2, 1)$ is equivalent to $\hat{x}_1 \leq 1 - \phi_2(u_2(0), 1)$, or $u_2(0) > U_2(1, 1)$, in which case $\phi_2(u_2(0), 1) = 1$ together with inequality (16) (for $i = 1$) implies that \hat{A}_1 contains no delayed agreements.

By equation (17), the initial proposer 1 can obtain at most the value v_1 from making a deviant accepted offer $q \geq \hat{q}_2$; making a deviant rejected offer $q < \hat{q}_2$ yields value $U_1(\pi_1(w_1), 1)$, which is no greater than v_1 due to inequality (16); hence making her supposed (rejected) offer of a zero share is optimal if and only if $v_1 \leq U_1(\hat{x}_1, 1)$. This is equivalent to $\hat{x}_1 \geq \phi_1(v_1, 1)$ unless $v_1 > U_1(\phi_1(v_1, 1), 1)$; however, the latter would imply $\phi_1(v_1, 1) = 1$ and together with inequality (16) (for $i = 2$) would yield that \hat{A}_1 contains no delayed-agreement outcomes. In summary of this case for $\hat{q}_2 > 0$, using that $\pi_1(w_1) < \phi_1(v_1, 1)$ from inequality (16), and noting that $\min\{1 - \phi_2(v_2, 0), 1 - \phi_2(u_2(0), 1)\}$ equals $1 - \max\{\phi_2(v_2, 0), \phi_2(u_2(0), 1)\}$, we obtain there is no profitable deviation if and only if $(\hat{x}_1, 1) \in \hat{A}_1$.

Finally, consider **further delayed agreement outcomes** (\hat{x}, \hat{t}) such that $\hat{t} > 1$; $\alpha^{Q,1}(\hat{x}, \hat{t})$ is a play if and only if $\hat{q}_1 \cdot \hat{q}_2 > 0$. From the previous case we know that if $\hat{q}_2 = 0$ then \hat{A}_1 would not contain any delayed agreement; now note that $\hat{q}_1 = 0$ is equivalent to $\phi_2(v_2, 0) = 1$, by equation (17), and in combination with inequality (16) (for $i = 1$) would imply that \hat{A}_1 contains no agreements delayed by more than one period. Hence it remains to establish the claim for this case under the assumption that $\hat{q}_1 \cdot \hat{q}_2 > 0$.

In the last round of play $\alpha^{Q,1}(\hat{x}, \hat{t})$, which is round $\hat{t} + 1$, we can use the previous findings

to conclude that there is no profitable deviation if and only if

$$\begin{cases} \pi_1(w_1) \leq \hat{x}_1 \leq 1 - \phi_2(v_2, 0) & \hat{t} \text{ odd} \\ \phi_1(v_1, 0) \leq \hat{x}_1 \leq 1 - \pi_2(w_2) & \hat{t} \text{ even} \end{cases}.$$

Consider then play $\alpha^{Q,1}(\hat{x}, \hat{t})$ for any round $n < \hat{t} + 1$, in which player P makes an offer to player R . Optimality of R 's response rule is characterized in a manner similar to optimality of initial respondent 2's response rule when we considered agreement-outcomes with one round of delay; it is therefore characterized by $u_R(0) \leq U_R(\hat{x}_R, \hat{t} + 1 - n)$. Since $U_R(\hat{x}_R, \hat{t} + 1 - n)$ is non-decreasing in n , this yields only two restrictions, namely those for the first two rounds' respondent stages, which are $u_2(0) \leq U_2(\hat{x}_2, \hat{t})$ and $u_1(0) \leq U_1(\hat{x}_1, \hat{t} - 1)$, respectively. These two inequalities are equivalent to

$$\phi_1(u_1(0), \hat{t} - 1) \leq \hat{x}_1 \leq 1 - \phi_2(u_2(0), \hat{t})$$

whenever both $u_2(0) \leq U_2(1, \hat{t})$ and $u_1(0) \leq U_1(1, \hat{t} - 1)$ hold true; otherwise, however, \hat{A}_1 contains no outcome that has agreement delayed by \hat{t} periods.

Again, similar to optimality for initial proposer 1 when we considered one round of delay, proposer P 's zero offer is here optimal if and only if $v_P \leq U_P(\hat{x}_P, \hat{t} + 1 - n)$. Since $U_P(\hat{x}_P, \hat{t} + 1 - n)$ is non-decreasing in n , this yields only two restrictions, namely those for the first two rounds' proposer stages, which are $v_1 \leq U_1(\hat{x}_1, \hat{t})$ and $v_2 \leq U_2(\hat{x}_2, \hat{t} - 1)$, respectively. These two inequalities are equivalent to

$$\phi_1(v_1, \hat{t}) \leq \hat{x}_1 \leq 1 - \phi_2(v_2, \hat{t} - 1)$$

whenever both $v_1 \leq U_1(1, \hat{t})$ and $v_2 \leq U_2(1, \hat{t} - 1)$ hold true; otherwise, however, \hat{A}_1 contains no outcome that has agreement delayed by \hat{t} periods. Now observe that $\phi_1(v_1, \hat{t})$ is at least as large as any of $\pi_1(w_1)$, $\phi_1(v_1, 0)$ or $\phi_1(u_1(0), \hat{t} - 1)$, due to 1's impatience and inequality (16); moreover, also $\phi_2(v_2, \hat{t} - 1)$ is at least as large as both $\phi_2(v_2, 0)$ and $\pi_2(w_2)$ due to 2's impatience and inequality (16). Hence we can summarize this case for $\hat{q}_1 \cdot \hat{q}_2 > 0$ by the condition that (\hat{x}, \hat{t}) is such that

$$\phi_1(v_1, \hat{t}) \leq \hat{x}_1 \leq 1 - \max\{\phi_2(v_2, \hat{t} - 1), \phi_2(u_2(0), \hat{t})\},$$

which is again equivalent to $(\hat{x}_1, \hat{t}) \in \hat{A}_1$.

A similar proof applies to the case of $i = 2$, hence $\{(1 - y_j^{(j)}, 0), (x_i^{(i)}, t^{(i)})\} \subseteq A_i(v_1, w_1, v_2, w_2) \subseteq$

A_i^* , and the lemma's claimed equations are easily verified. \square

Lemma 7. *The following relationships hold true for each $i \in I$:*

$$\tilde{v}_i = u_i (1 - \pi_j (U_j (1 - \pi_i (\tilde{w}_i), 1))) \quad (18)$$

$$\tilde{w}_i \geq \inf \{U_i (\phi_i (\tilde{v}_i, t), t + 1) \mid t \in T, t \leq t_i^*\} \quad (19)$$

$$t_i^* \leq \sup \{t \in T \mid \kappa_i (t, \tilde{v}_i, \tilde{v}_j) \leq 1\} \quad (20)$$

Proof. First, observe that, for each i ,

$$(q, t) \in A_i^* \Rightarrow (1 - \pi_i (U_i (q, t + 1)), 0) \in A_j^*. \quad (21)$$

Let σ be an equilibrium of game G_i which supports i 's personal outcome (q, t) , denote the share $1 - \pi_i (U_i (q, t + 1))$ by \hat{q} and the division such that j 's share equals \hat{q} by \hat{x} . The strategy profile $\hat{\sigma}$ in game G_j such that $\hat{\sigma} (h^0) = (\hat{x}, X_{i,\hat{q}})$ and $\hat{\sigma} (x, h) = \sigma (h)$ for any division x and history h , is an equilibrium supporting j 's personal outcome $(1 - \hat{q}, 0)$: following any initial rejection, $\hat{\sigma}$ specifies equilibrium σ , which induces personal outcome (q, t) for player i and thus implies that the initial response rule of accepting with threshold \hat{q} is optimal for i ; the initial proposer j best-responds by offering this share, because this is the lowest accepted offer and, moreover, satisfies $u_j (1 - \hat{q}) \geq U_j (1 - q, t + 1)$, due to $\hat{q} \leq q$, which follows from i 's impatience, together with the desirability and impatience properties of j 's preferences.

Using this observation, I will now prove all three conditions (18)-(20) for the case of $i = 1$; mere relabeling yields them for $i = 2$.

To show that the pair $(\tilde{v}_1, \tilde{w}_1)$ satisfies equation (18), combine (21) (for $i = 2$) with the fact that any equilibrium of game G_1 must have the initial respondent 2 accept all offers greater than $\sup \{\pi_2 (U_2 (q, t + 1)) \mid (q, t) \in A_2^*\}$, to obtain

$$\tilde{v}_1 = u_1 (1 - \sup \{\pi_2 (U_2 (q, t + 1)) \mid (q, t) \in A_2^*\}).$$

It then remains to prove that $\pi_2 (U_2 (1 - \pi_1 (\tilde{w}_1), 1)) = \sup \{\pi_2 (U_2 (q, t + 1)) \mid (q, t) \in A_2^*\}$. For this, also combine (21) (now for $i = 1$) with the fact that any equilibrium of G_2 must have the initial respondent 1 reject all offers less than $\pi_1 (\tilde{w}_1)$, which yields that

$$1 - \pi_1 (\tilde{w}_1) = \sup \{q \in [0, 1] \mid (q, 0) \in A_2^*\}.$$

Now observe that any $(q, t) \in A_2^*$ with $t > 0$ satisfies $U_1 (1 - q, t) \geq \tilde{w}_1$, which implies $1 - q \geq \pi_1 (U_1 (1 - q, t)) \geq \pi_1 (\tilde{w}_1)$ by 1's impatience and the non-decreasingness of π_1 , and

therefore

$$\pi_2(U_2(q, t+1)) \leq \pi_2(U_2(1 - \pi_1(\tilde{w}_1), t+1)) \leq \pi_2(U_2(1 - \pi_1(\tilde{w}_1), 1))$$

by the desirability and impatience properties of 2's preferences, together with the non-decreasingness of π_2 .

Regarding the proof that $(\tilde{v}_1, \tilde{w}_1, t_1^*)$ satisfies inequality (19), simply note that $(q, t) \in A_1^*$ implies $U_1(q, t) \geq \tilde{v}_1$ by the definition of \tilde{v}_1 , and thus $q \geq \phi_1(\tilde{v}_1, t)$; the claim then follows from the desirability property of 1's preferences.³⁷

Inequality (20) certainly holds true if $t_1^* = 0$; for the case of $t_1^* > 0$, note that $(q, t) \in A_1^*$ implies both $U_1(q, t) \geq \tilde{v}_1$ and $U_2(1 - q, t) \geq u_2(0)$. These two inequalities imply, respectively, that $q \geq \phi_1(\tilde{v}_1, t)$ and $1 - q \geq \phi_2(u_2(0), t)$. Moreover, if $(q, t) \in A_1^*$ with $t > 0$ also implies that $(q, t - 1) \in A_2^*$, hence $U_2(1 - q, t - 1) \geq \tilde{v}_2$, and thus $1 - q \geq \phi_2(\tilde{v}_2, t - 1)$. Altogether, for any $t > 0$ there exists a share q such that $(q, t) \in A_1^*$ only if $\kappa_1(t, \tilde{v}_1, \tilde{v}_2) \leq 1$, concluding the proof. \square

Lemma 8. *There exist values $(v_i, w_i, t_i)_{i \in I} \in E$ such that $v_i \leq \tilde{v}_i$, $w_i \leq \tilde{w}_i$ and $t_i \geq t_i^*$ for both $i \in I$.*

Proof. Consider the following sequence $(v_i^n, w_i^n, t_i^n)_{i \in I}$: $(w_1^1, w_2^1) \equiv (\tilde{w}_1, \tilde{w}_2)$ and, for any $n \in \mathbb{N}$ and each i ,

$$\begin{aligned} v_i^n &\equiv u_i(1 - \pi_j(U_j(1 - \pi_i(w_i^n), 1))) \\ t_i^n &\equiv \sup \{t \in T \mid \kappa_i(t, v_i^n, v_j^n) \leq 1\} \\ w_i^{n+1} &\equiv \inf \{U_i(\phi_i(v_i^n, t), t+1) \mid t \in T, t \leq t_i^n\}. \end{aligned}$$

Note that $v_i^1 = \tilde{v}_i$ and $t_i^1 \geq t_i^*$, by lemma 7. It is straightforward that $w_i^{n+1} \leq w_i^n$, $v_i^{n+1} \leq v_i^n$ and $t_i^{n+1} \geq t_i^n$. I will establish the claim by proving that the sequence $(v_i^n, w_i^n, t_i^n)_{i \in I}$ possesses a limit in E .

The first step is to prove that the sequence (w_1^n, w_2^n) converges: since each component sequence w_i^n is non-increasing and bounded from below by $U_i(0, \infty) \in \mathbb{R}$, it converges. Denoting this limit by (\hat{w}_1, \hat{w}_2) , the continuity properties of the functions involved imply the

³⁷Under the weakening of desirability suggested in fn. 12, the observation $\tilde{v}_1 > u_1(0)$ from (16) means that no equilibrium delay t can be such that player 1 does not care about her share: otherwise, there would exist $(q, t) \in A_1^*$ with $U_1(q, t) = U_1(0, t)$, but $U_1(0, t) \leq u_1(0)$ by impatience; hence $U_1(q, t) < \tilde{v}_1$, a contradiction.

following convergence properties of the sequences v_i^n and t_i^n , for each i :

$$\begin{aligned} v_i^n &\rightarrow u_i(1 - \pi_j(U_j(1 - \pi_i(\hat{w}_i), 1))) \equiv \hat{v}_i \\ t_i^n &\rightarrow \sup \{t \in T \mid \kappa_i(t, \hat{v}_i, \hat{v}_j) \leq 1\} \equiv \hat{t}_i; \end{aligned}$$

i.e., $(\hat{v}_i, \hat{w}_i, \hat{t}_i)_{i \in I} \in E$. □

A.4 Corollary 1

Proof. Lemma 6 implies that E 's being a singleton is necessary for equilibrium uniqueness.

Concerning its sufficiency, the proof of lemma 4 shows that whenever E is a singleton, its unique element $(v_i^*, w_i^*, t_i^*)_{i \in I}$ equals $(u_i(\hat{q}_i), U_i(\hat{q}_i, 1), 0)_{i \in I}$, for \hat{q}_1 the unique fixed point of f_1 and $\hat{q}_2 \equiv 1 - \pi_1(U_1(\hat{q}_1, 1))$. Characterization theorem 1 then implies that each A_i^* equals the singleton $\{(\hat{q}_i, 0)\}$. Consider then any round in which player P makes an offer to responding player R : since any equilibrium has the outcome that offer \hat{q}_P is accepted, it must be that P indeed offers \hat{q}_P , and that R accepts this offer. Since any equilibrium has the same continuation outcome with R 's associated rejection value equal to $U_R(\hat{q}_R, 1)$, any optimal response rule must have R accept any offer $q > \pi_R(U_R(\hat{q}_R, 1))$ as well as reject any offer $q < \pi_R(U_R(\hat{q}_R, 1))$. This pins down a unique equilibrium that is, moreover, stationary. □

A.5 Lemma 3

This lemma will be proven based on the following characterization of stationary equilibrium, which establishes a one-to-one relationship between stationary equilibria and fixed points of f_1 (defined by equation (12) to prove lemma 4, as part of theorem 1). Note that in terms of the players' impatience (3.) the characterization of stationary equilibrium relies only on property (3.b), players' attitudes to delay beyond a single (first) period are irrelevant.

Lemma 9. *The profile of stationary strategies $(x^{(i)}, Y^{(i)})_{i \in I}$ is an equilibrium if and only if*

$$\left\{ \begin{array}{l} x_1^{(1)} = f_1(x_1^{(1)}) \\ x_2^{(2)} = 1 - \pi_1(U_1(x_1^{(1)}, 1)) \end{array} \right\}, \text{ and for each } i \in I, Y^{(i)} = X_{i, x_i^{(j)}}.$$

A stationary equilibrium exists, and it is unique if and only if f_1 has a unique fixed point.

Proof. First, note that any equilibrium, hence any stationary equilibrium, has agreement, since $v_i^* > u_i(0) \geq U_i(0, \infty)$ from inequality (16). Consider then a stationary equilibrium $(x^{(i)}, Y^{(i)})_{i \in I}$. If $x^{(1)} \notin Y^{(2)}$ then its outcome in G_1 must be $(x^{(2)}, 1)$. Because this outcome

obtains irrespective of play in the initial round of G_1 , responding player 2 must accept any proposal x with $x_2 > \pi_2(U_2(x_2^{(2)}, 1))$. Player 2's impatience property (3.b) implies that either (i) $\pi_2(U_2(x_2^{(2)}, 1)) < x_2^{(2)}$ or (ii) $\pi_2(U_2(x_2^{(2)}, 1)) = x_2^{(2)} = 0$. In case of (i) there exist values $\epsilon > 0$ such that $\epsilon < x_2^{(2)} - \pi_2(U_2(x_2^{(2)}, 1))$, and any of them satisfy

$$u_1(1 - \pi_2(U_2(x_2^{(2)}, 1)) - \epsilon) > U_1(1 - \pi_2(U_2(x_2^{(2)}, 1)) - \epsilon, 1) \geq U_1(1 - x_2^{(2)}, 1)$$

by impatience property (3.b) and desirability of player 1's preferences, applied in this sequence. In case of (ii), impatience property (3.b) together with continuity of player 1's preferences imply existence of $\epsilon > 0$ such that $u_1(1 - \epsilon) > U_1(1, 1)$. In any case player 1 can therefore propose immediately accepted divisions that yield a value greater than that from proposing $x^{(1)}$, contradicting equilibrium. After a symmetric argument, it is then proven that $x^{(i)} \in Y^{(j)}$ for both $i \in I$.

Given this immediate-agreement property of stationary equilibrium, by desirability, (i) a responding player j must accept any offered share $x_j > \pi_j(U_j(x_j^{(j)}, 1))$ as well as reject any $x_j < \pi_j(U_j(x_j^{(j)}, 1))$, and (ii) there cannot exist a proposal x by player i with $x_i > x_i^{(i)}$ such that $x \in Y^{(j)}$, whereby

$$x_i^{(i)} = 1 - \pi_j(U_j(x_j^{(j)}, 1)) \text{ and } Y^{(i)} = X_{i, x_i^{(i)}},$$

and substituting the expression for $x_2^{(2)}$ into that for $x_1^{(1)}$ yields $x_1^{(1)} = f_1(x_1^{(1)})$, establishing necessity. Sufficiency is easily verified, and its proof omitted here.

Existence of a fixed point of f_1 and hence stationary equilibrium is established by the proof of lemma 4, and the characterization shows that there are as many distinct stationary equilibria as there are fixed points of f_1 . \square

Lemma 3 follows from combining the above characterization with the next result.

Lemma 10. *If both players' preferences exhibit immediacy, then f_1 has a unique fixed point.*

Proof. Suppose that player i 's preferences exhibit immediacy, take any share q and any $\epsilon > 0$ such that $q + \epsilon \leq 1$, and consider various possible cases to establish that $l_i(q) \equiv q - \pi_i(U_i(q, 1))$ is increasing. First, if $U_i(q + \epsilon, 1) \leq u_i(0)$, then also $U_i(q, 1) \leq u_i(0)$ and $l_i(q) = q < q + \epsilon = l_i(q + \epsilon)$. Second, if $U_i(q, 1) \leq u_i(0) < U_i(q + \epsilon, 1)$, then continuity and impatience imply existence of a share $q' \in [q, q + \epsilon)$ such that $U_i(q', 1) = u_i(0)$; letting $\epsilon' \equiv q + \epsilon - q'$, immediacy implies $u_i(\epsilon') > U_i(q' + \epsilon', 1) \equiv U_i(q + \epsilon, 1)$, and hence $l_i(q + \epsilon) > q + \epsilon - \epsilon' \geq q = l_i(q)$. Finally, if $u_i(0) < U_i(q, 1)$, then continuity and impatience imply

existence of a share $q' \in (0, q)$ such that $u_i(q') = U_i(q, 1)$; immediacy implies $u_i(q' + \epsilon) > U_i(q + \epsilon, 1)$, and hence $l_i(q + \epsilon) > q + \epsilon - (q' + \epsilon) = l_i(q)$.

Consider then the following difference:

$$\begin{aligned} q - f_1(q) &= q - 1 + \pi_2(U_2(1 - \pi_1(U_1(q, 1)), 1)) \\ &= [q - \pi_1(U_1(q, 1))] - [(1 - \pi_1(U_1(q, 1))) - \pi_2(U_2(1 - \pi_1(U_1(q, 1)), 1))]. \\ &\equiv l_1(q) - l_2(1 - \pi_1(U_1(q, 1))). \end{aligned}$$

If l_i is increasing for both i , then l_1 is increasing in q and l_2 is increasing in $1 - \pi_1(U_1(q, 1))$. Since $1 - \pi_1(U_1(q, 1))$ is non-increasing in q , overall the two terms' difference is increasing in q , and $q - f_1(q)$ has at most one root; by existence of a fixed point, established earlier, it has exactly one. \square

A.6 Proposition 1

Proof. As a first step, I will show the following: if $w_i^* = U_i(\phi_i(v_i^*, 0), 1)$ for both $i \in I$, then equilibrium is unique if and only if stationary equilibrium is unique. Theorem 1 implies that the outcome $(x^{R,i}, 0)$ such that $x_i^{R,i} = \phi_i(v_i^*, 0)$ is an optimal respondent punishment outcome for player i , and that her optimal proposer punishment therefore has outcome $(x^{P,i}, 0)$ such that $x_i^{P,i} = \pi_i(U_i(\phi_i(v_i^*, 0), 1))$. Using equation (4),

$$\begin{aligned} \phi_i(v_i^*, 0) &= 1 - \pi_j(U_j(1 - \pi_i(U_i(\phi_i(v_i^*, 0), 1)), 1)) \\ &= f_i(\phi_i(v_i^*, 0)), \end{aligned}$$

which, by lemma 9, reveals that $x_i^{R,i} = f_i(x_i^{R,i})$ as well as $x_j^{P,i} \equiv 1 - x_i^{P,i} = 1 - \pi_i(U_i(x_i^{R,i}, 1))$ are the two players' respective proposer shares in one particular stationary equilibrium. If there is a unique stationary equilibrium, then $(x^{R,1}, 0) = (x^{P,2}, 0)$ and $(x^{P,1}, 0) = (x^{R,2}, 0)$ such that $x_1^{R,1} = x_1^{P,2} = \hat{q}_1$ and $x_2^{P,1} = x_2^{R,2} = 1 - \pi_1(U_1(\hat{q}_1, 1))$ for $\hat{q}_1 = \phi_1(v_1^*, 0)$ the unique fixed point of f_1 . Letting $\hat{q}_2 \equiv 1 - \pi_1(U_1(\hat{q}_1, 1))$, theorem 1 then says that $(v_i^*, w_i^*, t_i^*)_{i \in I} = (u_i(\hat{q}_i), U_i(\hat{q}_i, 1), 0)_{i \in I}$, and $A_i^* = \{(\hat{q}_i, 0)\}$, so uniqueness of equilibrium follows from the argument in the proof of corollary 1. This proves sufficiency. Necessity holds trivially.

The second step shows that $w_i^* = U_i(\phi_i(v_i^*, 0), 1)$ follows whenever a player i 's preferences exhibit a weak present bias. This establishes the proposition, because under immediacy stationary equilibrium is indeed unique. The proof of lemma 5 and theorem 1 imply a finite delay \bar{t}_i such that

$$w_i^* = \min \left\{ U_i(\phi_i(v_i^*, t), t + 1) \mid t \in T, t \leq \bar{t}_i \right\},$$

where $U_i(\phi_i(v_i^*, t), t) = v_i^*$ holds true for any $t \leq \bar{t}_i$. A weak present bias then implies that $U_i(\phi_i(v_i^*, 0), 1) \leq U_i(\phi_i(v_i^*, t), t+1)$ for all such t , and hence $w_i^* = U_i(\phi_i(v_i^*, 0), 1)$, proving the claim. \square

A.7 Proposition 2

Proof. The proposition holds trivially for immediate-agreement equilibria. Suppose therefore that (\hat{x}, \hat{t}) with $\hat{t} > 0$ is an equilibrium outcome of game G_1 ; the case of game G_2 follows from mere relabeling. Theorem 1 implies that \hat{x} is an interior division, since

$$0 < \phi_1(v_1^*, \hat{t}) \leq \hat{x}_1 \leq \max\{\phi_2(v_2^*, \hat{t}-1), \phi_2(u_2(0), \hat{t})\} < 1. \quad (22)$$

For every round $n \leq \hat{t} + 1$, define each player i 's reservation share for the rejection value corresponding to agreement on \hat{x} with remaining delay $\hat{t} + 1 - n$: $\pi_i^n \equiv \pi_i(U_i(\hat{x}_i, \hat{t} + 1 - n))$. The inequalities in (22) imply $u_i(\pi_i^n) = U_i(\hat{x}_i, \hat{t} + 1 - n)$ because of $U_i(\hat{x}_i, \hat{t} + 1 - n) \geq u_i(0)$, and the stronger impatience property assumed in the proposition yields that π_i^n is increasing, since $\hat{x}_i > 0$.

Define a play as follows: in round 1, player 1 offers a share of $b_1^1 = 0$, and player 2 accepts with threshold $1 - b_2^1$ such that $b_2^1 = \phi_1(v_1^*, 0)$; in round n such that $1 < n < \hat{t} + 1$, player $P(n)$ offers a share of $b_{P(n)}^n = \frac{1}{2}(b_{P(n)}^{n-1} + \pi_{R(n)}^n)$ and player $R(n)$ accepts with threshold $1 - b_{R(n)}^n$ such that $b_{R(n)}^n = \frac{1}{2}(b_{R(n)}^{n-1} + \pi_{P(n)}^n)$, with the sole exception that $b_1^2 = \phi_2(v_2^*, 0)$; in round $n = \hat{t} + 1$, player $P(n)$ offers a share $b_{P(n)}^n = \hat{x}_{P(n)}$ and player $R(n)$ accepts with threshold $1 - b_{R(n)}^n$ such that $b_{R(n)}^n = \hat{x}_{R(n)}$.

First, verify that each sequence $(b_i^n)_{n=1}^{\hat{t}+1}$ is increasing since $b_i^{n-1} < \pi_j^n$: this is true for $n - 1 = 1$, because $b_i^1 \leq \pi_j^1 < \pi_j^2$, and if it is true for $n - 1 \geq 1$ such that $n < \hat{t} + 1$, it is true for n , because $b_i^n = \frac{1}{2}(b_i^{n-1} + \pi_j^n) < \pi_j^n < \pi_j^{n+1}$. Second, observe that $b_{P(n)}^n < 1 - b_{R(n)}^n$ for all $n < \hat{t} + 1$: since $\pi_1^n + \pi_2^n < 1$ for all such n , this follows from $b_i^n \leq \pi_j^n$; hence this indeed defines a play with outcome (\hat{x}, \hat{t}) .

The final step is to show that this defines equilibrium play. Taken then any strategy profile σ of game G_1 such that $\langle \sigma \rangle$ equals the above play (clearly, one exists) and define the strategy profile $\hat{\sigma} \equiv \sigma^* \left(\sigma \Big|_{(\sigma^{P,i}, \sigma^{R,i})_{i \in I}} \right)$, where $(\sigma^{P,i}, \sigma^{R,i})_{i \in I}$ is an OPC, as in proposition 4, part (i). Hence $\langle \hat{\sigma} \rangle = \langle \sigma \rangle$ and $\hat{\sigma}$ is an equilibrium if and only if there are no profitable one-stage deviations from its play $\langle \hat{\sigma} \rangle$.

Consider then any round $n \leq \hat{t} + 1$ of play $\langle \hat{\sigma} \rangle$. Rejecting an offer $q \geq 1 - b_{R(n)}^n$ is no better than accepting it for $R(n)$, since it yields the minimal credible rejection value $w_{R(n)}^*$

due to optimal punishment, but

$$w_{R(n)}^* \leq U_{R(n)}(\hat{x}_{R(n)}, \hat{t} + 1 - n) = u_{R(n)}(\pi_{R(n)}^n) \leq u_{R(n)}(1 - \pi_{P(n)}^n) \leq u_{R(n)}(1 - b_{R(n)}^n),$$

using that $(\hat{x}, \hat{t} - n)$ is a continuation equilibrium outcome (by assumption), that $\pi_1^n + \pi_2^n \leq 1$ and that $b_{R(n)}^n \leq \pi_{P(n)}^n$; accepting an offer $q < 1 - b_{R(n)}^n$ such that $q \neq b_{P(n)}^n$ is no better than rejecting it, since

$$u_{R(n)}(1 - b_{R(n)}^n) \leq u_{R(n)}(1 - \phi_{P(n)}(v_{P(n)}^*, 0)) = U_{R(n)}(1 - \pi_{P(n)}(w_{P(n)}^*), 1),$$

using that any *responding* player i 's concession is at least $\phi_j(v_j^*, 0)$, by construction, and theorem 1, which shows that continuation with optimal punishment of a proposing player i has rejection value $U_j(1 - \pi_i(w_i^*), 1)$ for respondent j , and that this is equal to $u_j(1 - \phi_i(v_i^*, 0))$; finally, accepting offer $q = b_{P(n)}^n < 1 - b_{R(n)}^n$, which can only be the case for $n < \hat{t} + 1$, is no better than rejecting it, since

$$u_{R(n)}(b_{P(n)}^n) \leq u_{R(n)}(\pi_{R(n)}^n) = U_{R(n)}(\hat{x}_{R(n)}, \hat{t} + 1 - n).$$

Consider then the proposer's incentives, given the respondent's behavior and punishments for deviations: the minimal offer which the respondent accepts equals $b_{R(n)}^n$, which is no greater than $\pi_{P(n)}^n$, whereby

$$u_{P(n)}(b_{R(n)}^n) \leq u_{P(n)}(\pi_{P(n)}^n) = U_{P(n)}(\hat{x}_{P(n)}, \hat{t} + 1 - n),$$

so there is no profitable deviation to any (alternative) accepted offer; any other deviant offer has (rejection) value $U_{P(n)}(\pi_{P(n)}(w_{P(n)}^*), 1)$ which is no greater than $v_{P(n)}^*$ by theorem 1, and since $U_{P(n)}(\hat{x}_{P(n)}, \hat{t} + 1 - n) \geq v_{P(n)}^*$, because (\hat{x}, \hat{t}) is an equilibrium outcome, there is no profitable deviation to a rejected offer either. \square

A.8 Proposition 3

Proof. Omitting player indices due to symmetry, by theorem 1, if there exists an equilibrium with agreement delayed $t > 0$ periods, then $\kappa(t, v^*, v^*) \leq 1$. This implies that $\phi(v^*, t') \leq \frac{1}{2}$

for all $t' < t$, since

$$\begin{aligned} \kappa(t, v^*, v^*) \leq 1 &\Leftrightarrow \underbrace{\phi(v^*, t)}_{\geq \phi(v^*, t-1)} + \underbrace{\max\{\phi(v^*, t-1), \phi(u(0), t)\}}_{\geq \phi(v^*, t-1)} \leq 1 \\ &\Rightarrow \phi(v^*, t-1) \leq \frac{1}{2}, \end{aligned}$$

and $\phi(v^*, \cdot)$ is non-decreasing. Using again theorem 1, recalling also that $\pi(w^*) \leq \phi(v^*, 0)$, $(\frac{1}{2}, t') \in A^*$ follows for all $t' < t$. \square

B Supplementary Material

B.1 Empirical Evidence on Time Preferences

Early evidence on time preferences comes mainly from psychological research and is summarized by [Frederick, Loewenstein, and O’Donoghue \(2002\)](#). They conclude that “virtually every assumption underlying the [exponential-discounting] model has been tested and found to be descriptively invalid in at least some situations” (p. 352). The most compelling refutation of ED is a direct violation of its stationarity axiom. It requires that, holding amounts constant, choice between two delayed rewards depends only on their *relative* delay; yet, a typical experimental subject would, e.g., choose \$45 now over \$52 in 20 days but also \$52 in 130 over \$45 in 110 days ([Kirby and Herrnstein, 1995](#)).

Great effort has gone into uncovering what alternative forms of impatience humans display. For primary rewards (consumption, in a broad sense), a “present bias” towards instantaneous gratification appears rather uncontested; more specifically, this seems to be the consequence of some form of hyperbolic discounting ([Chung and Herrnstein, 1967](#); [Ainslie, 1975](#)), and the present bias is well-captured by the (β, δ) -model of quasi-hyperbolic discounting ([Phelps and Pollak, 1968](#); [Laibson, 1997](#)). The large survey of mainly psychologists’ studies by [Frederick et al. \(2002\)](#) makes this point (for humans as well as other animals), and the more recent evidence from experimental economics confirms it ([McClure, Ericson, Laibson, Loewenstein, and Cohen, 2007](#); [Brown, Chua, and Camerer, 2009](#); [Augenblick, Niederle, and Sprenger, 2014](#)). The implied dynamic preference reversal takes the following form: a person may prefer a larger later reward (LL) over a smaller sooner one (SS) when both are in the future, but once sooner becomes now, she will prefer SS; the rare longitudinal designs of [Ainslie and Haendel \(1983\)](#), [Read and van Leeuwen \(1998\)](#) and [Augenblick, Niederle, and Sprenger \(2014\)](#) indeed find such “impatient switches” for primary rewards.

Conclusion 1. Regarding primary rewards, most subjects exhibit present bias.

By far, most of the empirical evidence on time preferences has been collected from the study of inter-temporal trade-offs in monetary amounts, however. [Cubitt and Read \(2007\)](#) theoretically explain why the link between revealed time preferences over such monetary rewards and those over consumption is likely to be tenuous, given people have access to external credit markets; indeed, they should only reveal the market interest rates which individuals face, in an unbiased manner (cf., however, [Harrison and Swarthout, 2011](#)).³⁸ Indeed, although the overall degree of impatience seems positively correlated across the consumption and money domains ([Reuben, Sapienza, and Zingales, 2010](#)), the evidence regarding present bias is much weaker for monetary rewards. Most remarkably, [Augenblick et al. \(2014\)](#) compare time preferences over effort and money within a single experimental paradigm and find much stronger evidence of present bias regarding effort than money. However, neither is present bias entirely absent, nor is it the only significant bias on the money domain.

Table 1 lists representative experimental studies of *individual* time preferences over monetary rewards not included in [Frederick et al. \(2002\)](#).³⁹ It reveals a striking amount of individual heterogeneity in terms of basic *qualitative* preference properties, apparent in any of the various experimental designs. In particular, all studies—employing very different subject pools as well as methods—find a significant proportion of subjects who violate stationarity or dynamic inconsistency in the opposite direction of present bias, “future bias”. Overall, we obtain the following conclusion.

Conclusion 2. Regarding monetary rewards, subjects split into three similarly sized groups: a third exhibits no bias, a third exhibits present bias, and a third exhibits (near-) future bias.

In longitudinal designs future bias also turns into actual dynamic preference reversals: e.g., 19 out of the 38 participants in [Sayman and Öncüler \(2009, experiment 1\)](#) chose 7 euros the next day over 10 euros in three days, but reversed their choice the next day (when it was 7 euros now v. 10 euros in two days). While the incidence of future bias fluctuates across designs, it appears particularly strong when both the delay to SS and that between SS and LL are relatively short ([Sayman and Öncüler, 2009](#), suggest less than a week, p. 470);

³⁸Relatedly, [Chabris, Laibson, and Schuldt \(2008\)](#) present a whole list of potential confounds in the estimation of utility-discount rates when studying monetary rewards.

³⁹The table is not exhaustive of the large number of recent studies. Rather, it is meant to be representative. However, I exclude studies (e.g., [Read, 2001](#); [Meier and Sprenger, 2015](#)) or parts of studies (e.g., [Eil, 2012](#)) where *qualitative* results would be distorted by the failure to control for utility curvature (see [Andersen, Harrison, Lau, and Rutström, 2008](#)). Moreover, I exclude studies which provide too little information on qualitative individual heterogeneity; e.g., between-subjects designs (e.g., [Rubinstein, 2003](#); [Cohen, Tallon, and Vergnaud, 2011](#)) or fits of mixture models (e.g., [Andersen, Harrison, Lau, and Rutström, 2014](#)) as well as models with individual random effects on parameters (e.g., [Abdellaoui, Bleichrodt, and l’Haridon, 2013](#)).

Study	N	Method	Delays (t/Δ /unit)	% PB	% FB	Remarks
Classifications from Choice						
Ahlbrecht and Weber (1997, part 2)	132	BC	0-24/6/Ms	25	25	Also study loss domain
Ashraf, Karlan, and Yin (2006)	1777	BC*	0, 6/1/Ms	28	20	Clients of rural Philippine banks
Sayman and Öncüler (2009, study 1)	38	BC, L	0-7/2-7/Ds	13	29	Up to 50% FB for t & Δ small
Sayman and Öncüler (2009, study 2a)	72	BC*	0-14/2-14/Ds	6	13	
Attema, Bleichrodt, Rohde, and Wakker (2010)	55	M-D*	0/n.a./Ms	15	65	
Meier and Sprenger (2010)	541	BC	0, 6/1/Ms	36	9	
Takeuchi (2011)	55	M-D	0/n.a./Ds	17	66	Also estimates "concave" discounting
Dohmen, Falk, Huffman, and Sunde (2012)	344	BC	0, 6/6, 12/Ms	34	32	% of those who were classified
Eil (2012, "WTP" task)	95	BC	0, 6/1, 6/Ms	36	43	Comparison for fixed $\Delta = 1$
Eil (2012, "WTW" task)	95	BC-D	0, 6/0.5-48/Ms	31	69	Comparison for fixed $t = 0$
Read, Frederick, and Airoldi (2012, exp. 1)	128	BC, L	0-5/1/Ws	11	11	
Read, Frederick, and Airoldi (2012, exp. 2)	201	BC, L	0, 3/2/Ws	9	10	US residents (email recruits)
Dupas and Robinson (2013)	185	BC*	0, 1/1/Ms	35	20	Poor working Kenyans
Augenblick, Niederle, and Sprenger (2014)	75	CTB, L	0, 3/3, 6/Ws	37	20	Contrast results with those for effort
Giné, Goldberg, Silverman, and Yang (2014)	661	CTB, L	1-61/30/Ds	34	31	Malawian farmers
Halevy (2015, larger stakes)	176	BC, L	0, 4/1/Ws	34	20	Many violate (time-) invariance
Halevy (2015, smaller stakes)	176	BC, L	0, 4/1/Ws	31	17	Many violate (time-) invariance
(β, δ)-Estimations (PB as $\beta < 1$, FB as $\beta > 1$)						
Benhabib, Bisin, and Schotter (2010)	27	M-A	0/3-181/Ds	Median $\beta > 1$		Suggest fixed cost of delay
Andreoni and Sprenger (2012)	97	CTB	0-35/35-98/Ds	Median $\beta > 1$		
Augenblick, Niederle, and Sprenger (2014)	75	CTB, L	0, 3/3, 6/Ws	≥ 33	≥ 17	Contrast results with those for effort
Olea and Strzalecki (2014)	336	BC*	0/1-60/Ys	> 10	> 30	Options have three payoff dates

N: number of participants analyzed.

Method: BC binary choices (*D* means choose delay), *M* matching (choose indifferent amount *A* or delay *D*), *CTB* convex time budgets, *L* longitudinal, * hypothetical.

Delays: ranges if more than two (*t* delay to *SS*; Δ delay *LL* minus delay *SS*; *D* day, *W* week, *M* month, *Y* year).

PB present bias, *FB* future bias; for *L* design choice classification % refers to dynamic preference reversals.

Table 1: Experimental studies of individual time preferences over monetary rewards not included in Frederick et al. (2002) (discounting estimations are only included if they control for curvature and allow for future bias).

such designs have been investigated only more recently but are most relevant for bargaining applications. Salience of the time dimension seems to be another strongly promoting factor (see [Eil, 2012](#)).

For the purpose of using this evidence in the present bargaining application, two difficulties in qualitatively classifying participants' choices should be mentioned. First, due to the discreteness of the choice problems posed, small biases go undetected. A rather extreme example is the longitudinal design of [Read et al. \(2012\)](#): all participants received only a few identical binary choice problems, and 79% of the participants either always chose SS or always chose LL. Second, future bias over a short horizon might very well be coded as present bias when "immediate" ($t = 0$ in the table) does not refer to the delay in *receiving* the payment, as is the case under designs with "front-end delay". This is indeed very common to equalize the credibility of receiving "immediate" and later payments; e.g., in [Meier and Sprenger \(2010\)](#) "no delay" ($t = 0$) means having a mail order sent off the same day, not receiving cash immediately. As important as this procedure is for ruling out confounds regarding present bias, it also makes it impossible to distinguish a bias for immediate gratification from one regarding gratification within only a few days (cf. examples [1](#) and [2](#)). Given that small biases and very-short run attitudes matter for the present application under sufficiently frequent offers, the above conclusion is very conservative regarding the incidence of (near-) future bias.

Finally, a few authors have also investigated the separability into discounting and instantaneous utility functions (on the money domain). [Benhabib et al. \(2010\)](#) find that a fixed-cost of delay in addition to discounting greatly improves their estimation results. [Echenique et al. \(2014\)](#) analyze the data of [Andreoni and Sprenger \(2012\)](#) and reject separability for almost one half of the participants on the basis of its revealed preference implications; using their own method and data, [Ericson and Noor \(2015\)](#) reject separability for almost 70% of their participants.

Conclusion 3. On the money domain, the separability of preferences into discounting and instantaneous utilities tends to be violated when tested.

Why time preferences over monetary rewards reveal various biases when, theoretically, they should not reveal any bias at all is not well understood. Current work in experimental economics (e.g., [Carvalho, Prina, and Sydnor, 2014](#); [Ambrus, Ásgeirsdóttir, Noor, and Sándor, 2015](#); [Carvalho, Meier, and Wang, 2015](#)) explores the role of fluctuations in liquidity (present or anticipated), which has theoretically been shown to be potentially important ([Noor, 2009](#); [Gerber and Rohde, 2010, 2015](#)). The reason may also have to do with how the human brain processes monetary rewards: e.g., the recent meta-analysis by [Sescousse, Caldú, Segura, and Dreher \(2013\)](#) finds monetary rewards to engage areas of the brain which

are active also for different primary rewards, but at the same time also a distinct, evolutionarily more recent, one.⁴⁰ Money may therefore act, at least partially, as a learned cue for immediate or near-future consumption (cf. Fudenberg and Levine, 2006, pp. 1457-8); in a similar vein, it may produce anticipatory utility, which can lead to initially concave discounting (Loewenstein, 1987).

In any case, given the three conclusions of this section, I impose only minimal assumptions on time preferences in the present study. Essentially, these are only that more is better for any given delay, and sooner is better for any given (positive) amount. Thus I cover the entire spectrum of suggested time preferences and can investigate differential implications of very broad qualitative features, in particular present and (near-) future bias.⁴¹

B.2 Multiplicity under Exponential Discounting

The following example is one of ED that exhibits multiple stationary equilibria and, possibly, delay, due to a violation of immediacy. It was presented already by Rubinstein (1982, concl. I), but to the best of my knowledge, its set of equilibria has not yet been explicitly characterized.

Example 4. Let the two parties' preferences be given by $U_i(q, t) = q - ct$, for $c \in (0, 1)$. Due to preference symmetry, player indices are omitted in what follows. The preferences are covered by assumption 1 once $U(0, \infty) \equiv -\infty$ is specified; in particular, impatience property (3.c) is satisfied: $U(1, t)$ tends to minus infinity, whereas $u(0) = 0$.⁴² In the assumed absence of uncertainty, they actually satisfy ED, albeit with “strongly” convex instantaneous utility: $U(q, t) = \ln(\delta^t u(q))$ for $\delta \equiv \exp(-c)$ and $u(q) \equiv \exp(q)$. Hence they exhibit a weak present bias but violate immediacy (increasing shares by the same amount leaves indifferent).⁴³

This results in a multiplicity of stationary equilibrium: any $q \in [c, 1]$ is a proposer's equilibrium share in some stationary equilibrium (with immediate agreement, of course). Applying the characterization of theorem 1, $v^* = c$ and $w^* = 0$, where both of these minimal proposer and rejection values correspond to a player's least preferred stationary equilibrium.

⁴⁰In prior work Kable and Glimcher (2007) had demonstrated that individual time preferences over monetary rewards have a neural correlate.

⁴¹Some findings (e.g., Read, 2001; Rubinstein, 2003; Dohmen et al., 2012) suggest that transitivity may yet be violated in comparisons across different delays (cf. Manzini and Mariotti, 2007). Ok and Masatlioglu (2007) propose a model of “relative” discounting, which maintains separability but accommodates those violations. Fn. 14 explains why also these preferences are covered here.

⁴² U violates the requirement of assumption 1 that $U(0, \infty) \in \mathbb{R}$, but the positive monotonic transformation $\exp(U)$ represents the same preferences and satisfies also this property.

⁴³One may interpret such preferences as there being a cost to bargaining. To justify the non-negativity of each player's share in any proposal, assume then that players have an “outside option” of leaving the bargaining table forever, which is equivalent to obtaining a zero share immediately.

Using these two least preferred stationary equilibria as optimal punishments, non-stationary delay equilibria can be constructed, and equation 6 offers a formula to compute the maximal such delay for any $c \in (0, 1)$:

$$\begin{aligned}
\kappa(t, c, c) &= \min\{c + ct, 1\} + \min\{ct, 1\} \\
&= \begin{cases} (2t + 1)c & t \leq \frac{1-c}{c} \\ 1 + ct & \frac{1-c}{c} \leq t \leq \frac{1}{c} \\ 2 & \frac{1}{c} \leq t \end{cases} \\
\Rightarrow t^* &= \sup\{t \in T \mid \kappa(t, c, c) \leq 1\} \\
&= \max\left\{t \in T \mid t \leq \frac{1}{2} \cdot \frac{1-c}{c}\right\} \\
&= \left\lfloor \frac{1}{2} \cdot \frac{1-c}{c} \right\rfloor.
\end{aligned}$$

For instance, if $c = \frac{1}{100}$, so that the cost per bargaining round equals one percent of the surplus per player, then the maximal equilibrium delay is 49 periods, with an associated efficiency loss of 98 percent of the surplus. To determine the values of c for which delayed agreement is an equilibrium outcome, simply solve $\kappa(1, c, c) \leq 1$ for c , yielding $c \leq \frac{1}{3}$. The set of equilibrium divisions with a given delay $t \leq t^*$ in game G_1 equals $\{x \in X \mid c + ct \leq x_1 \leq 1 - ct\}$ and is monotonically shrinking in t .

B.3 Unbounded Equilibrium Delay

The following example slightly modifies example 3 to exhibit unbounded equilibrium delay.

Example 5. Let the two players' preferences be symmetrically given by $U_i(q, t) = d(t) \cdot q$ with

$$d(t) = \begin{cases} \delta^t & t \leq \tau \\ \gamma\delta^{\tau+1} & t > \tau \end{cases}, \quad (\delta, \gamma) \in (0, 1)^2 \text{ and } \tau > 0.$$

Due to preference symmetry, the player subscript is again omitted in what follows.

The difference to example 3 is that delays beyond horizon $\tau + 1$ are not discounted. Observe, however, that $\Delta(t)$ equals δ for all $t \leq \tau$ and $\gamma\delta$ for all $t > \tau$, exactly as in example 3. Hence, whenever there is an equilibrium in which agreement is delayed by τ periods, $v^* = \frac{1-\delta}{1-\gamma\delta^2}$ and $w^* = \gamma\delta v^*$, as was found there.

The absence of discounting beyond a delay of $\tau + 1$ periods implies that equilibrium delay

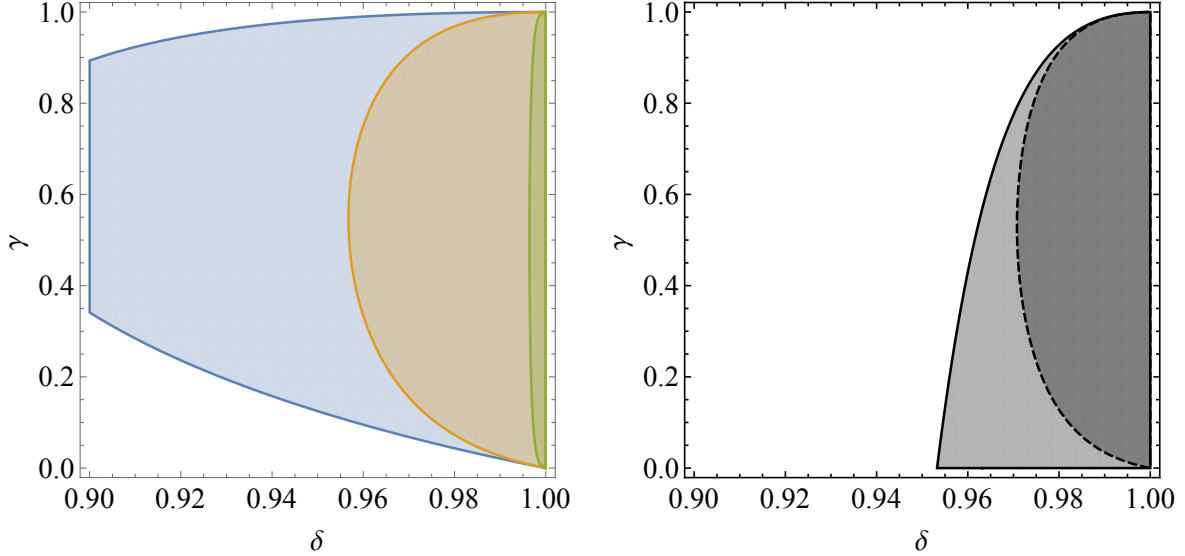


Figure 3: Graphs regarding unbounded equilibrium delay in example 3. The panel on the left shows the parametric regions (δ, γ) such that equilibrium delay is unbounded for three given values of τ , which are 1 (blue, brown and green), 25 (brown and green) and 1000 (green). The panel on the right illustrates how the respective parametric regions for existence of delay equilibria (superset, bounded by solid line) and unbounded equilibrium delay (subset, bounded by dashed line) are related for the case of $\tau = 50$.

is unbounded if and only if $1 \geq \kappa(\tau + 2, v^*, v^*) = 2 \frac{v^*}{\gamma \delta^{\tau+1}}$, which reduces to

$$\delta^\tau \geq \frac{2}{\gamma \delta} \cdot \frac{1 - \delta}{1 - \gamma \delta^2} \quad (23)$$

after substituting for v^* . Notice that this inequality is more stringent than example 3's inequality (8), which shows when delay equilibria exist; in particular, $\gamma > 0$ is here required. Indeed, γ might be too low: despite existence of an equilibrium with delay τ , which fully determines the optimal punishments, proposing players would then require too large a compensation for longer delays, as those would involve additional discounting through γ . Nonetheless, for any given $\tau > 0$ and $\gamma < 1$, there again exist large enough values of δ such that also inequality (23) is satisfied, with the set of parameters γ and τ such that equilibrium delay is unbounded expanding as δ increases. Figure 3 illustrates this.

B.4 Other Sources of Dynamic Inconsistency

B.4.1 Imperfect Altruism and Inter-generational Bargaining

Suppose there are two communities with access to a productive resource. They decide over how to share it by means of bargaining over usage rights. As long as these have not been

settled, each period some surplus, normalized to one, is forgone. Upon failure to agree, both communities nominate a new delegate to engage in the bargaining on their behalf. Once they agree, however, all future generations enjoy the agreed surplus every period. I now sketch a simple version of this general problem with imperfect altruism of community members towards future ones.

Denote the two communities by $i \in I$. Each has a population of two members in any period $n \in \mathbb{N}$: a young member (i, y) and an old member (i, o) . Each such member lives for two periods, where in the first half of her life she is called young, and in the second half she is called old; reproduction is therefore such that the old member at time t is replaced by a young one at time $t + 1$. All young members of a community are identical, and so are all old ones (though they live at different times).

Each community member discounts future payoffs exponentially (discount factor δ_i) and is altruistic towards all future generations, but with an extra discount (factor γ_i) for *payoffs* beyond her own lifetime: at any point in time, for any agreement with a delay of $t \in T$ periods, where community i 's share is equal to q ,

$$U_{i,o}(q, t) = \begin{cases} q + \gamma_i \delta_i \frac{1}{1-\delta_i} q & t = 0 \\ \gamma_i \delta_i^t \frac{1}{1-\delta_i} q & t > 0 \end{cases}, \quad U_{i,y}(q, t) = \begin{cases} q + \delta_i U_{i,o}(q, 0) & t = 0 \\ \delta_i U_{i,o}(q, t-1) & t > 1 \end{cases}, \quad (\delta_i, \gamma_i) \in (0, 1)^2.$$

It is straightforward to show that these preferences can be represented as $U_{i,g}(q, t) = d_{i,g}(t) \cdot q$, $g \in \{y, o\}$, such that

$$d_{i,o}(q, t) = \begin{cases} 1 & t = 0 \\ \alpha_i \delta_i^t & t > 0 \end{cases}, \quad d_{i,y}(q, t) = \begin{cases} 1 & t = 0 \\ \beta_i \delta_i & t = 1, \\ \alpha_i \beta_i \delta_i^t & t > 1 \end{cases}, \quad \left\{ \begin{array}{l} \alpha_i \equiv \frac{\gamma_i}{(1-\delta_i) + \gamma_i \delta_i} \\ \beta_i \equiv \frac{(1-\delta_i) + \gamma_i \delta_i}{(1-\delta_i^2) + \gamma_i \delta_i^2} \end{array} \right\}.$$

Notice that $0 < \delta_i, \gamma_i < 1$ implies $0 < \alpha_i < \beta_i < 1$. Since $\alpha_i < 1$, old members' preferences exhibit a weak present bias, discounting the first period of delay with factor $\alpha_i \delta_i$ and thus more heavily than any other, which is discounted with constant factor δ_i . However, young members discount the second period of delay more than the first, namely with a factor $\alpha_i \delta_i$ less than that for the first one, which equals $\beta_i \delta_i$; all periods further in the future are discounted less, with constant factor δ_i .

Suppose that at the beginning of each period n , a round of bargaining takes place, where if n is odd, community 1's delegate gets to propose, and otherwise it is community 2's delegate. Since this means alternating offers in terms of communities, any bargaining protocol where

each community either sends only young or only old members as delegates to the bargaining table results in a stationary game of the type analyzed in this paper, and the results of this paper, in particular characterization theorem 1, apply in a straightforward manner.

B.4.2 Non-linear Probability Weighting and Bargaining Under the Shadow of Breakdown Risk

One motive for impatience in the sense of discounting future payoffs is uncertainty, such as mortality risk. Halevy (2008) and Saito (2015) show how dynamically inconsistent discounting can be related to non-linear probability weighting.

Suppose that two parties bargain over a surplus of normalized size one, with alternating offers, where, after each round without agreement, there is a constant probability $1-p \in (0, 1)$ that bargaining (exogenously) breaks down before the next round, leaving players without any surplus. Applying the aforementioned authors' results in this context, in any round n a player i 's preferences over shares q that is agreed upon with delay $t \in T$ have the following representation, which—for the sake of simplicity—involves breakdown risk as the sole source of impatience:

$$U_i(q, t) = g_i(p^t) u_i(q),$$

where $g_i : [0, 1] \rightarrow [0, 1]$ is a so-called probability-weighting function, assumed continuous and increasing from $g_i(0) = 0$ to $g_i(1) = 1$, and $u_i : [0, 1] \rightarrow \mathbb{R}_+$ is an instantaneous utility function, assumed continuous and increasing from $u_i(0) = 0$. These preferences are dynamically consistent if and only if g_i is the identity, in which case i maximizes expected utility.

Redefining, for a given “survival rate” p , $g_i(p^t) \equiv d_i(t)$, all results of this paper can be applied in a straightforward manner. Thus the players' probability weighting, which determines their dynamic inconsistency, can be related to the set of bargaining outcomes.

References for Appendix

- Abdellaoui, M., H. Bleichrodt, and O. l'Haridon (2013). Sign-dependence in intertemporal choice. *Journal of Risk and Uncertainty* 47, 225–253.
- Ahlbrecht, M. and M. Weber (1997). An empirical study on intertemporal decision making under risk. *Management Science* 43(6), 813–826.
- Ainslie, G. (1975). Specious reward: A behavioral theory of impulsiveness and impulse control. *Psychological Bulletin* 82(4), 463–496.
- Ainslie, G. and V. Haendel (1983). The motives of the will. In E. Gottheil, K. Durley,

- T. Skodola, and H. Waxman (Eds.), *Etiologic Aspects of Alcohol and Drug Abuse*, pp. 119–140. Charles C. Thomas.
- Ambrus, A., T. L. Ásgeirsdóttir, J. Noor, and L. Sándor (2015, March). Compensated discount functions - an experiment on the influence of expected income on time preferences.
- Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2008). Eliciting risk and time preferences. *Econometrica* 76(3), 583–618.
- Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2014). Discounting behavior: A reconsideration. *European Economic Review* 71(1), 15–33.
- Andreoni, J. and C. Sprenger (2012). Estimating time preferences from convex budgets. *The American Economic Review* 102(7), 3333–3356.
- Ashraf, N., D. Karlan, and W. Yin (2006). Tying Odysseus to the mast: Evidence from a commitment savings product in the Philippines. *The Quarterly Journal of Economics* 121(2), 635–672.
- Attema, A. E., H. Bleichrodt, K. I. M. Rohde, and P. P. Wakker (2010). Time-tradeoff sequences for analyzing discounting and time inconsistency. *Management Science* 56(11), 2015–2030.
- Augenblick, N., M. Niederle, and C. Sprenger (2014, May). Working over time: Dynamic inconsistency in real effort tasks.
- Benhabib, J., A. Bisin, and A. Schotter (2010). Present-bias, quasi-hyperbolic discounting, and fixed costs. *Games and Economic Behavior* 69(2), 205–223.
- Brown, A. L., Z. E. Chua, and C. F. Camerer (2009). Learning and visceral temptation in dynamic saving experiments. *The Quarterly Journal of Economics* 124(1), 197–231.
- Carvalho, L. S., S. Meier, and S. W. Wang (2015, May). Poverty and economic decision-making: Evidence from changes in financial resources at payday.
- Carvalho, L. S., S. Prina, and J. Sydnor (2014, August). The effect of saving on risk attitudes and intertemporal choices.
- Chabris, C. F., D. I. Laibson, and J. P. Schuldt (2008). Intertemporal choice. In S. N. Durlauf and L. E. Blume (Eds.), *The New Palgrave Dictionary of Economics* (2 ed.). Palgrave Macmillan.
- Chung, S.-H. and R. J. Herrnstein (1967). Choice and delay of reinforcement. *Journal of the Experimental Analysis of Behavior* 10(1), 67–74.
- Cohen, M., J.-M. Tallon, and J.-C. Vergnaud (2011). An experimental investigation of imprecision attitude and its relation with risk attitude and impatience. *Theory and Decision* 71(1), 81–109.
- Cubitt, R. P. and D. Read (2007). Can intertemporal choice experiments elicit time preferences for consumption? *Experimental Economics* 10(4), 369–389.
- Dohmen, T., A. Falk, D. Huffman, and U. Sunde (2012, February). Interpreting time horizon effects in inter-temporal choice. IZA Discussion Paper No. 6385.

- Dupas, P. and J. Robinson (2013). Why don't the poor save more? evidence from health savings experiments. *American Economic Review* 103(4), 1138–1171.
- Echenique, F., T. Imai, and K. Saito (2014, May). Testable implications of quasi-hyperbolic and exponential time discounting.
- Eil, D. (2012, June). Hypobolic discounting and willingness-to-wait. GMU Working Paper in Economics No. 12-28.
- Ericson, K. M. and J. Noor (2015, April). Delay functions as the foundation of time preference: Testing for separable discounted utility.
- Frederick, S., G. Loewenstein, and T. O'Donoghue (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature* 40(2), 351–401.
- Fudenberg, D. and D. Levine (2006). A dual-self model of impulse control. *The American Economic Review* 96(5), 1449–1476.
- Gerber, A. and K. I. M. Rohde (2010). Risk and preference reversals in intertemporal choice. *Journal of Economic Behavior and Organization* 76(3), 654–668.
- Gerber, A. and K. I. M. Rohde (2015). Eliciting discount functions when baseline consumption changes over time. *Journal of Economic Behavior and Organization* 116, 56–64.
- Giné, X., J. Goldberg, D. Silverman, and D. Yang (2014, May). Revising commitments: Field evidence on the adjustment of prior choices.
- Halevy, Y. (2008). Strotz meets Allais: Diminishing impatience and the certainty effect. *The American Economic Review* 98(3), 1145–1162.
- Halevy, Y. (2015). Time consistency: Stationarity and time invariance. *Econometrica* 83(1), 335–352.
- Harrison, G. W. and J. T. Swarthout (2011, May). Can intertemporal choice experiments elicit time preferences for consumption? Yes.
- Kable, J. W. and P. W. Glimcher (2007). The neural correlates of subjective value during intertemporal choice. *Nature Neuroscience* 10(12), 1625–1633.
- Kirby, K. N. and R. J. Herrnstein (1995). Preference reversals due to myopic discounting of delayed reward. *Psychological Science* 6(2), 83–89.
- Laibson, D. (1997). Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics* 112(2), 443–478.
- Loewenstein, G. (1987). Anticipation and the valuation of delayed consumption. *The Economic Journal* 97(387), 666–684.
- Manzini, P. and M. Mariotti (2007). Sequentially rationalizable choice. *The American Economic Review* 97(5), 1824–1839.
- McClure, S. M., K. M. Ericson, D. I. Laibson, G. Loewenstein, and J. D. Cohen (2007). Time discounting for primary rewards. *The Journal of Neuroscience* 27(21), 5796–5804.
- Meier, S. and C. Sprenger (2010). Present-biased preferences and credit card borrowing. *American Economic Journal: Applied Economics* 2(1), 193–210.

- Meier, S. and C. D. Sprenger (2015). Temporal stability of time preferences. *The Review of Economics and Statistics* 97(2), 273–286.
- Noor, J. (2009). Hyperbolic discounting and the standard model: Eliciting discount functions. *Journal of Economic Theory* 144(5), 2077–2083.
- Ok, E. A. and Y. Masatlioglu (2007). A theory of (relative) discounting. *Journal of Economic Theory* 137(1), 214–245.
- Olea, J. L. M. and T. Strzalecki (2014). Axiomatization and measurement of quasi-hyperbolic discounting. *The Quarterly Journal of Economics* 129(3), 1449–1499.
- Phelps, E. S. and R. A. Pollak (1968). On second-best national saving and game-equilibrium growth. *The Review of Economic Studies* 35(2), 185–199.
- Read, D. (2001). Is time-discounting hyperbolic or subadditive? *Journal of Risk and Uncertainty* 23(1), 5–32.
- Read, D., S. Frederick, and M. Airoldi (2012). Four days later in Cincinnati: Longitudinal tests of hyperbolic discounting. *Acta Psychologica* 140(2), 177–185.
- Read, D. and B. van Leeuwen (1998). Predicting hunger: The effects of appetite and delay on choice. *Organizational Behavior and Human Decision Processes* 76(2), 189–205.
- Reuben, E., P. Sapienza, and L. Zingales (2010). Time discounting for primary and monetary rewards. *Economics Letters* 106(2), 125–127.
- Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica* 50(1), 97–109.
- Rubinstein, A. (2003). “Economics and psychology”? The case of hyperbolic discounting. *International Economic Review* 44(4), 1207–1216.
- Saito, K. (2015, April). A relationship between risk and time preferences.
- Sayman, S. and A. Öncüler (2009). An investigation of time inconsistency. *Management Science* 55(3), 470–482.
- Sescousse, G., X. Caldú, B. Segura, and J.-C. Dreher (2013). Processing of primary and secondary rewards: A quantitative meta-analysis and review of human functional neuroimaging studies. *Neuroscience and Biobehavioral Reviews* 37(4), 681–696.
- Takeuchi, K. (2011). Non-parametric test of time consistency: Present bias and future bias. *Games and Economic Behavior* 71(2), 456–478.